CHAPTER 6

NUCLEAR FISSION AND FUSION

6.A Nuclear Binding Energies

A nucleus is characterized by the number of protons Z and neutrons N that it contains. The mass number A is the total number of nucleons (that is, protons and neutrons) in the nucleus:

$$A = Z + N. \qquad (6.1)$$

Only two of A, Z or N are needed to describe a nucleus, and the commonly used pair is A and Z. Because the chemical properties of an element are related to the charge on its nucleus, each atomic element corresponds to a unique value of Z: hydrogen has $Z = 1$, helium has $Z = 2$, lithium has $Z = 3$ *etc.* Although the properties of a nucleus are not affected by the properties of the atom in which it may reside, nevertheless it is conventional to label a nucleus, in part, by the elemental symbol corresponding to the Z of the nucleus. For example, all $Z = 1$ nuclei are called hydrogen nuclei, because they are found in hydrogen atoms or ions. All $Z = 2$ nuclei are called helium nuclei, all $Z = 3$ nuclei are called lithium nuclei, *etc.*

In spite of the notational emphasis on the nuclear charge, the mass number of a nucleus plays a large role in its binding energy. The convention for showing the mass number of a nucleus is to place a superscript A before the elemental symbol, as in

$$^{A}[elemental\ symbol]. \qquad (6.2)$$

For example, a nucleus with 3 protons (lithium) and 4 neutrons has $A = 7$, and is denoted by ^{7}Li. Some further items from the nuclear lexicon: nuclei with the same Z and differing N are called *isotopes* (e.g., ^{6}Li and ^{7}Li), nuclei with the same N and differing Z are called *isotones* (e.g., ^{7}Li and ^{8}Be), and nuclei with the same A and differing Z are called *isobars* ("equal weight", e.g. ^{7}Li and ^{7}Be). Lastly, some notation still in use today has historical roots. The three different types of radiation observed in early studies of radioactivity were labelled α, β, and γ-rays (or alpha, beta and gamma rays). We now know that α-particles are ^{4}He nuclei, β-particles are electrons and γ-rays are very short wavelength electromagnetic waves.

The binding energy of a nucleus is given by Eq. (4.4) as

$$B.E. = Zm_p c^2 + Nm_n c^2 - m(Z, A)c^2, \tag{6.3}$$

where $m(Z, A)$ is the mass of a bound nucleus with Z protons, N neutrons and A nucleons. In subatomic physics, binding energies are extracted from mass energies via Eq. (6.3). One of the most common techniques for determining masses involves accelerating an *ion* (charged atom) to high speed and then injecting it into a magnetic field, with the velocity vector of the ion perpendicular to the direction of the field. Like any charged object in a magnetic field, the ion will undergo an acceleration that is perpendicular to its direction of motion, resulting in the trajectory of the ion sweeping out an arc, as in Fig. 6.1. The mass can be calculated from the radius of the arc, and the charge of the ion.

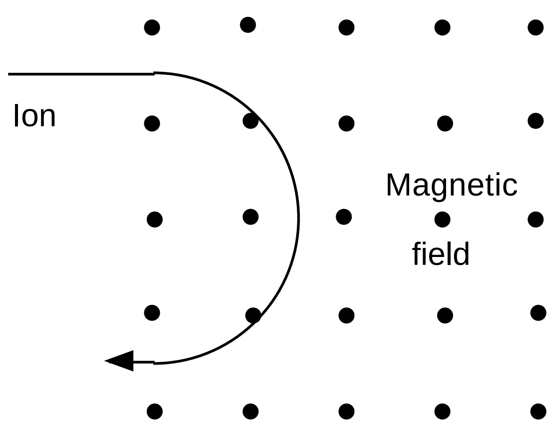

Fig. 6.1. Trajectory of an ion introduced into a magnetic field (indicated by the dots) that is perpendicular to the plane of the paper and to the ion's velocity.

The binding energy per nucleon *B.E./A* for the most deeply bound nuclei is roughly constant and is a more convenient quantity to work with than *B.E.* itself. A summary of *B.E./A* for most of the stable nuclei is provided in Table C.7. It is common in nuclear physics texts to work with a quantity called the mass defect Δ, which provides an easy and highly accurate way of calculating *Q*-values. Mass defects also are tabulated in Table C.7, and a brief description of how to use them is included in the table caption.

Example 6.1: *Find the mass energy of an ^{16}O nucleus using* B.E./A.

From Table C.7, the binding energy per nucleon, *B.E./A* for ^{16}O is 7.98 MeV, which corresponds to a nuclear binding energy of

B.E. = 16 x 7.98 = 127.68 MeV.

From Eq. (6.3), the mass energy of the ^{16}O nucleus is

$$m(^{16}\mathrm{O})c^2 = 8m_pc^2 + 8m_nc^2 - B.E.$$

$$= 8 \times 938.280 + 8 \times 939.573 - 127.68$$

$$= 14985.1 \text{ MeV.}$$

Note the accuracy with which the calculation must be carried out.

