Angular motion - multiple choice

To increase the moment of inertia of a body about an axis, you must

(a) increase the angular acceleration.
(b) increase the angular velocity.
(c) decrease the angular velocity.
(d) make the body occupy less space.
(e) place part of the body farther from the axis.

The moment of inertia is proportional to R^2; hence, to increase I at fixed mass, the weighted value of R^2 must be increased.

A solid ball and a cylinder roll down an inclined plane. Which reaches the bottom first?

(a) The ball reaches the bottom first.
(b) The cylinder reaches the bottom first.
(c) They arrive at the bottom at the same time.
(d) It depends on the relative masses of the two.
(e) It depends on the relative diameter of the two.

The smaller the moment of inertia (divided by MR^2), the less rotational kinetic energy and the greater translational energy (and speed). A ball has the smallest I / MR^2, namely 2/5, and hence it reaches the bottom first.

The moment of inertia of a body depends on

(a) the angular velocity.
(b) the angular acceleration.
(c) the mass distribution.
(d) the torque acting on the body.
(e) the linear acceleration.

Angular velocity, angular acceleration and torque are kinematic quantities - they do not effect the resistance against angular motion. The moment of inertia depends only on the mass distribution.

A point P is at a distance R from the axis of rotation of a rigid body whose angular velocity and angular acceleration are ω and α ρespectively. The linear speed, centripetal acceleration and tangential acceleration of the point can be expressed as:

	Linear speed	Centripetal acceleration	Tangential acceleration
(a)	$R\omega$	$R\omega^2$	$R\alpha$
(b)	$R\omega$	$R\alpha$	$R\omega^2$
(c)	$R\omega^2$	$R\alpha$	$R\omega$
(d)	$R\omega$	$R\omega^2$	$R\omega$
(e)	$R\omega^2$	$R\alpha$	$R\omega^2$

Only choice (a) is consistent with $v = \omega R$, $a_c = v^2/R = \omega^2 R$, $a_{tan} = \alpha R$.

A woman sits on a spinning stool with her arms folded. When she extends her arms, which of the following occurs?

(a) She increases her moment of inertia, thus increasing her angular speed.
(b) She increases her moment of inertia, thus decreasing her angular speed.
(c) She decreases her moment of inertia, thus increasing her angular speed.
(d) She decreases her moment of inertia, thus decreasing her angular speed.
(e) Her angular speed remains constant by conservation of angular momentum.

The moment of inertia I is proportional to R^2; hence, if her weighted value of R^2 increases, so does I. At fixed L, if I increases, her angular speed must decrease.

When a wheel is rotating at a constant angular acceleration, which of the following statements must be true?

(a) The net force acting on the wheel is constant.
(b) The acceleration of a point at the rim of the wheel is constant.
(c) The speed of a point at the rim of the wheel is constant.
(d) The magnitude of the angular velocity is constant.
(e) The tangential acceleration of a point at the rim of the wheel is constant.

The tangential acceleration a_{tan} is equal to αR. Thus, if the angular acceleration α is constant, so must be a_{tan}.

A man turns with an angular velocity on a rotating table, holding two equal masses at arms' length. If he drops the two masses without moving his arms,

(a) his angular velocity decreases.
(b) his angular velocity remains the same.
(c) his angular velocity increases.
(d) he stops rotating.
(e) his angular velocity changes direction.

The moment of inertia of the man does not change when he drops the masses, so his angular velocity remains the same.

Two points, A and B, are on a disk that rotates about an axis. Point A is three times as far from the axis as point B. If the speed of point B is v, then what is the speed of point A?

(a) v (b) $9v$ (c) $v/3$ (d) $3v$ (e) $\sqrt{3}\ v$

All points on a rigid body rotate at the same ω, so that the tangential speed of a point is proportional to R according to $v = \omega R$. Thus, the speed of A is $3v$ if its position is $3R$.

What is the angular frequency of rotation of the Earth as it rotates about its axis (in angular units)?

(a) $\pi / 12$ (b) $\pi / 43{,}200$ (c) $1 / 86{,}400$ (d) $1 / 24$ (e) none of [a]-[d]

In 24 hours, the Earth completes one revolution, giving an angular frequency of $2\pi / (24 \cdot 3600) = \pi / 43{,}200$.

A turntable rotates through 6 radians in 3 seconds as it accelerates uniformly from rest. What is its angular acceleration in radians per square second?

