Problem Set #6 Answer Key

Economics 808: Macroeconomic Theory

Fall 2004

1 Overlapping generations with Cobb-Douglas production

a) The Lagrangean is:

$$L = \ln c_{1,t} + \beta \ln c_{2,t+1} + \lambda_t \left(w_t - c_{1,t} - k_{t+1} - b_{t+1}\right) + \theta_t \left(r_{t+1}k_{t+1} + R_{t+1}b_{t+1} - c_{2,t+1}\right)$$

So the first order conditions are:

$$\begin{aligned}
\frac{1}{c_{1,t}} - \lambda_t &= 0 \\
\frac{\beta}{c_{2,t+1}} - \theta_t &= 0 \\
-\lambda_t + \theta_t r_{t+1} &= 0 \\
-\lambda_t + \theta_t R_{t+1} &= 0
\end{aligned}$$

b) First we note that the first order conditions imply:

$$c_{2,t+1} = \beta r_{t+1} c_{1,t}$$

Since $c_{2,t+1} = r_{t+1}k_{t+1}$, we have:

$$k_{t+1} = \beta c_{1,t}$$

Since $k_{t+1} + c_{1,t} = w_t$, we have

$$k_{t+1} = \frac{\beta}{1+\beta} w_t$$

so:

$$s = \frac{\beta}{1+\beta}$$

c) Steady state consumption is:

$$c_\infty = A k_\infty^\alpha - k_\infty$$

The golden rule is the value of k_∞ that maximizes c_∞. We take first order condtions and get:

$$k_{GR} = (\alpha A)^{\frac{1}{1-\alpha}}$$

d) The savings rate of young worker t is:

$$s_t = \frac{k_{t+1}}{w_t}$$

At the golden rule, this is:

$$s_{GR} = \frac{k_{GR}}{(1-\alpha)Ak_{GR}^{\alpha}} = \frac{\alpha}{1-\alpha} \tag{1}$$

e) The equilibrium savings rate will exceed the golden rule level if

$$\frac{\beta}{1+\beta} > \frac{\alpha}{1-\alpha}$$

Notice that the equilibrium savings rate is increasing in β, and the golden rule savings rate is increasing in α.

f) In order to find a Pareto dominant allocation, we must find an allocation that leaves everyone at least as well off and leaves someone better off.

Let s_E be the equilibrium savings rate and let s_{GR} be the golden rule savings rate. Since the planner is free to allocate consumption across the old and young, we only need to show that we can find an allocation that gives at least as much *total* consumption in each time period and more total consumption in some time period.

My proposed improvement over the equilibrium allocation is to save s_E of the worker's income up until k_t reaches k_{GR}. Once k_{GR} is reached, save s_{GR} from then on. In the first period that s_{GR} is the savings rate, consumption is higher than in the equilibrium allocation, and it is higher in every subsequent period. Before this period, consumption is the same as in the equilibrium allocation. This allocation Pareto dominates the equilibrium allocation, therefore the equilibrium allocation is not Pareto efficient.

Notice that we needed to wait until the golden rule capital stock was reached to impose the golden rule savings rate. Otherwise, consumption might not be as high for some agents in the first few periods, and our allocation would fail to be Pareto dominant.

g) Dynamic inefficiency is more likely if capital's share is low and if young people care more about their future consumption.

h) The savings rate is constant. As a result, we can write k_{t+1} as a function of k_t. Therefore the equilibrium is unique.

i) The model doesn't exhibit history dependence because we found a unique steady state (not counting the zero-capital steady state).

2 Overlapping generations with linear preferences

a) Worker t's problem is to select $c_{1,t}$, $c_{2,t+1}$ and k_{t+1} to maximize:

$$U_t = c_{1,t} + \beta c_{2,t+1}$$

subject to the constraints:

$$\begin{aligned} c_{1,t} + k_{t+1} + b_{t+1} &\leq w_t + \pi_t \\ c_{2,t+1} &\leq r_{t+1}k_{t+1} + R_{t+1}b_{t+1} \\ k_{t+1} &\geq 0 \\ c_{1,t} &\geq 0 \\ c_{2,t+1} &\geq 0 \end{aligned}$$

Notice that we need to add a nonnegativity constraint on consumption here.

The firm's problem is to select k_t and L_t to maximize:

$$\pi_t = Ak_t^\alpha L_t^{1-\alpha} - r_t k_t - w_t L_t$$

An equilibrium in this economy is a sequence of prices $\{w_t, r_t, R_t\}$ and allocations $\{c_{1,t}, c_{2,t}, k_t, b_t, \pi_t, L_t\}$ such that:

1. Taking prices and firm profits as given, $c_{1,t}$, $c_{2,t+1}$, and k_{t+1} solve worker t's problem.
2. Taking prices as given, k_t and L_t solve the firm's problem.
3. Markets clear, i.e., $L_t = 1$, $b_t = 0$, and $c_{1,t} + c_{2,t} + k_{t+1} = Ak_t^\alpha$.

b) Suppose the worker has one unit of output when young. He can consume it when young, gaining one unit of utility, or save it, gaining βr_{t+1} units of utility The young worker's savings rate is, therefore:

$$s(r_{t+1}) = \begin{cases} 0 & \beta r_{t+1} < 1 \\ s \in [0,1] & \beta r_{t+1} = 1 \\ 1 & \beta r_{t+1} > 1 \end{cases}$$

Clearly, savings is increasing in the interest rate.

c) The evolution of the capital stock will be governed by the following difference equation:

$$k_{t+1} = \min\{(1-\alpha)Ak_t^\alpha, (\alpha A\beta)^{\frac{1}{1-\alpha}}\}$$

d) The steady state capital stock is:

$$k_\infty = (\alpha A\beta)^{\frac{1}{1-\alpha}}$$

3 Overlapping generations with Leontief preferences

a)

b) Leontief preferences imply that $c_{1,t} = c_{2,t+1}$. Since $k_{t+1} = w_t - c_{1,t}$ and $c_{2,t+1} = r_{t+1}k_{t+1}$ we can find that

$$k_{t+1} = \frac{1}{1 + r_{t+1}} w_t$$

The savings rate is thus

$$s(r_{t+1}) = \frac{1}{1 + r_{t+1}}$$

which is clearly decreasing in the interest rate.

c) Since production is Cobb-Douglas, $w_t = (1-\alpha)Ak_t^{\alpha}$ and $r_t = \alpha A k_t^{\alpha-1}$. Substituting in, we get:

$$(1 + \alpha A k_{t+1}^{\alpha-1})k_{t+1} = (1-\alpha)Ak_t^{\alpha}$$

d) We find the steady state capital stock by setting $k_{t+1} = k_t = k_\infty$ in the above equation. This yields:

$$k_\infty = ((1-2\alpha)A)^{\frac{1}{1-\alpha}}$$