Lecture 29x – Viscosity (extended version)

What's important:
- viscosity
- Poiseuille's law
- Stokes' law

Demo: dissipation in flow through a tube

Viscosity

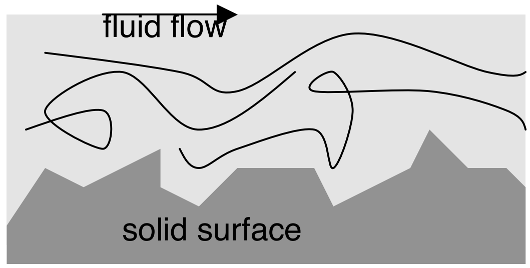
We introduced fluids as materials that cannot resist a shear stress, where a shear stress might involve a couplet of forces which have opposite direction and are applied to opposite faces of an object. Alternately, squeezing jello in your hand also has a shear component if the jello can escape around the top and bottom of the cylinder formed by your fingers. Passing your hand through air or water also demonstrates the fluid nature of these phases of matter, in that the air or water does not restore itself to initial state once your hand has passed by - rather, there has been mixing and rearranging of the gas or liquid.

We know intuitively that there is a characteristic time for a fluid to respond to an applied stress. Water responds fairly fast when we pour it from a container by changing its orientation, whereas salad dressing usually responds more slowly, and sugar-laced molasses slower still. What determines the response time is the strength and nature of the interactions among the fluids molecular components. For example, the molecules could be long and entangled (as in a polymer) or they could be small, but strongly interacting (as in water or molten glass). The response time of a fluid is determined by a characteristic called its viscosity η. In Lecture 8, this viscosity appears in Stokes' Law for the drag force on a sphere of radius R moving at a speed v

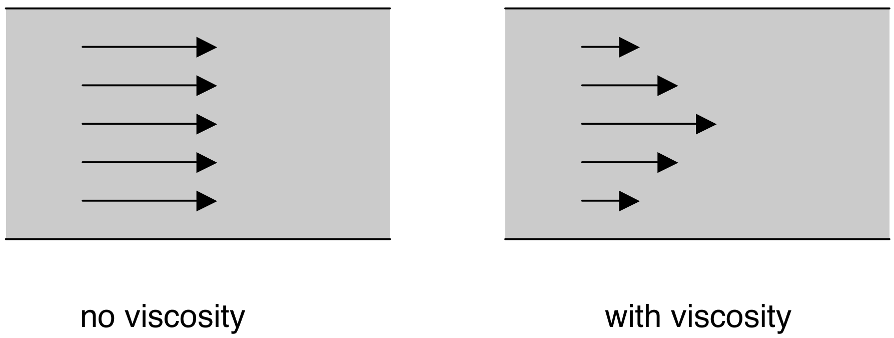
$$F_{\text{DRAG}} = 6\pi\eta Rv.$$

The larger η, the larger the drag force, all other things being equal.

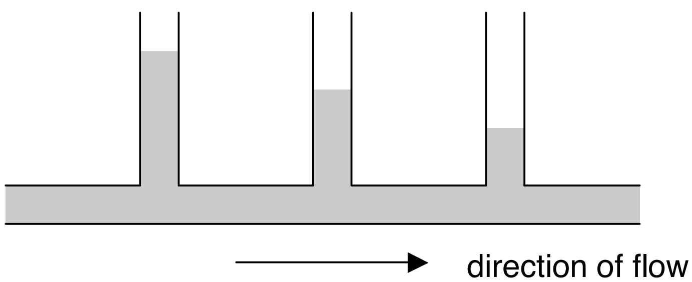
Like other frictional forces, viscosity doesn't cause an object to spontaneously move, but rather it determines the magnitude of the resistance to an applied force. One place where this has an effect is on the motion of fluids near a stationary boundary. In previous lectures, we drew diagrams of streamline flow and assumed that all fluid flowing into a region moved at the same speed. But near a stationary boundary, such as the walls of a pipe, the fluid interacts with a surface that is rough, or perhaps attractive, as a result of which the boundary exerts drag on the fluid motion. This is similar to the frictional force between two rough surfaces.



At small length scales, the motion would appear to be locally turbulent, as drawn above. At a larger length scale, one would observe the local speed of the fluid to decrease towards zero the closer it flows near the boundary. In any of these cases, the effect of the drag force is to dissipate energy.



Demo: The effects of dissipation show up as a pressure gradient at constant height y



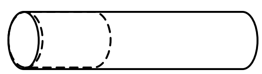
In this demo, fluid from a reservoir enters from the left with sufficient pressure that some fluid is forced up the first tube against atmospheric pressure. However, as the fluid continues to the right, its local pressure drops, and the fluid in the side tubes cannot rise as strongly against gravity. For a liquid, this is not an effect of the fluid slowing down: the equation of continuity demands that what goes in on the left must come out on the right, and if the density of the fluid is constant, the mean speed of the fluid must be the same along the horizontal pipe (otherwise, liquid would "pile up" in the middle of the pipe).