(a) 1 (b) 6 (c) 2 (d) 1/3 (e) 4/3

Using the expression for constant angular acceleration

$$\theta = \alpha t^2 / 2,$$

we have

$\alpha = 2\theta / t^2 = 2 \times 6 / 3^2 = 4/3$ rad/sec^2.

Old-fashioned clocks and watches have an hour hand, a minute hand and a second hand. What is the angular frequency of the second hand?

(a) 1/60 rad/s (b) 120π rad/s (c) π/60 rad/s (d) 60 rad/s
(e) π/30 rad/s

The second hand completes one revolution in 60 seconds, *i.e.* $T = 60$ s. Thus, the angular frequency is

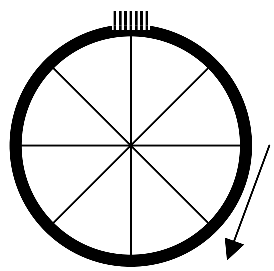
$\omega = 2\pi/T = 2\pi / 60 = \pi/30$ rad/s.

For angular motion, if **R** lies along the *x*-axis with a length 1 m, and **v** points in the negative *z* direction with a magnitude of 1 m/s, what is the angular velocity in rad / s?

(a) (0,1,0) (b) (0,-1,0) (c) (0,-2π,0) (d) (0,2π,0) (e) none of [a-d]

Because **v**, ω and **r** must be perpendicular according to **v** = ω x **r**, then ω must lie along the *y*-axis. Of this choice, ω along +*y* produces **v** along -*z* according to the right-hand rule.

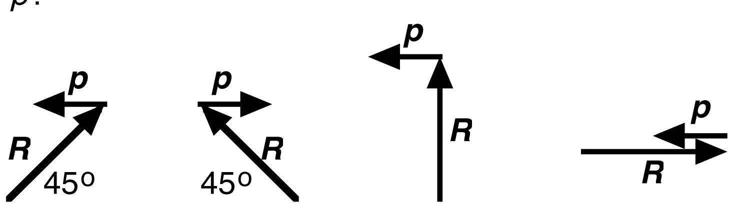
A bike wheel rotates in the direction shown with a frequency of 1 revolution per second. It is brought to a stop in 1 second by friction applied through a block at the top of the wheel. What is the angular acceleration α (in angular units)?



(a) 2π (b) 1 (c) -2π (d) -1
(e) none of [a]-[d]

The initial angular frequency of the bike wheel is -2π radians/second, and this changes uniformly to zero in 1 second. Thus, the angular acceleration is $\omega_f - \omega_i = 0 - (-2\pi) = +2\pi$, and the angular acceleration is (+2π)/1 = +2π.

Which of the following configurations has the largest angular momentum for a given *R* and *p*?



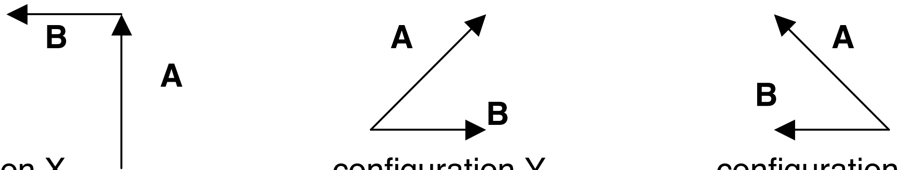
[a] [b] [c] [d] [e] all [a]-[d] are the same

The angular momentum ***L*** for each of these configurations is:

[a] $L = RP/\sqrt{2}$ [b] $L = -RP/\sqrt{2}$ [c] $L = RP$ [d] $L = 0$.

Thus, the largest angular momentum is *RP*.

Vectors **A** and **B** lie in the *xy*-plane of the paper. Their cross product is a vector **C** = **AxB**. For the three orientations of **A** and **B**, order the values of **C** (including its sign) from largest to smallest.

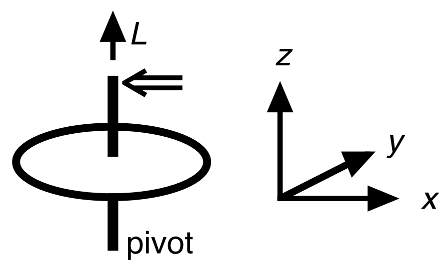


configuration X configuration Y configuration Z

(a) XYZ (b) XZY (c) YZX (d) ZYX (e) none of [a]-[d]

Only configuration Y has a negative cross-product, so it must be the smallest. Having a right angle, configuration X has the largest cross product, so the order from largest to smallest is XZY.