Before leaving this section, we mention a numerical shortcut for finding *Q*-values from the binding energy for nuclear reactions that separately conserve the number of protons and neutrons. We take as an example

$$\gamma + {}^{16}\mathrm{O} \rightarrow {}^{12}\mathrm{C} + {}^{4}\mathrm{He}. \qquad (6.4)$$

Starting with Eq. (4.12) for *Q*, we add the value of *B.E.* for each of the nuclei in question to the appropriate multiples of m_pc^2 and m_nc^2. That is, Eq. (4.12) becomes:

$$Q = [8\,m_{\rm p}c^2 + 8\,m_{\rm n}c^2 - B.E.(^{16}{\rm O})] - [6\,m_{\rm p}c^2 + 6\,m_{\rm n}c^2 - B.E.(^{12}{\rm C})]$$

$$- [2\,m_{\rm p}c^2 + 2\,m_{\rm n}c^2 - B.E.(^{4}{\rm He})]. \qquad (6.5)$$

Summing all of the proton and neutron masses together gives

$$Q = - B.E.(^{16}{\rm O}) + B.E.(^{12}{\rm C}) + B.E.(^{4}{\rm He}). \qquad (6.6)$$

Clearly, all of the proton and neutron masses have cancelled out, since the total number of protons and the total number of neutrons is the same before and after the reaction. For this type of reaction, the *Q*-value only involves differences in binding energies. If the number of protons changes during the reaction, then the resulting expression for the *Q*-value will not be as simple as Eq. (6.6), since there will be an unbalanced number of proton and neutron masses.

What we learn from Eq. (6.6) is that it is better to do a little algebra first when evaluating *Q*-values, since many of the mass terms cancel. Also, note the signs in front of the binding energies, which are reversed from the signs in front of the nuclear masses. The signs tell us that the more deeply bound the initial nucleus (^{16}O) is, the more negative the *Q*-value is. This is as expected, since it will take more kinetic energy to make the reaction occur. Similarly, the more deeply bound the final nuclei (^{12}C and ^{4}He) are, the more positive the *Q*-value. Again, this makes sense since there is more kinetic energy liberated if the final nuclei are deeply bound.

6.B Fission and Fusion

Most of the reactions of interest in Chaps. 10 - 11 are nuclear, and therefore involve protons, neutrons and perhaps other particles. The presence of protons and neutrons means that nuclear reactions always involve conservation of baryon number, Eq. (5.12). Since the baryon number of a nucleus is equal to its mass number *A*, then Eq. (5.12) can be rewritten for nuclei as

$$A_{\rm A} + A_{\rm B} = A_{\rm C} + A_{\rm D} \qquad (6.7)$$

for the reaction $A + B \rightarrow C + D$. The charge *Q* of a nucleus is given by

$$Q = Ze \tag{6.8}$$

where Z is the number of protons in the nucleus and e is the elementary charge, 1.6 x 10^{-19} C. Hence, Eq. (5.11) for conservation of charge can be written as

$$Z_A + Z_B = Z_C + Z_D. \tag{6.9}$$

Example 6.2: *Find the nucleus X in* $^{14}N + ^{4}He \rightarrow ^{1}H + ^{A}X$.

The solution is an application of Eqs. (6.7) and (6.9). From (6.7)

$$14 + 4 = 1 + A,$$

so that A = 17. The Z of each nucleus involved must be determined in order to use Eq. (6.9). From Table C.7, Z has values of 7 (nitrogen), 2 (helium) and 1 (hydrogen). Hence

$$7 + 2 = 1 + Z,$$

resulting in Z = 8. Oxygen nuclei have Z = 8, so the unknown nucleus is ^{17}O.

Having established some nuclear notation, we now examine the binding energy per nucleon in more detail, and explore its consequences in nuclear reactions and decays. The binding energy per nucleon for ^{16}O is about 8 MeV. In fact, over a very wide mass range the value of $B.E./A$ for the most deeply bound nuclei is within 10% of 8 MeV! In other words, in spite of the formidable range of data shown in Table C.7, the binding energy can be expressed as a fairly smooth function of Z and A. The qualitative behaviour of the binding energy data in Table C.7 is shown in Fig. 6.2

Two general features of Fig. 6.2 are expected from the properties of the strong and electromagnetic interactions. It is the strong interaction that binds the nucleons together, and this interaction extends over just a short range. Accordingly, nucleons sense the presence only of their immediate neighbouring nucleons - they are unaffected by the presence of

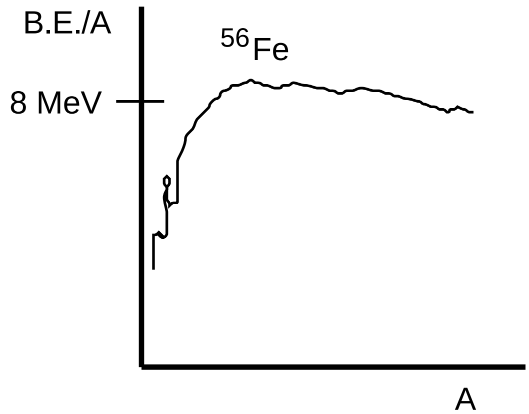

Fig. 6.2. Qualitative behaviour of the binding energy per nucleon *B.E./A* shown as a function of *A* for the most deeply bound nucleus at a given value of *A*.

other nucleons further away in the nucleus. This implies that nucleons in the interior of the nucleus are deeply bound, with an individual binding energy that does not depend strongly on *A*, while nucleons on the surface are weakly bound, since they have fewer neighbouring nucleons.

