Problem Set 3 – Solutions

ECON 837

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1. (a)

$$\hat{\beta} = \frac{\sum_i (x_i - \bar{x}) y_i}{\sum_i (x_i - \bar{x})^2} = \frac{\sum_i x_i y_i - \bar{x}\sum_i y_i}{\sum_i x_i^2 - n\bar{x}^2} = \frac{4430 - \left(\frac{220}{20}\right) 440}{2260 - 22\left(\frac{220}{22}\right)^2} = \frac{1}{2}$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} = \frac{440}{22} - \frac{1}{2}\frac{220}{20} = 15$$

(b)

$$R^2 = \hat{\beta}^2 \frac{\sum_i (x_i - \bar{x})^2}{\sum_i (y_i - \bar{y})^2} = \hat{\beta}^2 \frac{\sum_i x_i^2 - n\bar{x}^2}{\sum_i y_i^2 - n\bar{y}^2} = \frac{1}{4}\frac{2260 - 22\left(\frac{220}{22}\right)^2}{8900 - 22\left(\frac{440}{22}\right)^2} = 0.15$$

(c)

$$\begin{aligned} s^2 &= \frac{1}{n-2}\sum_i e_i^2 = \frac{1}{20}\sum_i \left(y_i - \hat{\alpha} - \hat{\beta}x_i\right)^2 \\ &= \frac{1}{20}\left(\sum_i y_i^2 - 2\hat{\alpha}\sum_i y_i - 2\hat{\beta}\sum_i x_i y_i + \hat{\alpha}^2 + 2\hat{\alpha}\hat{\beta}\sum_i x_i + \hat{\beta}^2\sum_i x_i^2\right) \\ &= 4.25 \end{aligned}$$

Therefore, the standard error of $\hat{\beta}$ is

$$se\left(\hat{\beta}\right) = \left(\frac{s^2}{\sum_i x_i^2 - n\bar{x}^2}\right)^{1/2} = \left(\frac{4.25}{60}\right)^{1/2} = 0.266$$

and our test statistic is

$$t = \frac{\hat{\beta} - \beta}{se\left(\hat{\beta}\right)} = \frac{0.5}{0.266} = 1.8787.$$

The 0.975 critical value of the t distribution with $n-2 = 20$ degrees of freedom is 2.086, so we do not reject the null hypothesis.

(d) By the Gauss-Markov theorem, the BLUE of $\alpha - \beta$ is $\hat{\alpha} - \hat{\beta}$. Its variance is given by

$$\begin{aligned} Var\left[\hat{\alpha} - \hat{\beta}\right] &= Var\left[\hat{\alpha}\right] + Var\left[\hat{\beta}\right] - 2Cov\left[\hat{\alpha}, \hat{\beta}\right] \\ &= \sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_i (x_i - \bar{x})^2}\right] + \frac{\sigma^2}{\sum_i (x_i - \bar{x})^2} + 2\frac{\sigma^2 \bar{x}}{\sum_i (x_i - \bar{x})^2} \\ &= \sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2 + 1 + 2\bar{x}}{\sum_i (x_i - \bar{x})^2}\right]. \end{aligned}$$

An unbiased estimator of this variance is

$$\begin{aligned} \widehat{Var}\left(\hat{\alpha}, \hat{\beta}\right) &= s^2\left[\frac{1}{n} + \frac{\bar{x}^2 + 1 + 2\bar{x}}{\sum_i (x_i - \bar{x})^2}\right] \\ &= 4.25\left[\frac{1}{22} + \frac{10^2 + 1 + 2(10)}{60}\right] = 8.764. \end{aligned}$$

Under normality, we therefore know that

$$\hat{\alpha} - \hat{\beta} \sim N\left(\alpha - \beta, Var\left[\hat{\alpha} - \hat{\beta}\right]\right)$$

where normality of $\hat{\alpha} - \hat{\beta}$ follows from normality of $\hat{\alpha}$ and $\hat{\beta}$. Hence

$$\begin{aligned} \frac{\hat{\alpha} - \hat{\beta} - (\alpha - \beta)}{\sqrt{Var\left[\hat{\alpha} - \hat{\beta}\right]}} &\sim N(0,1) \\ (n-2)\frac{s^2}{\sigma^2} &\sim \chi^2_{n-2} \end{aligned}$$

and $\hat{\alpha}-\hat{\beta}$ is independent of s^2 (follows from independence of $\hat{\alpha}$ and s^2, and independence of $\hat{\beta}$ and s^2). Therefore