A spinning bike wheel has its angular momentum vector pointing in the + *z* direction. If you press gently on the axle in the - *x* direction, as shown in the diagram, in what direction does the axle move?



(a) +x (b) -x (c) +y (d) -y (e) none of (a) - (d)

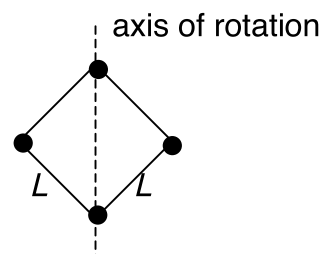
In the figure, the displacement from the pivot point to the point of application is in the +*z* direction, and the force is in the -*x* direction. Hence the torque **R x F** is in the -*y* direction.

A gyroscope rotates about a pivot displaced from its centre of mass location. It has an angular momentum *L* which regains its original orientation after a period *T*. What is the magnitude of the torque acting on the wheel?

(a) L / T (b) $2L / T$ (c) $L / 2T$ (d) $2\pi L / T$ (e) 0

The angular momentum vector sweeps out a total change of $2\pi L$ in a period *T*, so its rate of change is $2\pi L / T$. The torque is equal to the rate of change of angular momentum.

Consider four objects arranged at the corners of a square having dimensions *L* x *L*. What is the moment of inertia of this configuration with respect to an axis lying in the plane of the objects, as shown. Each mass is equal to *M* and is concentrated at a point.

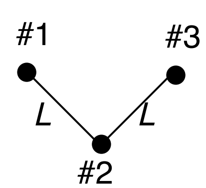


(a) $4ML^2$ (b) $2ML^2$ (c) ML^2 (d) $ML^2/2$ (e) none of [a-d]

The masses along the axis do not contribute to the moment of inertia, only the two masses a distance from the axis $L / \sqrt{2}$ make a contribution. Thus, the moment of inertia is

$$I = 2 \cdot M(L/\sqrt{2})^2 = 2ML^2/2 = ML^2.$$

Consider three objects (1, 2, 3) arranged at right angles,



with distances as shown. What is the moment of inertia of this configuration with respect to an axis passing through object #2 and perpendicular to the 1-2-3 plane. Each mass is equal to M and is concentrated at a point.

(a) $3ML^2$ (b) $2ML^2$ (c) ML^2 (d) $ML^2/2$ (e) none of [a-d]

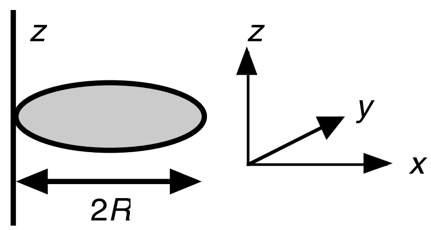
Masses along the axis do not contribute to the moment of inertia, only the two masses a distance from the axis L make a contribution. Thus, the moment of inertia is $I = 2 \cdot ML^2$.

What is the moment of inertia of a thin ring of mass M and radius R if the axis of rotation is in the plane of the ring and passes through its centre?

(a) 0 (b) $MR^2/4$ (c) $MR^2/3$ (d) $MR^2/2$ (e) MR^2

The moment of inertia perpendicular to the plane is MR^2. By the perpendicular axis theorem, this must be equal to the sum of the moments around the x and y-axes in the plane. Thus, the moments about an axis within the plane is $MR^2/2$.

A solid disk has a mass M and radius R. What is the moment of inertia along an axis which is perpendicular to the disk and passes through its edge?



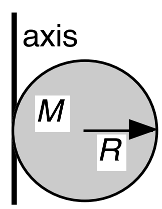
(a) MR^2 (b) $2MR^2$ (c) $MR^2/2$ (d) $(2/5)MR^2$ (e) $(3/2)MR^2$

About the centre-of-mass, the moment of inertia is

$I_{cm} = MR^2/2$ for a disk.

Using the parallel-axis theorem, $I_{||} = MR^2 + I_{cm} = MR^2 + MR^2/2 = 3MR^2/2$.

A solid disk has a mass M and radius R. What is the moment of inertia around an axis which lies in the plane of the disk and passes through its edge?



(a) MR^2 (b) $MR^2/4$ (c) $MR^2/2$ (d) $3MR^2/2$ (e) $5MR^2/4$

About an axis through the centre-of-mass, perpendicular to the plane, the moment of inertia is

$I_{cm,\perp} = MR^2/2$ for a disk.