Now, nuclei with small values of *A* have a large surface area compared to their volume: if you construct a model $A = 6$ nucleus by gluing six marbles together you see that all six marbles are on the surface of your model nucleus and none are in the interior. This means that light nuclei are not deeply bound: a surface nucleon has only a few nearest neighbours with which to interact and bind. As *A* increases, there are more nucleons in the interior of the nucleus. In a "marble" model of a large nucleus, each of the interior nucleons has 12 nearest neighbours, so that interior nucleons are more deeply bound than surface nucleons. Hence, we expect the binding energy per nucleon, *B.E./A*, to increase with the mass number *A* for light nuclei, then become relatively constant as the surface nucleons constitute a decreasing fraction of the population of nucleons. This is the behaviour seen in Fig. 6.2.

Generally speaking, however, the number of protons in a nucleus rises with the mass number, so that Coulomb repulsion among the protons plays an increasingly important role for large nuclei. Because of the long-ranged nature of the electromagnetic interaction, protons in different parts of the nucleus are affected by each other, especially at large *Z*. The measured binding energy per nucleon reaches a maximum at around

A = 60 (^{56}Fe, to be precise), and then starts to decrease. Ultimately, very large nuclei become unbound because of the Coulomb repulsion. In summary, both light and heavy nuclei have lower values of *B.E.*/*A* than do intermediate mass nuclei: binding is reduced in light nuclei because of the large surface-to-volume ratio, and binding is reduced in heavy nuclei because of the electrostatic repulsion among protons.

The curve shown in Fig. 6.2 stops at around *A* = 240. Massive nuclei still may be bound (*B.E.* > 0), but their binding energy is so low that they are unstable against breakup into smaller nuclei with a larger *B*/*A*. This phenomenon is referred to as *fission*, examples of which include the decay of very heavy nuclei in the Earth and the nuclear reactions in today's nuclear power reactors. The opposite phenomenon can happen for light (i.e., small-*A*) nuclei that have a smaller *B.E.*/*A* than intermediate mass nuclei. Two small-*A* nuclei can join together to produce a heavier nucleus and liberate energy at the same time. This process is referred to as *fusion*. The nuclear reactions which power the Sun are fusion reactions.

Example 6.3: *How much energy is released if two protons and two neutrons are brought together to form a helium nucleus?*

To calculate the *Q*-value for the reaction

$$2p + 2n \rightarrow {}^4He + \gamma,$$

we proceed as usual:

$$Q = 2m_pc^2 + 2m_nc^2 - m_{He}c^2$$

$$= 2m_pc^2 + 2m_nc^2 - [\, 2m_pc^2 + 2m_nc^2 - B.E.({}^4He)\,]$$

$$= B.E.({}^4He).$$

Substituting for the binding energy of ^{4}He, we find

$$Q = 28.2959 \text{ MeV} = 4.55 \times 10^{-12} \text{ J}.$$

Hence, the energy liberated in the production of a ^{4}He nucleus is 4.6 x 10^{-12} J.

In general, light nuclei (for example, ^{16}O) *cannot* spontaneously fission into even lighter nuclei (e.g. ^{12}C and ^{4}He) because the lighter nuclei are less well bound. The *Q*-value of Eq. (6.6) is negative, for example. Similarly, two very heavy nuclei (for example, two ^{197}Au [gold] nuclei) *cannot* fuse to form a stable heavy nucleus with $A = 394$, because nuclei with *A* more than 250 or so are loosely bound and hence unstable. Generally speaking, then, *fusion* is a process involving light nuclei and *fission* is a process involving heavy nuclei.

6.C Decay Lifetimes

There is an element of randomness in the decay of a particle or fission of a nucleus. Suppose we had a system of N_o identically prepared neutrons that can spontaneously decay and liberate kinetic energy, according to Example 4.4. What do we expect to see happen in a collection of neutrons: will the neutrons stay neutrons for some time, and then suddenly decay all at once into protons and other particles? Or will the neutrons decay individually in a random fashion?

The answer is that the neutrons decay randomly: one cannot *a priori* determine which neutron will decay at what time. The population of neutrons $N(t)$ present at any given time might look like the curve shown in Fig. 6.3. In the figure, the number of neutrons in the sample starts off at N_o and then decreases. The figure assumes that the number of neutrons does not drop linearly to zero: i.e. it does *not* have the form $N(t) = N_o - bt$. What form does $N(t)$ have?

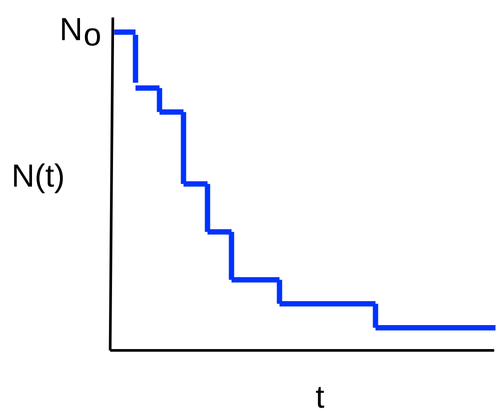

Fig. 6.3. Schematic representation of the number of neutrons $N(t)$ present in a sample as a function of time.

Consider the *rate* of neutron decay $R(t)$, where the rate is defined as the number of neutrons decaying per unit time. At $t = 0$, there are N_o neutrons present and the rate is equal to, say R_o. At some time later, when $N(t)$ has dropped by a factor of two (i.e. when $N(t) = N_o/2$), the rate also must have decreased by a factor of two, since only half the neutrons remain to decay. We refer to this time as $t_{1/2}$. Similarly, at $t_{1/4}$ when only 1/4 of the original neutrons are left: $N(t_{1/4}) = N_o/4$ and $R(t_{1/4}) = R_o/4$.

