Lecture 3 - Relative motion, projectile motion

What's important:
- relative motion
- projectile motion
- projectile range

Demos: bat and blocks on a ladder; toy car on a sheet of paper; Monkey-shoot

Motion with constant acceleration

Let's apply the vector notation to kinematics in three dimensions. Because Cartesian coordinates are orthogonal, a vector equation of motion

$$\mathbf{v} = \mathbf{v}_o + \mathbf{a}t$$

and the corresponding equation for position can be separated into x, y, and z components. For *constant acceleration*:

$$v_x = v_{o,x} + a_x t \qquad v_y = v_{o,y} + a_y t \qquad v_z = v_{o,z} + a_z t$$
$$x = x_o + v_{o,x}t + a_x t^2/2 \qquad y = y_o + v_{o,y}t + a_y t^2/2 \qquad z = z_o + v_{o,z}t + a_z t^2/2.$$

Demo: bat and blocks on a ladder to demonstrate independence of motion in orthogonal directions.

Relative motion

Based upon his observations in a friction-dominated world, Aristotle proposed that there is a natural state of rest. It took almost 2 millenia to understand why this view is incorrect. Rather, motion is relative: objects move with respect to each other or their environment and there is no absolute "state of rest". Some examples:
- toy car on a piece of paper
- Albion ferry crossing the Fraser River
- bacterium swimming in a fluid.

Consider a car moving on a straight-line path on a piece of paper, itself moving with respect to a table (demo). We define
$\mathbf{r}_{c\text{-}p}$ the position of the car with respect to the paper
$\mathbf{r}_{p\text{-}t}$ the position of the paper with respect to the table (no rotation - that's PHYS 211).

The position of the car with respect to the table $\mathbf{r}_{c\text{-}t}$ is then

$$\mathbf{r}_{c\text{-}t} = \mathbf{r}_{c\text{-}p} + \mathbf{r}_{p\text{-}t} \qquad \text{(vector equation)}$$

with a suitable choice of origin. If we examine the change of these vectors in time Δt, we find

$$\Delta\mathbf{r}_{c\text{-}t} = \Delta\mathbf{r}_{c\text{-}p} + \Delta\mathbf{r}_{p\text{-}t} \qquad \text{(vector equation).}$$

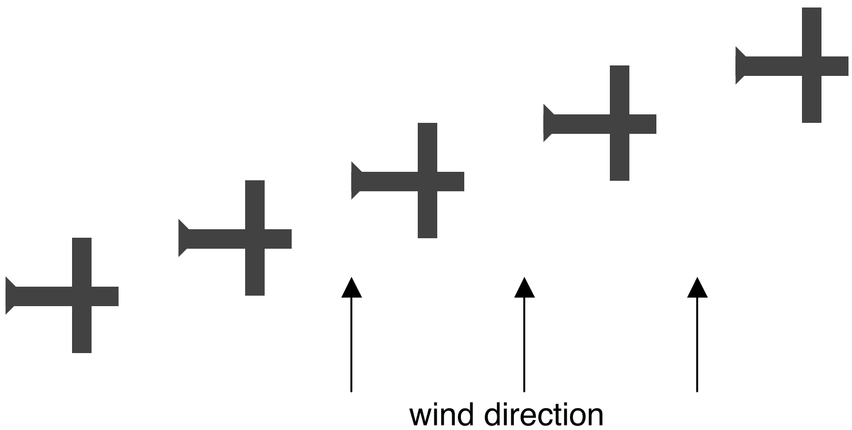
Dividing this equation by Δt gives a relationship among the velocities

$$\mathbf{v}_{c\text{-}t} = \mathbf{v}_{c\text{-}p} + \mathbf{v}_{p\text{-}t} \quad \text{(vector equation).}$$

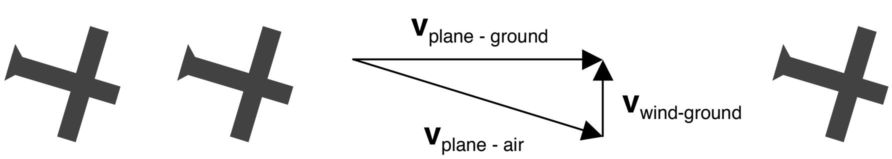
We could also watch how **v** changes with time, to obtain a relationship among the accelerations:

$$\mathbf{a}_{c\text{-}t} = \mathbf{a}_{c\text{-}p} + \mathbf{a}_{p\text{-}t} \quad \text{(vector equation).}$$

The Albion ferry provides a visual demonstration of relative motion, as do small planes landing in a crosswind at Boundary Bay airport.

