Chapter 4

The overlapping generations (OG) model

4.1 The model

Now we will briefly discuss a macroeconomic model which has most of the important features of the RA model, but one - people die. This small concession to reality will have a big impact on implications.

Recall that the RA model had a few special characteristics:

1. An unique equilibrium exists.
2. The economy follows a deterministic path.
3. The equilibrium is Pareto efficient.

The overlapping generations model will not always have these characteristics. So my presentation of this model will have two basic goals: to outline a model that appears frequency in the literature, and to use the model to illustrate some features that appear in other models and that some macroeconomists think are important to understanding business cycles.

4.1.1 The model

Time, information, and demography

Discrete, indexed by t.

In each period, one worker is born. The worker lives two periods, so there is always one young worker and one old worker.

We're going to assume for the time being that workers have perfect foresight, so we'll dispense with the E_t's.

Workers

Each worker is identified with the period of her birth - we'll call the worker born in period t "worker t". Let $c_{1,t}$ be worker t's consumption when young, and let $c_{2,t+1}$ be her consumption when old. She only cares about her own consumption. Her utility is:

$$U_t = u(c_{1,t}) + \beta u(c_{2,t+1}) \tag{4.1}$$

The worker is born with no capital or bond holdings. When young, she supplies one unit of labor inelastically to the market and receives wage w_t. She can consume or save (buy capital).

$$c_{1,t} + k_{t+1} + b_{t+1} \leq w_t \tag{4.2}$$

When old, she is retired and lives off of her capital and bond income:

$$c_{2,t+1} \leq r_{t+1}k_{t+1} + R_{t+1}b_{t+1} \tag{4.3}$$

At time zero, there is an old generation (worker -1) with an exogenous capital stock $k_0 > 0$ and bond holdings $b_0 = 0$. Notice that the only variable with a subscript to denote generation is consumption, since only the young work and invest, and only the old receive investment income.

Notice that the old are not allowed to issue bonds. Also notice that the depreciation rate is 100%. This is for two reasons: first, if the depreciation was lower we would have to specify how capital is passed on from the old to the young; and second, that since each period is supposedly a whole generation of 30-40 years 100% depreciation is empirically plausible.

Firms

Firms are just like before. Each firm chooses k_t, L_t to maximize:

$$F(k_t, L_t) - r_t k_t - w_t L_t \tag{4.4}$$

where F is a neoclassical production function.

Equilibrium

An equilibrium is a sequence of prices $\{r_t, w_t, R_t\}$ and allocations $\{c_{1,t}, c_{2,t}, k_t, L_t, b_t\}$ such that:

1. Taking prices as given, the allocations k_{t+1}, $c_{1,t}$, $c_{2,t+1}$ and b_{t+1} solve worker t's utility maximization problem for all $t \geq 0$.

2. Taking prices as given, the allocations k_t and L_t solve the firm's profit maximization problem.

3. Markets clear, i.e., for all t, we have $L_t = 1$, $b_t = 0$, and $c_{1,t}+c_{2,t}+k_{t+1} = F(k_t, 1)$

4.1.2 Solving the model

Consumers

The Lagrangean for the worker t's utility maximization problem is:

$$\begin{aligned} L &= u(c_{1,t}) + \beta u(c_{2,t+1}) \\ &+ \lambda_t(w_t - c_{1,t} - k_{t+1} - b_{t+1}) \\ &+ \theta_t(r_{t+1}k_{t+1} + R_{t+1}b_{t+1} - c_{2t+1}) \end{aligned} \tag{4.5}$$

The first order conditions are:

$$u'(c_{1,t}) - \lambda_t = 0 \tag{4.6}$$

$$\beta u'(c_{2t+1}) - \theta_t = 0 \tag{4.7}$$

$$-\lambda_t + \theta_t r_{t+1} = 0 \tag{4.8}$$

$$-\lambda_t + \theta_t R_{t+1} = 0 \tag{4.9}$$

Unlike the RA model, there is no transversality condition. Even though time goes on forever, no individual has infinite lifetime and the TVC is a condition that only applies to infinite-horizon optimization problems.

Putting the first order conditions together, we get:

$$\beta u'(c_{2,t+1})r_{t+1} = u'(c_{1,t}) \tag{4.10}$$

which looks very similar to the Euler equation from the RA model. However, notice that the consumption levels are not aggregate consumption for the period but rather consumption for a particular generation.

As before, prices will adjust so that no young agent wishes to buy or sell bonds.

$$R_t = r_t \tag{4.11}$$

for all $t > 0$.