Let's put into mathematics what we just said in English. The initial rate is R_o and the initial number is N_o. Their ratio is called the decay constant λ, which has units of $[time]^{-1}$:

$$R_o = \lambda N_o. \tag{6.10}$$

But since the decay rate is always proportional to the number of particles present, then it is true at all times that

$$R(t) = \lambda N(t). \tag{6.11}$$

That is, $R(t_{1/2}) = \lambda N(t_{1/2})$, $R(t_{1/4}) = \lambda N(t_{1/4})$ and so on. In other words, the shape of the $R(t)$ *vs.* time curve is the same as the $N(t)$ *vs.* time curve, except for an overall multiplicative constant, λ.

Example 6.4: *A sample of 40,000 neutrons initially decays at a rate of 40 sec $^{-1}$. How many decay per second when there are only 4,000 neutrons left?*

This problem corresponds to N_o = 40,000 and R_o = 40 s^{-1}. Hence, the decay constant λ is, from Eq. (6.11)

$$\lambda = R_o/N_o = 40 / 40{,}000 = 0.001 \text{ s}^{-1}.$$

From Eq. (7.21) we then predict that when there are 4000 neutrons

$$R = \lambda N = 0.001 \times 4000 = 4.0 \text{ s}^{-1}.$$

So, when the number of neutrons drops by a factor of ten, their rate of decay also drops by a factor of ten.

It turns out that Eq. (6.11) specifies how $N(t)$ changes as a function of time. The proof requires a knowledge of derivatives, so those readers who have not yet covered derivatives should skip to Eq. (6.16). The decay rate as we have defined it is a positive quantity. Defining the change in $N(t)$ in a time Δt to be

$$\Delta N \equiv N_{\mathrm{final}} - N_{\mathrm{initial}}, \tag{6.12}$$

then

$$R(t) = -\Delta N/\Delta t \tag{6.13}$$

where the minus sign is required since N is decreasing with time (i.e., ΔN is negative). In the limit of small time changes Δt the right hand side of Eq. (6.13) becomes a derivative and

$$R(t) = -\,dN(t)/dt = \lambda N(t). \tag{6.14}$$

Eq. (6.14) is a *differential equation* that relates $N(t)$ to its derivative $dN(t)/dt$ at any time t. We claim that the solution to this equation is a function of the form $N(t) = N_o \exp(-\lambda t)$. The proof is by direct substitution:

$$\begin{aligned} dN(t)/dt &= d[N_o \exp(-\lambda t)]/dt \\ &= N_o\, d[\exp(-\lambda t)]/dt \\ &= N_o[-\lambda \exp(-\lambda t)] \\ &= -\lambda[N_o \exp(-\lambda t)] = -\lambda N(t). \end{aligned} \tag{6.15}$$

Hence, the number of particles present at a given time t is

$$N(t) = N_o \exp(-\lambda t) \tag{6.16}$$

and their rate of decay is

$$R(t) = R_o \exp(-\lambda t) = \lambda N_o \exp(-\lambda t). \tag{6.17}$$

Clearly, Eqs. (6.16) and (6.17) start at a value of N_o or R_o at $t = 0$, and then decrease to zero as t becomes infinitely large. They do not decrease linearly to zero at a finite value of t.

Example 6.5: *A sample of 40,000 neutrons decay with a decay constant $\lambda = 0.001\ s^{-1}$. How many neutrons are left after 1000 seconds?*

From Eq. (6.16) we have $N(t) = N_o \exp(-\lambda t)$, which becomes

$N(t{=}1000) = 40000 \times \exp(-0.001 \times 1000)$

$= 40000 \times 0.368 = 14700.$

Hence, there are 14,700 neutrons left after 1000 seconds, which is approximately 1/3 of the original number of neutrons.

There are two other constants which are commonly quoted as alternatives to λ. One is the lifetime τ which is related to λ through

$$\tau = 1/\lambda \tag{6.18}$$

so that

$$N(t) = N_o \exp(-t/\tau). \tag{6.19}$$

The other constant is the half-life or $t_{1/2}$, which is related to λ by

$$t_{1/2} = \ln 2 \ / \ \lambda \tag{6.20}$$

so that

$$N(t) = N_o \exp(-\ln 2 \ t \ / \ t_{1/2}) \tag{6.21}$$

where ln2 is the logarithm in base e of 2, or 0.693... . Using the properties of logarithms, Eq. (6.21) can be rewritten as

$$N(t) = N_o \,(e^{\ln 2})^{-t/t_{1/2}} = N_o \, 2^{-t/t_{1/2}} \tag{6.22}$$

Substituting $t = t_{1/2}$ into Eq. (6.22), we find that

$$N(t_{1/2}) = N_o / 2. \quad (6.23)$$

In other words, the half-life is the time that is takes for the sample to decline to half of its original size, or to half of its original decay rate. So after one half-life, only half of the original sample remains, after two half-lives only a quarter of the original sample remains *etc*.