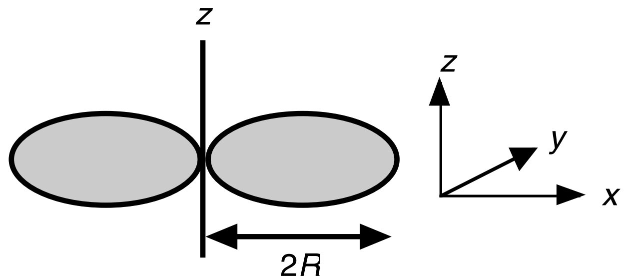
Using the perpendicular-axis theorem, the moment about an axis in the plane is

$I_{plane} + I_{plane} = I_{cm,\perp}$ ---> $I_{plane} = I_{cm,\perp}/2 = MR^2/4$.

From the parallel-axis theorem, the moment about an axis in the plane, along the edge, is

$I_{||} = I_{plane} + MR^2$ ---> $I_{plane} = MR^2/4 + MR^2 = 5MR^2/4$.

Two circular coins, each of mass M and radius R, are welded together at a point as shown. What is the moment of inertia of the pair about an axis perpendicular to the coins and passing through their contact point?



(a) MR^2 (b) $3MR^2$ (c) $2MR^2$ (d) $(3/2)MR^2$ (e) $4MR^2$

The moment of inertia through the centre of a disk, perpendicular to its plane, is $MR^2/2$. The moment through the edge of the disk can be found from the parallel axis theorem to be

$$MR^2/2 + MR^2 = (3/2)MR^2.$$

Adding the moments of the disks together, the total moment is $3MR^2$.

Two thin coins are made from identically the same metal, but one coin has triple the diameter of the other. What is the ratio of the moment of inertia of the large coin compared to the small coin? Take the axis of rotation to be perpendicular to the coin and through its centre; assume that the coins have the same thickness.

(a) 243 (b) 81 (c) 27 (d) 9 (e) 3

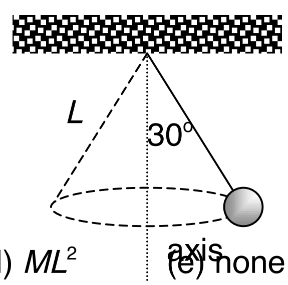
Because the coins are made from the same metal, then they have the same density. The larger coin has $3^2 = 9$ times the mass of the smaller coin. Since the moment of inertia is proportional to MR^2, then the moment of the larger coin will be $9 \cdot 3^2 = 81$ times that of the smaller coin.

Due to the rotation about its axis, the Earth has an angular momentum L. But the Earth also rotates about the Sun. What is the daily change in L?

(a) $L / 365$ (b) $L / 24$ (c) 0 (d) $2\pi L / 365$ (e) $L / 2\pi$

There is no torque acting on the Earth from the Sun, so there is no change in the Earth's angular momentum.

An object of mass M is attached by a string of length L to the ceiling, making an angle of 30° with respect to the vertical axis. The object moves in a horizontal circle, so that the string sweeps out a cone. What is the moment of inertia of the mass with respect to the vertical axis?

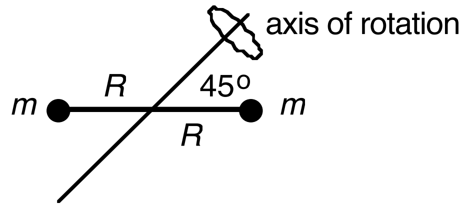


(a) $ML^2/4$ (b) $ML^2/2$ (c) $3ML^2/4$ (d) ML^2 (e) none of (a-d)

The moment of inertia is given by

$$I = MR_\perp^2 = M(L \sin 30^\circ)^2 = ML^2/4.$$

What is the moment of inertia of two objects of mass m separated by a rigid rod of length $2R$, as in the diagram? Take the axis for the moment to intersect the rod at an angle of 45°.



(a) $(4/3)mR^2$ (b) $(1/3)mR^2$ (c) mR^2 (d) $4mR^2$ (e) none of (a)-(d)

The perpendicular distance from the axis to each mass is $R / \sqrt{2}$, so the contribution of each mass to the total moment of inertia is $m(R / \sqrt{2})^2 = mR^2/2$. Then the total moment of inertia must be

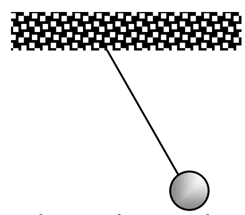
$$I = 2 \cdot (mR^2/2) = mR^2.$$

A uniform copper disk of radius R has a moment of inertia I around an axis passing through the centre of the disk perpendicular to its plane. If the radius of the disk were only $R/2$, but the thickness were the same, what would be the moment of inertia in terms of I?