$$\begin{aligned}
\frac{\left(\hat{\alpha}-\hat{\beta}-(\alpha-\beta)\right)\Big/\sqrt{Var\left[\hat{\alpha}-\hat{\beta}\right]}}{\sqrt{\frac{(n-2)s^2}{\sigma^2}\Big/(n-2)}} &= \frac{\left(\hat{\alpha}-\hat{\beta}-(\alpha-\beta)\right)\Big/\sqrt{\sigma^2\left[\frac{1}{n}+\frac{\bar{x}^2+1+2\bar{x}}{\sum_i(x_i-\bar{x})^2}\right]}}{\sqrt{\frac{s^2}{\sigma^2}}} \\
&= \frac{\hat{\alpha}-\hat{\beta}-(\alpha-\beta)}{\sqrt{\widehat{Var}\left[\hat{\alpha}-\hat{\beta}\right]}} \sim t_{n-2}
\end{aligned}$$

under the null. The null is $\alpha-\beta=10$, so our test statistic is

$$t=\frac{15-\frac{1}{2}-10}{\sqrt{8.764}}=1.5201.$$

The 0.95 critical value of the t distribution with 20 degrees of freedom is 1.725, so we do not reject the null hypothesis.

2. A good estimator of $\gamma=\alpha+\beta$ is $\hat{\gamma}=\hat{\alpha}+\hat{\beta}$ since it is unbiased and BLUE by the Gauss-Markov theorem. A point estimate is

$$\hat{\gamma}=\hat{\alpha}+\hat{\beta}=1+0.5=1.5.$$

To find its standard error, we need an estimate of $Cov\left[\hat{\alpha},\hat{\beta}\right]$ since

$$Var\left[\hat{\gamma}\right]=Var\left[\hat{\alpha}+\hat{\beta}\right]=Var\left[\hat{\alpha}\right]+Var\left[\hat{\beta}\right]+2Cov\left[\hat{\alpha},\hat{\beta}\right].$$

However, we know

$$Cov\left[\hat{\alpha},\hat{\beta}\right]=Corr\left[\hat{\alpha},\hat{\beta}\right]\times\sqrt{Var\left[\hat{\alpha}\right]}\times\sqrt{Var\left[\hat{\beta}\right]}=-0.5\times0.5\times0.25=-0.0625$$

and therefore

$$\begin{aligned}
\text{s.e.}\left(\hat{\gamma}\right) &= \sqrt{(0.5)^2+(0.25)^2-2\left(0.0625\right)} \\
&= \sqrt{0.1875}=0.43301.
\end{aligned}$$

3. [Note: my solution uses matrix notation, but the question can be solved without – just replace the quadratic forms with equivalent sums of squares.] Let $\mathbf{J}=\mathbf{I}_n-\mathbf{i}\left(\mathbf{i}'\mathbf{i}\right)^{-1}\mathbf{i}'$, where $\mathbf{I}_n$ is the identity matrix of order n and $\mathbf{i}$ is an n-vector of ones. Recall that $\mathbf{J}$ is the projection matrix that takes deviations from means. Let $\mathbf{x}$ and $\mathbf{y}$ denote the vector of observations on x_i and y_i, respectively, and let $\mathbf{e}_1$ and $\mathbf{e}_2$ denote the residual vectors from model 1 and model 2, respectively.

   (a) In matrix notation,

   $$R^2=R_1^2=R_2^2=\frac{\left(\mathbf{x}'\mathbf{J}\mathbf{y}\right)^2}{\left(\mathbf{x}'\mathbf{J}\mathbf{x}\right)\left(\mathbf{y}'\mathbf{J}\mathbf{y}\right)}$$

   so R^2 is equal in the two models.

   (b) In matrix notation,

   $$\hat{\beta}_1=\frac{\mathbf{x}'\mathbf{J}\mathbf{y}}{\mathbf{x}'\mathbf{J}\mathbf{x}}\quad\text{and}\quad\hat{\beta}_2=\frac{\mathbf{x}'\mathbf{J}\mathbf{y}}{\mathbf{y}'\mathbf{J}\mathbf{y}}$$

   so that $\hat{\beta}_1\hat{\beta}_2=R^2$.

   (c) Since $R_1^2=1-\mathbf{e}_1'\mathbf{e}_1/\left(\mathbf{y}'\mathbf{J}\mathbf{y}\right)=R^2$ and $\hat{\beta}_1=\mathbf{x}'\mathbf{J}\mathbf{y}/\mathbf{x}'\mathbf{J}\mathbf{x}$, we can write

   $$t_1=\frac{\hat{\beta}_1}{\sqrt{\left(\mathbf{x}'\mathbf{J}\mathbf{x}\right)^{-1}\left(\mathbf{e}_1'\mathbf{e}_1\right)/\left(n-2\right)}}=\frac{\mathbf{x}'\mathbf{J}\mathbf{y}}{\sqrt{\left(\mathbf{x}'\mathbf{J}\mathbf{x}\right)\left(\mathbf{y}'\mathbf{J}\mathbf{y}\right)\left(1-R^2\right)/\left(n-2\right)}}.$$

   Similarly, $R_2^2=1-\mathbf{e}_2'\mathbf{e}_2/\left(\mathbf{x}'\mathbf{J}\mathbf{x}\right)=R^2$ and $\hat{\beta}_2=\mathbf{x}'\mathbf{J}\mathbf{y}/\mathbf{y}'\mathbf{J}\mathbf{y}$, so that

   $$t_2=\frac{\hat{\beta}_2}{\sqrt{\left(\mathbf{y}'\mathbf{J}\mathbf{y}\right)^{-1}\left(\mathbf{e}_2'\mathbf{e}_2\right)/\left(n-2\right)}}=\frac{\mathbf{x}'\mathbf{J}\mathbf{y}}{\sqrt{\left(\mathbf{x}'\mathbf{J}\mathbf{x}\right)\left(\mathbf{y}'\mathbf{J}\mathbf{y}\right)\left(1-R^2\right)/\left(n-2\right)}}$$

   and hence $t_1=t_2$.