Firms

The firms face the same problems as before, so we get:

$$\begin{aligned} r_t &= \frac{\partial F}{\partial k_t} = f'(k_t) && (4.12) \\ w_t &= y_t - r_t k_t && (4.13) \\ & && (4.14) \end{aligned}$$

4.1.3 Comparison to the RA model

Distribution effects

First of all, the OLG model has "distribution effects". Recall that the Euler equation in the RA model related aggregate consumption and aggregate capital. In the OLG model, the primary dynamic equation relates agent t's consumption at time t to his consumption at time $t+1$. This difference is important - because labor's share of output goes to the young and capital's share goes to the old. Suppose, for example, we have an "Ak" production function

$$F(K, L) = AK \tag{4.15}$$

This implies

$$\begin{aligned} r_t &= A && (4.16) \\ w_t &= 0 && (4.17) \end{aligned}$$

At time zero, the old receive income Ak_0, which they consume, and the young receive no income at all. Since only the young invest, $k_1 = 0$, and $y_t = k_t = 0$ forever.

4.1.4 Efficiency

Now let's consider efficiency of equilibria here. Since there isn't a representative agent, we consider Pareto efficiency. Does this model satisfy the first welfare theorem? In other words, are all equilibria Pareto optimal?

The answer to that question is "no." We can show this by finding a single counterexample. Suppose that the production function is:

$$F(K, L) = L + aK \tag{4.18}$$

with $a < 1$. OK, so this production function is not strictly concave, but the problem set will have an example where it is and the argument goes through. Anyway, taking derivatives to get factor prices yields:

$$\begin{aligned} w_t &= 1 && (4.19) \\ r_t &= a && (4.20) \end{aligned}$$

An equivalent formulation is a pure exchange economy in which each worker is endowed with one unit of consumption when young, which can be stored for consumption when old. The good deteriorates at rate $1 - a$.

Utility is Cobb-Douglas:

$$U_t = \ln c_{1,t} + \beta \ln c_{2,t+1} \tag{4.21}$$

We'll find out shortly that this implies that the worker saves a constant fraction of income, $s = \frac{\beta}{\beta+1}$, yielding utility level:

$$\begin{aligned} U_t &= \ln 1 - s + \beta \ln sa \qquad \text{if } t \geq 0 && (4.22) \\ U_{-1} &= \ln ak_0 && (4.23) \end{aligned}$$

So is the equilibrium outcome efficient? No. Suppose that the social planner took all of the labor income and gave $1-s$ to the young generation and s to the older generation. The planner never purchases capital. Then:

$$\begin{aligned} U_t &= \ln 1 - s + \beta \ln s \qquad \text{if } t \geq 0 && (4.24) \\ U_{-1} &= \ln ak_0 + s && (4.25) \end{aligned}$$

This allocation is feasible and gives everyone more utility than the equilibrium allocation. Since there exists a feasible allocation that Pareto dominates the equilibrium allocation, the equilibrium allocation is not Pareto efficient. So the first welfare theorem does not apply to the OLG model.

What's going on here? Remember when we talked about the golden rule capital stock? Well, let's figure out what the golden rule capital stock is here. To find this we solve for long-run capital stock $k_\infty \geq 0$ that maximizes:

$$\begin{aligned} c_\infty &= F(k_\infty, 1) - k_\infty && (4.26) \\ &= 1 + (a-1)k_\infty \end{aligned}$$

Since $a - 1 < 0$ the solution is $k_\infty = 0$, and the golden rule savings rate is also zero. In the equilibrium of the RA model, the savings rate was always less than the golden rule level. In this model, the savings rate is above the golden rule. A savings rate which is above the golden rule level is called "dynamically inefficient" - workers accumulate too much capital. Why would workers do this? The short answer is that capital accumulation is the only way workers can save, and the net return on capital is $a - 1 < 0$

Savings rates

Next, let's find out what determines savings rates. For the rest of the discussion, let utility be CRRA:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} \tag{4.27}$$

Then (4.10) can be rewritten as:

$$\frac{c_{2,t+1}}{c_{1,t}} = (\beta r_{t+1})^{1/\sigma} \tag{4.28}$$

Now:

$$c_{2,t+1} = r_{t+1}k_{t+1} \tag{4.29}$$
$$k_{t+1} = c_{2,t+1}/r_{t+1} \tag{4.30}$$

which means:

$$k_{t+1} = \frac{(\beta r_{t+1})^{1/\sigma}}{r_{t+1}} c_{1,t} \tag{4.31}$$

Notice that, conditional on the interest rate, the savings rate is unaffected by the wage. For the time being, call that big number γ, so that $k_{t+1} = \gamma c_{1,t}$ Next we substitute back into the budget constraint.

$$(1+\gamma)c_{1,t} = w_t \tag{4.32}$$

The savings rate is k_{t+1}/w_t or $\frac{\gamma}{1+\gamma}$.

$$s_t = s(r_{t+1}) = \frac{\beta^{1/\sigma} r_{t+1}^{(1-\sigma)/\sigma}}{1 + \beta^{1/\sigma} r_{t+1}^{(1-\sigma)/\sigma}} \tag{4.33}$$

So is savings increasing or decreasing in the interest rate? It depends on the value of σ. If $\sigma < 1$ savings is increasing in the interest rate, and if $\sigma > 1$ savings is decreasing in the interest rate.