(a) I (b) $I/2$ (c) $I/4$ (d) $I/8$ (e) $I/16$

The mass of the smaller disk is only 1/4 that of the larger disk, since the mass is proportional to the area ~ R^2. Thus, the moment of inertia $(1/2)MR^2$ scales like R^4. So, the moment of the smaller disk is $(1/2)^4 = 1/16$ that of the larger disk.

A snapshot is taken of a freely-swinging pendulum hanging from a fixed support. What can be said about the motion of the pendulum in a room on Earth from this snapshot?



(a) $\tau < 0$ and $\omega > 0$ (b) $\tau > 0$ and $\omega < 0$
(c) $\tau > 0$ and ω is undetermined (d) $\tau < 0$ and ω is undetermined
(e) none of (a-d)

The speed of the pendulum is **undetermined** by a single snapshot. The torque from gravity gives the mass a clockwise angular acceleration: $\tau < 0$.

A uniform thin rod of mass m and length $4x$ pivots about a point a distance x from one of its ends. What is the moment of inertia about this point?

(a) $mx^2/12$ (b) $13mx^2/12$ (c) $4mx^2/3$ (d) $7mx^2/3$ (e) mx^2

About the centre-of-mass, the moment of inertia is

$$I_{cm} = m(4x)^2/12 = 16mx^2/12 = 4mx^2/3.$$

Using the parallel-axis theorem, $I_{||} = mx^2 + I_{cm} = mx^2 + 4mx^2/3 = 7mx^2/3$.

A hollow ring of mass m and radius R rolls without slipping along a table at speed v. What is its kinetic energy?

(a) $mv^2/4$ (b) $mv^2/2$ (c) mv^2 (d) $3mv^2/2$ (e) $2mv^2$

The speed of the ring is v, and its angular speed is $\omega = v/R$; its moment of inertia is $I = mR^2$. Thus, the total kinetic energy is

$$K.E. = mv^2/2 + I\omega^2/2 = mv^2/2 + (mR^2)(v/R)^2/2 = mv^2/2 + mv^2/2 = mv^2.$$

What is the kinetic energy of a solid cylinder of mass *m* which rolls without slipping on a level surface with velocity *v*?

(a) 0 (b) $mv^2/4$ (c) $mv^2/2$ (d) 3 $mv^2/4$ (e) mv^2

The linear part of the kinetic energy is $mv^2/2$.

The angular part of the kinetic energy is $I\omega^2/2$ with $I = mR^2/2$ for solid disks, and $\omega = v / R$.

Thus, the total kinetic energy is $mv^2/2 + (1/2)\cdot(mR^2/2)\cdot(v / R)^2 = 3mv^2/4$.

A solid sphere of mass *M* and radius *R* rolls without slipping along a table at speed *v*. What is its kinetic energy?

(a) $Mv^2/2$ (b) Mv^2 (c) $7Mv^2/10$ (d) $Mv^2/5$ (e) $3Mv^2/2$

The moment of inertia of a solid sphere is $I = (2/5)MR^2$. Combined with $\omega = v/R$, the kinetic energy is

$KE = mv^2/2 + I\omega^2/2 = mv^2/2 + (1/2)\cdot(2/5)mR^2(v/R)^2 = Mv^2(1/2 + 1/5) = (7/10)Mv^2$.

Two solid rods labeled *A* and *B* have the same mass but lengths 3*L* and 4*L*. If they rotate about their centers with the same angular frequency, what is the ratio of the angular kinetic energy of rod *A* compared to rod *B*?

(a) 4/3 (b) 3/4 (c) 16/9 (d) $(4/3)^{1/2}$ (e) 9/16

The angular kinetic energy is equal to $I\,\omega^2/2$, so the ratio of the kinetic energies must be $K_A/K_B = I_A / I_B = (3L)^2/(4L)^2 = 9/16$.

Three objects with the same mass - a solid sphere, a solid disk and a hollow cylinder - roll without slipping up an incline plane. Each object has the same initial speed at the bottom of the plane. What is the maximum height that each object can attain?

