Lecture 13 - Rotational kinematics

*What’s important:*
- definitions of angular variables
- angular kinematics

*Demonstrations:*
- ball on a string, wheel

*Text*: Walker, Secs. 10.1, 10.2, 10.3

*Problems:*

We now consider a type of motion in which the Cartesian description is fairly cumbersome. For example, if an object at the end of a string moves in a circular path at constant speed,

Then is has

\[
s = \text{speed} = \text{constant} \\
r = \text{radius} = \text{constant} \\
x, y \text{ and } v_x, v_y \text{ vary continuously}
\]

Since \( s \) and \( r \) are constant, independent of the object’s position, they do not specify the position or the velocity the object. While \( x, y \) and \( v_x, v_y \) do describe the object’s motion, they are time-dependent and cumbersome.

We seek a description that makes use of the constancy of \( s \) and \( r \), yet contains the time dependence in a functionally simple form. We start with uniform circular motion, and work towards general expressions for motion using angular variables.

**Uniform Circular Motion**

In two dimensions, the position of an object can be described using the polar coordinates \( r \) and \( \theta \). If the object is moving in uniform circular motion, then \( r \) is constant, and the time dependence of the motion is contained in \( \theta \). As a kinematic variable, \( \theta \) is the angular analogue of position.
The sign convention is that $\theta$ increases in a \textbf{counter-clockwise} direction. As a function of time, $\theta$ looks like (for uniform circular motion):

\begin{align*}
\theta \quad & \text{uniform speed} \\
\text{t} & \\
\end{align*}

Now, if $\theta$ is the angular analogue of position, then the slope of the $\theta$ vs. $t$ graph is the angular analogue of velocity. We define

\begin{align*}
\text{angular speed} & \quad \omega = \Delta \theta / \Delta t \quad \text{as } \Delta t \to 0 \\
\text{angular frequency of rotation} & \\
\end{align*}

(in other words, $\omega$ is the rate of change of $\theta$ with time)

We can go back and forth from $\theta$ to $\omega$ by slopes and areas, just like with $x$ and $v$. For example, from the area under the $\omega$ vs. $t$ curve, we find

\begin{align*}
\theta = \theta_0 + \omega t \quad \text{(at constant } \omega) \\
\end{align*}
Two other quantities used to describe uniform circular motion are the period \( T \) and the frequency \( f \). During the period \( T \), the object sweeps through one complete revolution, or \( 2\pi \) radians. Hence:

\[
\omega = \Delta \theta / \Delta t = 2\pi / T \quad \text{using } \Delta \theta = 2\pi \text{ radians during the period } T
\]

The frequency \( f \) is \( T^{-1} \), so we also have

\[
\omega = 2\pi f
\]

Note:

\[\omega = 1 \Rightarrow 1 \text{ radian / second} \]
\[f = 1 \Rightarrow 1 \text{ revolution / second} \]

**Angular and linear links**

So far, we have the following equations for linear and angular kinematics

\[
x = x_0 + vt \quad \theta = \theta_0 + \omega t
\]

\[
v = \Delta x / \Delta t \quad \omega = \Delta \theta / \Delta t \quad \text{as } \Delta t \rightarrow 0
\]

We have a relation between \( x \) and \( \theta \) already. To obtain a relation between \( v \) and \( \omega \), we return to the definition of speed:

\[
\text{speed} = |v| = \frac{\Delta \text{distance}}{\Delta \text{time}} = \frac{2\pi r}{T} \quad \leftrightarrow \text{circumference}
\]

\[
= \left( \frac{2\pi}{T} \right) r
\]

\[
= \omega r
\]

\[
\therefore v = \omega r \quad \text{links linear and angular speeds}
\]

Another linking relation is through the centripetal acceleration \( a_c \). From previous work, there is a centripetal acceleration even at constant speed:

\[
a_c = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r
\]

\[
\Rightarrow a_c = \omega^2 r \quad \text{(or } a_c = \omega v)\]
Example

We have a doughnut-shaped space station on which we wish to produce artificial gravity through rotation. What must \( \omega \) be so that \( a_c = g \) at 50 m from the centre of the station?

Using \( a_c = \omega^2 r \), we have

\[
\omega^2 = \frac{9.8}{50}
\]

or

\[
\omega = (\frac{9.8}{50})^{1/2} = 0.443 \text{ rad/s}
\]

(note that the units are OK)

Other quantities which we can calculate in this example are:

\[
T = \frac{2\pi}{\omega} = 2 \cdot 3.142 / 0.443 = 14.2 \text{ s.}
\]

which is moderately fast and would not encourage the occupants of the space station to look out the windows. The corresponding speed at the perimeter of the station is:

\[
v = \omega r \quad \longrightarrow \quad v = 0.443 \cdot 50 = 22.2 \text{ m/s} = 80 \text{ km/hr.}
\]