Why is this? Remember that σ indicates the strength of a person's desire to smooth consumption. When interest rates go up, two things happen. First, the relative cost of consumption today versus consumption tomorrow will increase. This is commonly called the substitution effect, and will push consumers towards buying more consumption tomorrow (investing). Second, the consumer experiences an effective increase in income - he can save less today and still consume the same amount tomorrow. This is commonly called the income effect. The income effect will cause the worker to want to consume more today and thus save less. As σ grows, the income effect becomes more important.

If $\sigma = 1$, utility is Cobb-Douglas. We know that when utility is CD, a person spends a constant fraction of income on each good regardless of prices or income. In other words, savings rates are constant at $s = \frac{\beta}{1+\beta}$. Notice that we actually have a fully-specified equilibrium model that implies a constant savings rate just like the Solow model.

Dynamics

The law of motion for capital can be described:

$$k_{t+1} = s(r_{t+1})w_t \tag{4.34}$$

Substituting in the market prices, we get:

$$k_{t+1} = s(f'(k_{t+1}))(f(k_t) - k_t f'(k_t)) \tag{4.35}$$

We can rearrange to get:

$$\frac{k_{t+1}}{s(f'(k_{t+1}))} = f(k_t) - k_t f'(k_t) \tag{4.36}$$

which we can rewrite as:

$$G(k_{t+1}) = H(k_t) \tag{4.37}$$

which implicitly defines a first-order difference equation. Let's see what we know about these functions:

- H is strictly increasing in k.
- If $s'(r) > 0$, then G is strictly increasing in k.

Because H is monotonic, we can take its inverse:

$$k_t = H^{-1}(G(k_{t+1})) \tag{4.38}$$

so k_t is a function of k_{t+1}. That means we can't have a relationship that looks like (graph).

However, we can have a relationship that looks like (graph). Notice that k_{t+1} is not necessarily a single-valued function of k_t. In other words, this economy has multiple equilibria. Here, workers expect high interest rates, so they save less. The low subsequent capital stock results in high interest rates. Here, workers expect low interest rates, so they save more. The high capital stock results in low interest rates.

In both cases these expectations are rational if people have them. This model exhibits "self-fulfilling prophecies."

Definition 4.1.1 (Self-fulfilling prophecies) *A model exhibits self-fulfilling prophecies if there are two or more distinct forecasts of the future each of which will be correct (in the rational expectations sense) if agents in the economy think it to be correct.*

By "in the rational expectations sense", I just mean that their forecasts could have some uncertainty, but that they know the correct probability distribution over future events.

Some macroeconomists believe that self-fulfilling prophecies are critical to understanding business cycles. John Maynard Keynes refers to the "animal spirits" of speculators in financial markets as an important source of instability in the economy.

So which equilibrium actually happens? We could simply say "no one knows". Or we could assume that people follow some rule of thumb and coordinate their activity on one of the equilibrium.

- We could simply all follow a rule like "pick the high-capital equilibrium".
- We could flip a coin. Heads we pick high-capital, tails we pick low-capital. A process like this in which the economy moves randomly between equilibria in coordination with some essentially irrelevant variable is called a "sunspot". Other examples of potential sunspots are
  - Today's weather in Toronto.
  - What Alan Greenspan said before Congress today.
  - Whether Microsoft announced higher or lower quarterly earnings than expected.

  Notice that a sunspot equilibrium may involve simply overreacting to information that is relevant. The important thing is that if everyone thinks a particular sunspot variable matters, it does.

Now suppose that this is a well-behaved world and that the savings rate increases with the interest rate. Then G^{-1} exists and is strictly increasing, so k_{t+1} is a strictly increasing function of k_t. Now there is a unique equilibrium for each initial capital stock. So the difference equation could look like (graph). In that case, we have a unique long-run steady state, and we converge directly to this steady state. However, it could also look like this. This is an example of history-dependence. Economies with different initial capital stocks will not converge. This property is sometimes called "path dependence" or "history dependence".

4.1.5 More on efficiency

First a little definition. An allocation is just a matrix X, where $X_{i,j}$ is agent i's consumption of good j. There is a set of feasible allocations Γ. Each agent receives utility $u_i(X)$ from a allocation X.

Suppose that we are a society that cares about the happiness of our citizens, and that we have to choose between two allocations. We would like some way of ranking allocations. One alternative is a social welfare function.