Example 6.6: *An initial sample of 1000 particles decays to 500 particles in 30 seconds. What are* $t_{1/2}$, λ and τ *for the decay?*

We start by calculating the half-life. From Eq. (6.22)

$$500 = 1000\ 2^{-30/t_{1/2}} \qquad \text{or} \qquad 2^{-1} = 2^{-30/t_{1/2}}.$$

Hence

$$t_{1/2} = 30 \text{ s}.$$

The decay constant is related to the half life via $\lambda = \ln 2 / t_{1/2}$, so

$$\lambda = \ln 2 / 30 = 0.023 \text{ s}^{-1}.$$

Finally, the lifetime is the reciprocal of the decay constant, so that

$$\tau = 1/\lambda = 1 / 0.023 = 43 \text{ s}.$$

Lastly, a word about units. Nuclear decay rates are measured in *Curies* (Ci) or *Becquerels* (Bq):

$$1 \text{ Ci} = 3.7 \times 10^{10} \text{ decays per second} \quad (6.24)$$

$$1 \text{ Bq} = 1 \text{ decay per second}. \quad (6.25)$$

It also should be mentioned that some authors prefer to use the word *activity* instead of decay rate.

Example 6.7: *A sample of 10^{20} particles has an initial activity (decay rate) of 2 Ci. What is the decay constant and lifetime of the particles in the sample?*

From Eqs. (6.11) and (6.24) we have

$\lambda = R/N$

$= 2 \times (3.7 \times 10^{10}) / 10^{20} = 7.4 \times 10^{-10}\ s^{-1}$.

The lifetime is then

$\tau = 1 / \lambda = 1 / (7.4 \times 10^{-10}) = 1.4 \times 10^{9}$ s.

6.D Radioactive Dating Techniques

The decay of radioactive or unstable nuclei can be used as a technique for dating the age of an object. In fact, Rutherford was the first to use the technique and show that a terrestrial piece of pitchblende had to be at least 700,000,000 years old. For the measurement to be accurate, the lifetime of the radioactive nucleus must be comparable to the age of the material of interest. Fortunately, nuclear lifetimes span a wide range, so dates have been obtained on everything from the Shroud of Turin to meteorites as old as the solar system. There are many variants of the technique, and the one that we discuss here is ^{14}C dating, which is used to date material from objects like plants or trees which once were alive.

While the most plentiful isotope of carbon on Earth is ^{12}C, there is another isotope of carbon, ^{14}C, that is produced continuously in the atmosphere. The reaction involves neutrons from cosmic ray sources bombarding nitrogen nuclei in the atmosphere:

$$^{1}n + {}^{14}N \rightarrow {}^{1}H + {}^{14}C. \qquad (6.26)$$

The ^{14}C nucleus is unstable and decays with a half-life of 5.73×10^{3} yrs, corresponding to a decay constant of $\lambda = \ln 2 / (5.73 \times 10^{3}\ yr) = 1.21 \times 10^{-4}\ yr^{-1}$. The observed ratio of ^{14}C:^{12}C in the atmosphere today is 1.3×10^{-12}.

If a tree or plant gets all of its carbon from the atmosphere (say, through carbon dioxide), then the ratio $^{14}C{:}^{12}C$ in the tree is the same as the atmospheric ratio as long as the tree is alive. Once the tree dies, then the ^{14}C nuclei will not be replaced as they decay in the tree. Hence, the ratio at any given time is

$$(^{14}C{:}^{12}C)_t = (^{14}C{:}^{12}C)_0 \exp(-\lambda_{14} t) \qquad (6.27)$$

where λ_{14} is the decay constant for ^{14}C decay. So, if we know the $^{14}C/^{12}C$ ratio in the atmosphere at $t = 0$ (when the tree dies), and we can measure it in the remains of the tree today, then we can use Eq. (6.27) to determine t. Conventionally, one makes the assumption that the ratio of ^{14}C to ^{12}C in the atmosphere has not changed since the tree or plant of interest was alive.

Example 6.8: *What is the $^{14}C{:}^{12}C$ ratio in a sample that is 10,000 years old? If the sample is pure carbon and has a mass of 10 g, what mass of ^{14}C is present today?*

Substituting into Eq. (6.27) gives

$$(^{14}C{:}^{12}C)_t = 1.3 \times 10^{-12} \exp(-1.21 \times 10^{-4} \times 10^4)$$

$$= 3.9 \times 10^{-13}$$

With so little ^{14}C present, we can approximate

$$^{14}C{:}^{12}C = {}^{14}C{:}C_{total}$$

and determine

$$^{14}C = 3.9 \times 10^{-13} \times 10 = 3.9 \times 10^{-12} \text{ g}.$$

We see that the amount of ^{14}C present in such an old sample is *very* small.

While the presumption of a constant $^{14}C/^{12}C$ ratio is a good first approximation, in fact the ratio does appear to change with time. The evidence for this comes from dating very old, but still living, trees. There are some trees that are several thousand years old, and one can count the rings in their trunks to get an absolute measure of the tree's age. One can then date the rings using ^{14}C and find just how much the ^{14}C:^{12}C ratio in the atmosphere has changed.

Because of its 5000-year half-life, ^{14}C is typically used to study objects that are less than 25,000 years old. In objects older than this, there is so little ^{14}C left today that the measurement is subject to large errors introduced by contaminants in the measurement or other processes. To date older objects, other radioactive nuclei have to be used. Heavy, long lived elements like uranium and thorium with half-lives in the 10^{10} year range are used to date the age of the Earth and meteoritic debris that strikes the Earth.

