Problem Set #7
Corrected version 11/15/2004

Economics 808: Macroeconomic Theory

Fall 2004

This problem set is not conceptually very difficult, but it is time-consuming and it is very easy to make costly mistakes in algebra. I suggest working in this problem set in groups, and checking answers with one another. I *will* check to make sure you have done everything, including the Excel portion.

1 A very simple RBC model

Consider the following extremely simple RBC model. It is a little too simple, as it assumes 100% depreciation and inelastic labour supply, but will give us a little practice with some of the methods we might want to use with better models.

The consumer has expected utility function:

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \ln C_t \tag{1}$$

subject to the budget constraint:

$$C_t + K_{t+1} \leq w_t + r_t K_t + \pi_t$$

plus the usual other conditions.

The firm has the Cobb-Douglas production function $Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha}$ where $Z_t = e^{z_t}$ and :

$$z_t = \rho z_{t-1} + \epsilon_t$$

where $\rho \in [0, 1)$ and ϵ_t is an IID sequence of random variables from the $N(0, \sigma_\epsilon^2)$ distribution.

a) Define the consumer's problem, the firm's problem, and equilibrium in this economy.

b) Use first order conditions to find a system of 3 equations relating K_t, Z_t and C_t to K_{t+1}, Z_{t+1} and C_{t+1}. There may be some conditional expectations in these equations.

c) The nonstochastic steady state of this economy is the steady state that would be reached if z_t were zero for all t. Find the nonstochastic steady state (C^*, K^*, Z^*) for this economy.

d) Let $c_t \equiv \ln \frac{C_t}{C^*}$ and let $k_t \equiv \ln \frac{K_t}{K^*}$. Use first-order Taylor series approximations to find a system of 3 linear equations relating (z_t, c_t, k_t) to $(E_t z_{t+1}, E_t c_{t+1}, E_t k_{t+1})$.

e) Note that k_{t+1} is chosen directly at time t so $E_t k_{t+1} = k_{t+1}$. Let

$$X_t \equiv \begin{bmatrix} k_t \\ c_t \\ z_t \end{bmatrix}$$

Write the log-linearized system in the matrix form:

$$AE_t X_{t+1} = BX_t$$

f) Multiply both sides of this equation by A^{-1}, then add the vector $X_{t+1} - E_t X_{t+1}$ to both sides. This will yield an equation of the form:

$$X_{t+1} = CX_t + (X_{t+1} - E_t X_{t+1})$$

Find the matrix C.

g) The next step is a little tougher. We know that $k_{t+1} - E_t k_{t+1} = 0$ and that $z_{t+1} - E_t z_{t+1} = \epsilon_{t+1}$. Let $v_{t+1} = c_{t+1} - E_t c_{t+1}$:

$$X_{t+1} - E_t X_{t+1} = \begin{bmatrix} 0 \\ v_{t+1} \\ \epsilon_{t+1} \end{bmatrix}$$

We know that the transversality condition implies that the system will eventually return to the steady state from any initial condition. In other words

$$\lim_{k\to\infty} E_t(X_{t+k}) = \lim_{k\to\infty} C^k X_t = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As we learned earlier, the transversality condition will pin down a unique value of c_t for a given state (k_t, z_t). To find that value, first calculate the eigenvector decomposition $C = Q\Lambda Q^{-1}$, where Λ is the diagonal matrix of C's eigenvalues, and Q is the matrix constructed from the corresponding eigenvectors (i.e., the first column of Q contains the eigenvector associated with the eigenvalue located in the first row, first column of Λ. Note that $C^k = Q\Lambda^k Q^{-1}$, so the above equation can be written:

$$\lim_{k\to\infty} C^k X_t = Q \lim_{k\to\infty} \Lambda^k Q^{-1} X_t = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since Λ is diagonal, Λ^k is just the diagonal matrix of each of the eigenvalues taken to the k power. In this particular problem, two of the eigenvalues are less than one in absolute value, and one eigenvalue is greater than one in absolute value. Suppose (without loss of generality) that the top row in Λ contains the unstable eigenvalue (i.e., the eigenvalue greater than one in absolute value). Then in order for the system to not "explode" we need the ith row in

$$Q^{-1} X_t$$

to equal exactly zero. Find Q^{-1} and use the above condition to solve for c_t as a linear function of (k_t, z_t). Feel free to use Maple, Mathematica, or any other resource to calculate the eigenvalues and eigenvectors of C.

h) Calibrate the model. Feel free to use any reasonable argument for setting parameter values.

i) Use Excel or any other software program to simulate the model, and then estimate variances and first-order autocorrelations for c_t, k_t, y_t, x_t, z_t, as well the correlation of each of these variables with y_t. Report these quantities, and email me the excel file (or Matlab/Maple/Mathematica/Gauss program) you used to generate them.

Be sure to generate a large enough number of observations to estimate these quantities accurately (10,000 is probably enough), and be sure to throw away the first few hundred observations (so that you are characterizing the long-run distribution).

j) Compare your results to the corresponding results from the US data described in class. In what ways does this simplified model do better or worse than the baseline RBC model? Why?