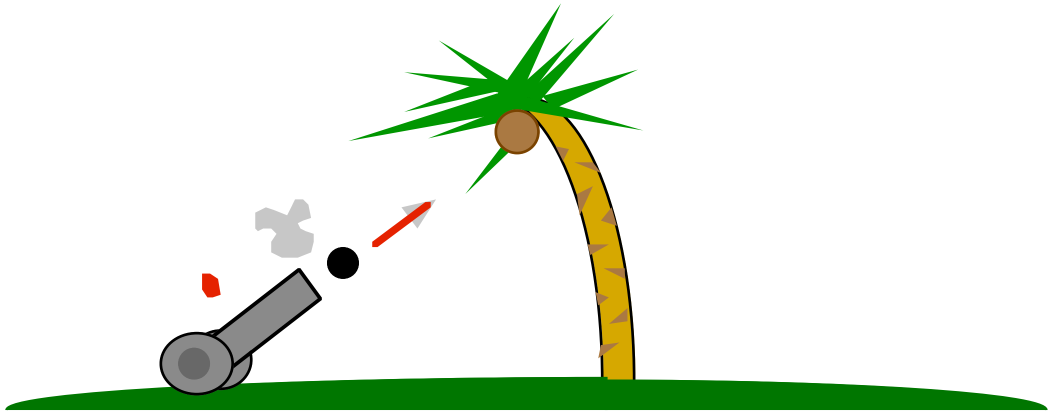


To fly due East, the plane would have to face slightly south:



Example

The Coconut Shoot problem (*a.k.a.* the Monkey Shoot problem). Suppose that you are trying to knock a coconut from a tree by firing a cannon at it. If you line up the cannon so that it is aimed directly at the coconut, then you know that the cannonball will, in principle, miss the coconut because of the acceleration of the cannonball in the vertical direction due to gravity. What happens if the coconut falls as soon as the cannonball is fired?



After the ball leaves the cannon, it experiences an acceleration in the vertical (y) direction due to gravity, which changes its *y*-velocity with time as:

ball: $v_y = v_{o,y} - gt$

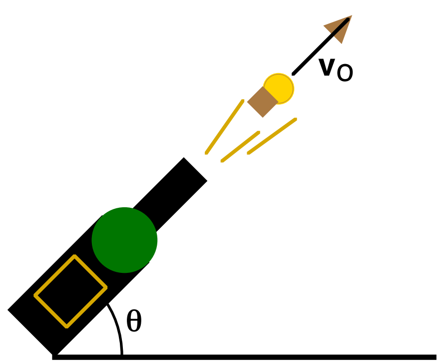
But the coconut experiences the same acceleration, so its velocity in the vertical direction is

coconut: $v_y = -gt$

The relative velocity is just $v_{o,y}$, independent of g. Hence, the cannonball will still hit the coconut as long as it falls as soon as the cannon is fired. If the gun were aimed horizontally, the motion of the ball and coconut would be the same as the motion of the two blocks in the bat-and-block demo.

Projectile Motion

Let's now solve the equations of motion for an object moving in two dimensions at constant acceleration. Consider a cork from a bottle of champagne:



The initial velocity is v_o. Once the projectile leaves the bottle, it is subject only to forces from gravity and air resistance. We neglect the latter. The initial velocity has components

horizontal (x) $v_{o,x} = v_o \cos\theta$
vertical (y) $v_{o,y} = v_o \sin\theta$

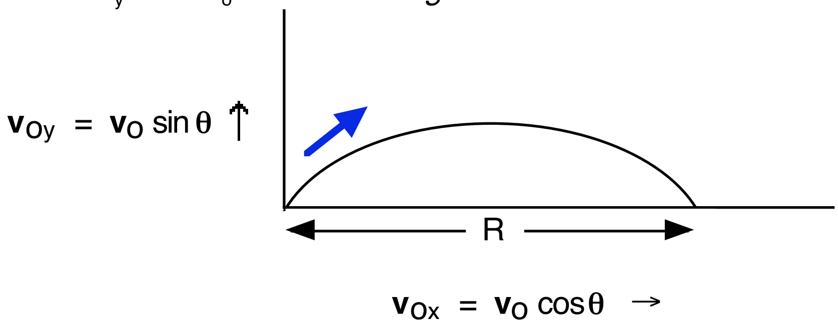
Only the vertical component is subject to change because of the acceleration due to gravity:

$a_x = 0$ $a_y = -g$ ($g = 9.8$ m/s^2)

The kinematic equations become:

$x = v_{o,x} t = v_o \cos\theta \; t$

$y = v_{o,y} t + 1/2 \, a_y t^2 = v_o \sin\theta \; t - 1/2 \, g t^2$.



If we want to find the range R of the cork, we need to solve for the time-of-flight t. When the cork hits the ground at time t, the y-coordinate is 0. Hence, we can solve for t from the equation of motion in the y-direction

$y = v_o \sin\theta \, t - 1/2 \, g t^2 = 0$

$\Rightarrow$ $v_o \sin\theta = 1/2 \, g t$

or

$t = 2 v_o \sin\theta / g$

This gives us the flight time. To get the range, which does not involve an acceleration in the x-direction, we use

$R = v_o \cos\theta \, t = v_o \cos\theta \cdot 2 v_o \sin\theta / g$

or

$R = (v_o^2 / g)$ (**$2\sin\theta \cos\theta$**)

But

$2\sin\theta \cos\theta = \sin 2\theta$

so that

$$R = (v_o^2 / g) \sin 2\theta$$

By inspection, the maximum value of R is at $\theta = 45°$, where $R = v_o^2 / g$. Alternatively, solve $dR/d\theta = 0$ to give $\cos 2\theta = 0$ or $\theta = 45°$.