6.E Binding Energy Formula (optional)

In this section, we use our qualitative picture of nuclear binding from Sec. 6.B to develop a simple equation describing the nuclear binding energy *B.E.* First, we ask what kind of functional dependence on Z and A should *B.E.* have. A clue can be found in the measured A-dependence of the nuclear radius R. This is discussed in Sec. 1.C, where the relationship

$$R = 1.2\ A^{1/3}\ (\text{fm}) \tag{6.28}$$

is given. The 1/3 exponent in Eq. (6.28) is expected if the nucleus behaves like a liquid or solid that is difficult to compress. To demonstrate this, consider a collection of hard spheres that cannot interpenetrate. The total volume of A such spheres packed together tightly is proportional to A times a reference volume per unit sphere, say V_o. That is

$$[\textit{total volume}] \approx AV_o. \tag{6.29}$$

If A is large enough, the collection of individual hard spheres can be made roughly spherical in shape, with an overall volume equal to:

$$[total\ volume] \approx (4\pi/3)R^3. \qquad (6.30)$$

Hence, for a closepacked incompressible liquid or solid, Eqs. (6.29) and (6.30) can be combined to give $R \quad A^{1/3}$ (see also Sec. 1.C).

Because the strong interaction between the nucleons is *short-ranged*, the strong interaction part of the binding energy of a given nucleon is proportional to its number of nearest neighbours (see also Sec. 6.B). If each nucleon in the interior of a nucleus has about the same number of nearest neighbours, then each individual nucleon in the nuclear interior should have about the same binding energy. To a first approximation, the total binding energy of all A nucleons from the strong interaction is proportional to A, with a proportionality constant c_v, where the subscript "v" stands for volume:

$$[volume\ contribution\ to\ B.E.] \approx c_v A. \qquad (6.31)$$

But how do we take into account the nucleons on the surface of the nucleus? A surface nucleon does not have as many neighbours as an interior nucleon and so its binding energy is less than that of an interior nucleon. The number of surface nucleons is proportional to the nuclear surface area, which is $4\pi R^2$ for a sphere. Since $R \quad A^{1/3}$, then the number of nucleons on the surface is proportional to $A^{2/3}$. We must then reduce the value of *B.E.* in Eq. (6.31), which assumes that all nucleons have the same number of nearest neighbours, by an amount proportional to the number of nucleons on the surface. We parametrize this as

$$[volume + surface\ contribution\ to\ B.E.] \approx c_v A - c_s A^{2/3}, \qquad (6.32)$$

where c_s is a parameter associated with the surface contribution.

We expect other contributions to the binding energy in addition to the attractive part of the strong interaction given in Eq. (6.32). The electromagnetic interaction between the nuclear protons is repulsive, and adds a new term to Eq. (6.32) to give

$$B.E. \approx c_v A - c_s A^{2/3} - (3/5)\ kZ^2 e^2/R. \qquad (6.33)$$

Substituting Eq. (6.28) for the nuclear radius into Eq. (6.33) yields

$$B.E. \approx c_v A - c_s A^{2/3} - c_{em} Z^2 / A^{1/3}, \tag{6.34}$$

with c_{em} = 0.720 MeV.

In these lectures, Eq. (6.34) is called the *simple binding energy formula*. For completeness sake, we should mention that other terms are expected to contribute to *B.E.* arising from the fact that nucleons are fermions. There are many binding energy formulae on the market, with varying numbers of parameters determined by fitting a broad range of nuclear masses. A set of parameters for Eq. (6.34) that works reasonably well for light, stable nuclei is

$$c_v = 16 \text{ MeV} \qquad c_s = 18 \text{ MeV} \qquad c_{em} = 0.720 \text{ MeV}. \tag{6.35}$$

Example 6.9: *Use Eq. (6.34) to calculate the binding energy per nucleon of ^{16}O (Z = 8) and ^{107}Ag (Z = 47).*

For oxygen:

B.E. = $(16 \times 16) - (18 \times 16^{2/3}) - (0.72 \times 8^2 / 16^{1/3})$ = 124 MeV

B.E./A = 7.7 MeV.

From Table C.7, the measured value of *B.E./A* for ^{16}O is 7.98 MeV.

For silver:

B.E. = $(16 \times 107) - (18 \times 107^{2/3}) - (0.72 \times 47^2 / 107^{1/3})$ = 971 MeV

B.E./A = 9.1 MeV.

From Table C.7, the measured value of *B.E./A* for ^{107}Ag is 8.55 MeV. One can see from the silver results that Eq. (6.33) has a 5% error at *A* = 100, since it predicts a binding energy per nucleon which is about 1/2 MeV too high.

Summary

A nucleus is represented by the notation (6.2)

$^{A}[elemental\ symbol]$,

where A is the mass number of the nucleus and the elemental symbol corresponds to the charge on the nucleus. Nuclei with the same Z and differing N are called *isotopes* , while nuclei with the same N and differing Z are called isotones. Finally, nuclei with the same A and differing Z are called isobars.

The binding energy per nucleon B/A for nuclei is observed to be roughly constant at 8 MeV over the range of stable nuclei up to A = 210. There is a maximum in B/A at ^{56}Fe, so that nucleons in light and very heavy nuclei are less well bound than in medium mass nuclei. Thus, light nuclei can combine to form a medium mass nucleus by *fusion*, and heavy nuclei can break apart into medium mass nuclei through *fission*.