Definition 4.1.2 *Social welfare function A social welfare function is a function* $W(X) = \hat{W}(u_1(X), u_2(X), u_3(X), \ldots, u_n(X))$ *such that* $\hat{W}$ *is increasing in all of its arguments.*

So a social welfare function would allow us to rank all possible allocations. However, a social welfare function requires us to make some form of interpersonal comparison - we have to decide whether giving Bill a dollar is better or worse than giving Ted a dollar.

Even if we prefer not to make this type of choice, we can come up with weaker criteria for a desirable allocation.

Definition 4.1.3 *Pareto dominance An allocation* X *weakly Pareto dominates allocation* Y*, if all agents receive at least as much utility under* Y *as they do under* Y *(*$u_i(X) \geq u_i(Y)$ *for all* i*), and at least one agent receives more (*$u_i(X) > u_i(Y)$ *for some* i*).* X *strictly Pareto dominates* Y *if all agents are better off under* X *than* Y*.*

Pareto dominance is a weaker way of comparing allocations. For example, a world in which I set your house on fire is Pareto dominated by a world in which I don't (because you lose and no one gains), but a world in which I steal your house is not (because you lose and I gain).

Definition 4.1.4 *Pareto efficient A feasible allocation X is Pareto efficient if there does not exist a feasible allocation which Pareto dominates it.*

Why do we care about Pareto efficiency? Because if the outcome we experience in equilibrium is not Pareto efficient, then a government can potentially step in and make everyone better off.

In your microeconomics course you will learn something called the first and second theorems of welfare economics. I will give you the basics. Let there be a finite number of agents and a finite number of goods. Assume that preferences are continuous and monotonic, and that we have a pure exchange economy.

Definition 4.1.5 *Walrasian equilibrium in exchange economies An allocation-price pair (x, p) is a Walrasian equilibrium if*

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \omega_i \tag{4.39}$$

and for every x_i' that is preferred by i to x_i, $px_i' > p\omega_i$.

Theorem 1 (First welfare theorem in exchange economies) *If (x, p) is a Walrasian equilibrium then x is Pareto efficient.*

Proof: Suppose not. Let x' be a feasible allocation which is strictly Pareto superior, i.e., that is all agents prefer x_i' to x_i. Then by the definition of Walrasian equilibrium we have:

$$px_i' > p\omega_i$$

Summing over all i and using the fact that x' is feasible, we have:

$$p\sum \omega_i = p\sum x_i' > \sum p\omega_i$$

which is a contradiction.

Now, let's see how this applies to our two models. First we need to note that the RA model has a finite number of agents (one) and an infinite number of consumption goods. In contrast the OLG model has an infinite number of agents and goods.

Let's look at the lifetime budget constraint for one agent in the the RA model, with c_0 as the numeraire:

$$\sum_{t=0}^{\infty} q_t c_t \leq R_0 \hat{b}_0 + r_0 k_0 + \sum_{t=0}^{\infty} q_t w_t \tag{4.40}$$

Now let's define the lifetime budget constraint for agent t in the OLG model, also with c_0 as the numeraire:

$$q_t c_{1,t} + q_{t+1} c_{2,t+1} \leq q_t w_t \tag{4.41}$$

Aggregating across all of the consumers we get an aggregate budget constraint:

$$\sum_{t=0}^{\infty} q_t c_{1,t} + q_{t+1} c_{2,t+1} \leq \sum_{t=0}^{\infty} q_t w_t \tag{4.42}$$

Now, consider our previous example, in which $F(k, L) = L + ak$. In that case $q_t = a^{-t}$. When $a < 1$, q_t is growing without bound. So the aggregate budget constraint in equation (4.42) is undefined. In the proof for the first welfare theorem above, we made use of this to show that any pareto improvement over the Walrasian equilibrium must be feasible.

In an economy with production, the set of conditions for the welfare theorems are a little stronger. Here's an informal description of these results, you will get a more formal picture in Econ 802.

1. Perfect competition. All agents (consumers and firms) take prices as given.

2. Constant returns to scale.

3. Complete markets. Every good has a price and can be traded freely at that price. An example of an economy with incomplete markets is one where no one is allowed to exchange labor in period 10.

4. No externalities. An example of an externality is the Romer 1986 model I described two weeks ago.

5. A finite number of agents and a finite number of goods.

as well as a few other technical assumptions, we have the folowing results:

1. First welfare theorem: Every competitive equilibrium is Pareto efficient.

2. Second welfare theorem: Every Pareto efficient allocation can be supported by a competitive equilibrium. In other words, there's a set of prices so that if you make that allocation and set prices at that level, no one will want to trade.

When we analyze an equilibrium model of the economy, we will always want to know if these two theorems hold - if the equilibrium is efficient.

So is our economy Pareto efficient? Notice that we fail to have a finite number of goods, so it might not be. However, it is. Since there is only one agent, the solution to the planner's problem is the only Pareto optimal allocation. We showed that the equilibrium allocation solves the planner's problem, so both welfare theorems hold.