Volume rate of flow

Rigid body kinematics describes the motion of a massive object in response to forces. For fluids, the analogous quantity is the volume rate of flow Q is defined as

$$Q \equiv \Delta V / \Delta t.$$

That is, Q is the rate at which volume moves past a fixed location. We've examined this quantity before as part of determining the fluid mass moving through a cross sectional area A at speed v:



The distance moved horizontally by a particular fluid element in time Δt is $v\,\Delta t$ in the diagram above so that the change in volume in a given time Δt is

$$\Delta V = vA\,\Delta t,$$

where v is the fluid velocity and A is the cross sectional area of the fluid element. Thus,

$$Q = vA.$$

If v is not a constant, this equation still applies, but now we must use the average velocity v_{av},

$$\Delta V \equiv v_{av}\, A\, \Delta t.$$

and

$$Q = v_{av}A.$$

Poiseuille's law

The problem of laminar (non-turbulent) fluid flow through a pipe can be solved exactly. In the presence of dissipation, Poiseuille showed that

$$Q = (\pi R^4 / 8\eta) \cdot (\Delta P / L) \qquad \textit{Poiseuille's law}$$

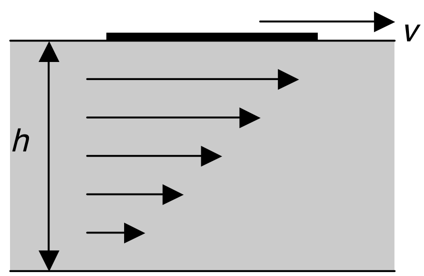
where ΔP is the pressure drop over the length L of the pipe. Note that the flow increases like R^4, a rather large power of R. This is the reason that gas pipelines across Canada do not have particularly large diameters in spite of the volume of gas they move to major cities: 1 m diameter pipelines can carry a prodigious amount of gas.

We measure the viscosity η of a liquid in Lec. 8 by means of a Stokes' Law analysis of the terminal speed of a falling object; η has units of J•s/m^2 ≡ 10 P, where P is a Poise. Again, some sample values:

Fluid	η (kg/m•sec at 20 °C)
Air	1.8×10^{-5}
Water	1.0×10^{-3}
Mercury	1.56×10^{-3}
Olive oil	0.084
Glycerine	1.34
Glucose	10^{13}
mixtures: blood	2.7×10^{-3}

An alternate approach to measuring η uses a shear stress applied to the free, horizontal surface of a fluid:

plate with area *A*
dragged by a force *F*



speed is zero at the boundary

$$F/A = \eta v/h$$

A plate of area *A* is pulled along the surface of the fluid with a force *F*, giving a stress of *F/A*. Stress has the same units as pressure, and pressure is one kind of stress. If the material in the diagram were a solid, it would resist the stress until it reached some equilibrium deformed configuration. But a fluid doesn't resist stress, and the floating plate continues to move as long as there is an applied force. The quantity *v/h* is the response of the fluid to the applied stress, and the viscosity is the proportionality constant in:

$$F/A = \eta\,(v/h).$$

Example (from Walker): blood flow in the pulmonary artery, which carries blood from the heart to the lungs). The pulmonary artery is 8.5 cm long, 2.4 mm in radius and supports a pressure difference along its length of 450 Pa. What is the average speed of the blood in the artery?

From Poiseuille's law,

$$Q = \pi R^4\,\Delta P / 8\eta L \qquad + \qquad Q = v_{av} A$$

gives:

$$v_{av} A = \pi R^4\,\Delta P / 8\eta L$$

Dividing by $A = \pi R^2$, we have

$$v_{av} = R^2 \Delta P / 8\eta L$$

Upon numerical substitution

$$v_{av} = (2.4 \times 10^{-3})^2\ 450 / (8 \bullet 2.7 \times 10^{-3} \bullet 8.5 \times 10^{-2})$$
$$= 1.4 \text{ m/s}.$$

Reynolds number

A rule of thumb for classifying flow about an object of length ℓ is the so-called Reynolds number, a dimensionless quantity given by

$$\text{Re} = \rho\, v\, \ell / \eta$$

where v is the velocity of the object, ρ is the density of the medium and η is its viscosity. Flow is streamline at small Re and turbulent at large Re; the cross-over between the regions depends on the geometry of the environment, and is in the 10-100 range. For a cell moving in a medium with a few times the viscosity of water, say 3×10^{-3} kg/m•s, one has

$$\text{Re} = 10^3 \text{ (kg/m}^3\text{) } 10^{-6} \text{ (m/s) } 10^{-6} \text{ (m) / } 3\text{x}10^{-3} \text{ (kg/m•s)}$$
$$= 3\text{x}10^{-7}.$$

Well, this is certainly much less than 1.

Rotational drag

The effect of viscosity on a rotating object has the same general form as the linear drag discussed earlier in the course (see also Lec. 31x and references therein to Howard Berg's book *Random Walks in Biology*). Here's a typical example:

In a viscous medium, an object rotates at an angular frequency ω when subject to a torque τ according to

$$\tau = \omega / C_{drag}$$

where C_{drag} is a drag parameter. For a sphere of radius R, the drag parameter is

$$C_{drag} = 8\pi\eta R^3,$$

where η is the viscosity of the medium. Find τ for a typical cell, if the rotational speed is 10 revolutions per second, $R = 1\ \mu$m, and $\eta = 10^{-3}$ kg/ m•s.