Conservation of energy, momentum, charge and baryon number all apply to nuclear reactions and decays. In the reaction $A + B \rightarrow C + D$, conservation of baryon number and charge are written via Eqs. (6.7) and (6.8) as

$$A_{\mathrm{A}} + A_{\mathrm{B}} = A_{\mathrm{C}} + A_{\mathrm{D}}$$

$$Z_{\mathrm{A}} + Z_{\mathrm{B}} = Z_{\mathrm{C}} + Z_{\mathrm{D}}.$$

In a decay, the decay rate $R(t)$ is proportional to the number of particles available for decay $N(t)$ via Eq. (6.11)

$$R(t) = \lambda N(t).$$

where λ is the decay constant. As seen in Eqs. (6.16) and (6.17), the number of particles and their decay rate decrease with time exponentially:

$$N(t) = N_{\mathrm{o}} \exp(-\lambda t)$$

$$R(t) = R_{\mathrm{o}} \exp(-\lambda t) = \lambda N_{\mathrm{o}} \exp(-\lambda t).$$

The lifetime τ and the half-life $t_{1/2}$ are related to the decay constant according to Eqs. (6.18) and (6.20)

$$\tau = 1/\lambda$$

$$t_{1/2} = \ln 2 \, / \, \lambda.$$

Decay rates are frequently quoted in units of Curies (1 Ci = 3.7 x 10^{10} decays per second) or Becquerels (1 Bq = 1 decay per second).

Carbon dating can be used to measure the age of carbon-based organisms which are up to several tens of thousands of years old. The $^{14}C{:}^{12}C$ ratio is given by Eq. (6.27)

$$(^{14}C{:}^{12}C)_t = (^{14}C{:}^{12}C)_0 \exp(-\lambda_{14} t)$$

where $\lambda_{14} = 1.21 \times 10^{-4}$ yr^{-1} and $(^{14}C{:}^{12}C)_0 = 1.3 \times 10^{-12}$.

Nuclear binding energies have a smooth functional dependence on A and Z, and have a direct physical interpretation. The binding energy $B.E.$ of a nucleus with mass number A and charge Z can be described approximately by the simple binding energy formula (6.34)

$$B \approx c_v A - c_s A^{2/3} - c_{em} Z^2 / A^{1/3} \qquad (optional)$$

where c_v = 16 MeV, c_s = 18 MeV and c_{em} = 0.720 MeV.

Further Reading

F. Blatt, *Modern Physics* (McGraw-Hill, New York, 1992), Chap. 15.

W. F. Lippy, *Radiocarbon Dating* (Chicago, 1955), Chap. 1.

J. S. Trefil, *From Atoms to Quarks* (Scribners, New York, 1980), Chap. 2.

S. Weinberg, *The Discovery of Subatomic Particles* (Scientific American, New York, 1983), Chap. 4, App. H.

Problems

1. Calculate the *Q*-values for the following reactions or decays and determine if they can proceed spontaneously:

(a) $^{4}He + ^{4}He \rightarrow ^{8}Be + \gamma$

(b) $^{8}Be + ^{4}He \rightarrow ^{12}C + \gamma$

(c) $^{1}H + n \rightarrow ^{2}H + \gamma$

(d) $n + ^{235}U \rightarrow ^{141}Ba + ^{92}Kr + 3\ n$

(e) $n + ^{3}He \rightarrow ^{3}H + p$.

2. Suppose that the proton could decay via the three step process

(a) $p \rightarrow \pi^{o} + e^{+}$ (b) $\pi^{o} \rightarrow \gamma + \gamma$ (c) $e^{+} + e^{-} \rightarrow \gamma + \gamma$.

In step (c), the positron produced from step (a) annihilates an electron from the medium surrounding the proton. What is the *Q*-value (in MeV) of each reaction, and how much energy is released in total?

*3. The so-called CNO cycle is one means by which ^{1}H is converted to ^{4}He in the Sun. The six steps are

(i) $^{12}C + ^{1}H \rightarrow ^{13}N + \gamma$

(ii) $^{13}N \rightarrow ^{13}C + e^{+} + \nu_e$

(iii) $^{13}C + ^{1}H \rightarrow ^{14}N + \gamma$

(iv) $^{14}N + ^{1}H \rightarrow ^{15}O + \gamma$

(v) $^{15}O \rightarrow ^{15}N + e^{+} + \nu_e$

(vi) $^{15}N + ^{1}H \rightarrow ^{12}C + ^{4}He$.

Find the energy released in each reaction and show that the total is the same as $4 \times ^{1}H \rightarrow ^{4}He + 2\ e^{+}$ (*ignore annihilations*).

4. Calculate the minimum energy (called the separation energy) required to remove a proton from ^{16}O. Compare this result with the average binding energy per nucleon for ^{16}O. Why is the separation energy larger than the binding energy per nucleon?

5. Find the nucleus or particle X in the following reactions:

(a) $^{238}U \rightarrow X + \alpha$

(b) $p + p \rightarrow X + \pi^{+}$

(c) $X \rightarrow ^{230}Th + \alpha$

(d) $^{137}Cs \rightarrow X +$ anti-ν $+ ^{137}Ba$

(e) $^{14}N + X \rightarrow ^{17}O + p$.

6. Find the nucleus or particle X in the following reactions:

(a) $^{235}U + ^{1}n \rightarrow X + \alpha$

(b) $^{12}C + X \rightarrow ^{15}N + p$

(c) $\nu_e + X \rightarrow ^{15}O + e^-$

(d) $^{90}Sr \rightarrow X + e^- +$ anti-ν_e

(e) $^{138}La + X \rightarrow \nu_e + ^{138}Ba$.