4. (a) Since the data are in deviations from means, the least squares estimator of β from the regression of y_i on z_i is

$$\begin{aligned}\hat{\beta} &= \frac{\sum z_i y_i}{\sum z_i^2} = \frac{\sum (x_i+\nu_i)(x_i\beta+\varepsilon_i)}{\sum (x_i+\nu_i)^2} = \beta\frac{\sum x_i^2}{\sum (x_i+\nu_i)^2} + \beta\frac{\sum x_i v_i}{\sum (x_i+\nu_i)^2} + \frac{\sum (x_i+\nu_i)\varepsilon_i}{\sum (x_i+\nu_i)^2} \\ &= \beta\frac{\sum x_i^2}{\sum (x_i+\nu_i)^2} + \frac{\sum (x_i+\nu_i)\varepsilon_i}{\sum (x_i+\nu_i)^2}\end{aligned}$$

since x_i and ν_i are orthogonal and hence $\sum x_i \nu_i = 0$. Now we need to be careful about evaluating the expected value of $\hat{\beta}$, since the second term has random elements in both the numerator and denominator. That is,

$$E\left[\frac{\sum (x_i+\nu_i)\varepsilon_i}{\sum (x_i+\nu_i)^2}\right] \neq \frac{E\left[\sum (x_i+\nu_i)\varepsilon_i\right]}{E\left[\sum (x_i+\nu_i)^2\right]}.$$

However, the law of iterated expectations comes in handy because

$$E\left[\frac{\sum (x_i+\nu_i)\varepsilon_i}{\sum (x_i+\nu_i)^2}\right] = E\left[E\left[\frac{\sum (x_i+\nu_i)\varepsilon_i}{\sum (x_i+\nu_i)^2}|\nu_i\right]\right] = E[0] = 0$$

and hence

$$E\left[\hat{\beta}\right] = \beta E\left[\frac{\sum x_i^2}{\sum (x_i+\nu_i)^2}\right] = \beta E\left[\frac{\sum x_i^2}{\sum x_i^2 + \sum \nu_i^2}\right] < \beta.$$

That is, the coefficient is biased towards zero.

(b) Notice that we can rewrite the expected value of the OLS estimator as

$$E\left[\hat{\beta}\right] = \beta\frac{\sum x_i^2}{\sum z_i^2} = \beta\frac{\widehat{Var}[x_i]}{\widehat{Var}[z_i]}.$$

We can compute $\widehat{Var}[z_i]$ from the data, and we could correct the bias in the OLS estimator if we had an estimate of $Var[x_i]$.
Now suppose we have a third variable $w_i = x_i + \xi_i$. Notice that

$$Cov[w_i, z_i] = E[(x_i+\xi_i)(x_i+\nu_i)] = E\left[x_i^2\right] + E[x_i\nu_i] + E[x_i\xi_i] + E[\nu_i\xi_i] = E\left[x_i^2\right] = Var[x_i].$$

Hence we can estimate the variance of x_i from the sample covariance of z_i and w_i and use this to correct the bias in $\hat{\beta}$.

5. Under normality, the likelihood function is

$$\begin{aligned}L\left(\alpha,\beta,\sigma^2\right) &= \left(2\pi\sigma^2\right)^{-n/2}\exp\left(-\frac{1}{2\sigma^2}\sum_i (y_i-\alpha-\beta x_i)^2\right) \\ &= \left(2\pi\sigma^2\right)^{-n/2}\exp\left(-\frac{1}{2\sigma^2}\left(\sum_i y_i^2 - 2\alpha\sum_i y_i - 2\beta\sum_i y_i x_i + 2\alpha\beta\sum_i x_i + n\alpha^2 + \beta\sum_i x_i^2\right)\right)\end{aligned}$$

and by the factorization theorem, the collection of statistics:

$$T(X_i, Y_i) = \left[\sum_i y_i^2, \sum_i y_i, \sum_i y_i x_i, \sum_i x_i, n, \sum_i x_i^2\right]$$

is sufficient for α, β and σ^2. You will note, of course, that the least squares estimators $\hat{\alpha}$ and $\hat{\beta}$ are functions of $T(X_i, Y_i)$, as are s^2 and σ_{ML}^2.