7. A particular type of particle decays with a lifetime of 2 seconds. What is the particle's half-life in seconds?

8. It takes 120 days for a radioactive source to decay to 1/4 of its initial activity. What is the decay rate of the source (in Bq) when there are 10^{20} radioactive nuclei in the sample?

*9. In some theories of elementary particle physics, the proton is predicted to have a decay constant of 10^{-39} s^{-1}. If this rate be true,
(a) how many protons decay per second in the Sun?
(b) what is the percentage contribution of the proton's decay to the solar luminosity if the decay is of the form $p \rightarrow \pi^o + e^+$ followed by
$\pi^o + e^+ + e^- \rightarrow 4\gamma$.
Assume the Sun is 100% hydrogen in (a) and (b).

*10. One technique in lubrication research is to implant trace amounts of radioactive material in the surface under study. The amount of wear is then determined by monitoring the residual activity in the sample or the radioactivity in the lubricant. Suppose that ^{59}Fe (decay constant = 1.78 x 10^{-7} s^{-1}) is incorporated uniformly in a planar plate to a known depth *D*, and a 30 day test is performed on the plate. The initial activity of the plate is 50 μCi but this decreases to 20 μCi during the test. What depth of the plate's surface (in terms of *D*) has worn off?

11. Radioactive sources are used extensively in medicine, but the nuclear reactors or accelerators used to produce the isotopes are usually located some distance from the hospitals they serve. Suppose that a source arrives at a hospital with only 80% of its initial activity, and this activity decreases by a factor of 4 over the next 10 days. How long did it take for the source to be delivered to the hospital from its place of manufacture?

*12. A pure 1 g sample of ^{214}Bi decays via

$^{214}\text{Bi} \rightarrow {}^{214}\text{Po} + e^- + \text{anti-}\nu_e$ ($\lambda = 5.9 \times 10^{-4}\ s^{-1}$)

followed by

$^{214}\text{Po} \rightarrow {}^{210}\text{Pb} + \alpha$ ($\lambda = 4.3 \times 10^{3}\ s^{-1}$)

(a) Determine the *Q*-value of the decay chain.
(b) How long (in hours) will the energy release of the sample exceed 100 J/s, the power of a typical light bulb? (Do this problem by approximation noting that the reaction sequence is dominated by the slowest step in the chain.) To calculate the number of ^{214}Bi nuclei present, assume that the mass of ^{214}Bi is 214 times the nucleon mass.

13. Suppose that all of the hydrogen in the Sun could be converted to ^{56}Fe.
(a) Calculate the *Q*-value for the nuclear reaction $56\ {}^{1}\text{H} \rightarrow {}^{56}\text{Fe} + 30\ e^+$. Include the subsequent annihilation $e^+ + e^- \rightarrow 2\gamma$ in your calculation.
(b) Calculate the total energy available in H → Fe conversion in the Sun assuming that the Sun is 100% hydrogen.
(c) How long will the Sun be able to shine at its present luminosity with this energy source (quote your answer in years)?

14. Assume that the Earth is 10^{-8} by weight ^{238}U, which decays by α-emission with a half-life of 4.5×10^9 years. (a) How much energy is released every second through this decay mode? (b) Compare this energy release with the total energy arriving on the surface of the Earth per second from the Sun (which is 0.14 J/s-cm^2 for an element of area directly facing the Sun). Assume that the mass of a uranium nucleus is 238 times the nucleon mass.

15. You wish to date a piece of paper which is said to be 2000 yrs old. The paper weighs only one gram and is 80% carbon. Can you accurately date the sample if the natural radiation background causes an uncertainty in your count of ±20 counts per minute?

16. A sample of wood found in an Egyptian archaeological site shows a ^{14}C activity of 0.16 Bq per gram of carbon. What is the age of the sample?

17. In this question, we use the radiant energy from the radioactive decay of material in the Earth to obtain an estimate of the proton lifetime.
(a) A possible proton decay sequence, which violates a number of conservation laws, is shown in question 2. Similar to question 2, evaluate

the Q-value for the total reaction sequence: p + e^- → 4γ. Quote your answer in Joules.

(b) Assuming that the Earth is made entirely of protons, calculate the rate of energy release per m^2 of the Earth's surface in terms of an unknown decay constant λ for proton decay.

(c) If the surface of the Earth radiates from its interior less energy than it receives from the Sun, determine a bound on λ by equating your answer from (b) to the energy received from the Sun, 1,400 watts/m^2.

(d) Calculate the proton lifetime in years from the decay constant in (c).

18. Suppose that the proton decays with a half-life $t_{1/2} = 10^{32}$ years. An experiment to detect proton decay involves watching a large mass M of water for signs of individual decays. What mass of water is required for one proton to decay, on average, during a year of observation? A single water molecule contains 10 protons and weighs approximately $18 \times 1.67 \times 10^{-27}$ kg.

The following questions use material from Sec. 6.E.

19. Calculate the binding energy of ^{12}C, ^{31}P and ^{56}Fe from the simple binding energy formula Eq. (6.34). Compare your results with the tabulated values in Appendix C.

20. Calculate the binding energy of ^{20}Ne, ^{40}Ar and ^{127}I from the simple binding energy formula Eq. (6.34). Compare your results with the tabulated values in Appendix C.