(a) $h_{sphere} < h_{disk} < h_{cylinder}$ (b) $h_{sphere} < h_{disk} = h_{cylinder}$

(c) $h_{sphere} > h_{disk} > h_{cylinder}$ (d) $h_{sphere} = h_{disk} = h_{cylinder}$

(e) none of (a)-(d)

The total kinetic energy of a rolling object is given by

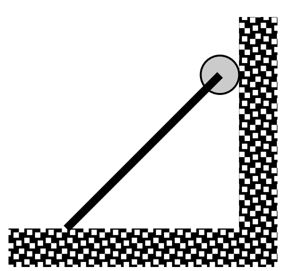
$$K = mv^2/2 + I\omega^2/2$$
$$= mv^2/2 + xMR^2\cdot(v/R)^2/2 \qquad \text{where } x = I / MR^2$$
$$= (1 + x)mv^2/2$$

Thus, the object with the largest *x* has the largest kinetic energy and will rise up the plane the furthest. From the info at the top of the exam

$x_{sphere} = 2/5$ $x_{disk} = 1/2$ $x_{cylinder} = 1$.

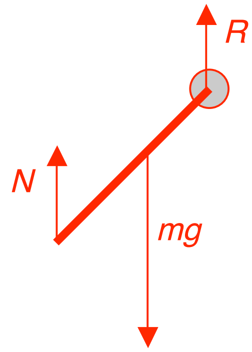
Thus, the cylinder should rise the highest and the sphere the least.

The upper end of a rod of mass *m* is attached to a wall by a frictionless hinge, as shown, while its lower end rests on a frictionless surface. What is the magnitude of the force on the lower end of the rod?



(a) *mg* (b) larger than *mg* (c) 0 (d) less than *mg* (e) none of (a-d)

The free-body diagram of the rod looks like



Taking the hinge to be the location of a rotational axis, we have for zero torque:

$$mg R_{\perp} = N\,2R_{\perp}$$

where $R_{\perp}$ is the perpendicular distance from the hinge to the line of action of *mg*. Thus

$$N = mg / 2.$$

and

$$N < mg.$$

Angular motion - problems

A CD accelerates uniformly from rest to its operating frequency of 310 revolutions per minute in a time of 4 seconds. During this time:
(i) What is its angular acceleration?
(ii) Through what angle does it spin?

(a) The operating frequency of f = 310 rpm is an angular speed of

$\omega = 2\pi f = 2\pi \cdot 310 / 60 = 62\pi/6 = 31\pi/3$.

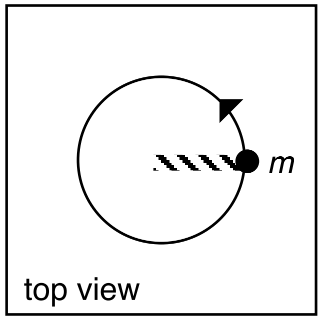
The angular acceleration can then be obtained from

$\alpha = (\omega_f - \omega_i) / \Delta t$
$= (31\pi/3 - 0) / 4$
$= 31\pi/12$ rad/s^2.

(b) For constant angular acceleration

$\theta = \omega_i t + \alpha t^2 /2$
$= 0 + (31\pi/12) \cdot 4^2 / 2$
$= 2 \cdot 31\pi / 3$ radians (or 65 rad)

A massless spring has an unstretched length L and spring constant k The spring is placed horizontally on a frictionless surface. One end of the spring is attached to a fixed pivot, while a mass m is attached to the other end. The mass is given an impulse, causing it to execute a circular path with a period T. What is the extension x of the spring (from equilibrium), expressed in terms of m, L, T and k?



Once the mass is in motion, the restoring force of the spring

$$F = kx$$

is equal to the centripetal force on the mass

$$F = ma = mv^2/(L+x).$$

To eliminate the velocity in favour of the period, we note that

$$v = 2\pi(L+x)/T$$

so that

$$F = m\,[2\pi(L+x)/T\,]^2 \,/\, (L+x)$$
$$= 4\pi^2 m(L+x) \,/\, T^2.$$

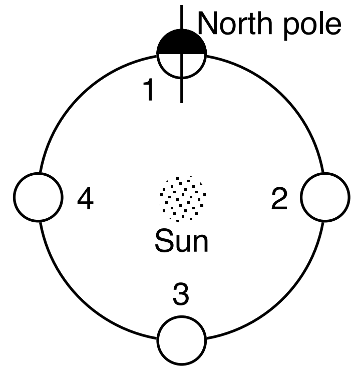
Equating the forces gives

$$kx = 4\pi^2 m(L+x) \,/\, T^2$$
$$kT^2 x \,/\, (4\pi^2 m) = L+x$$
$$x\,[kT^2/(4\pi^2 m) - 1] = L$$

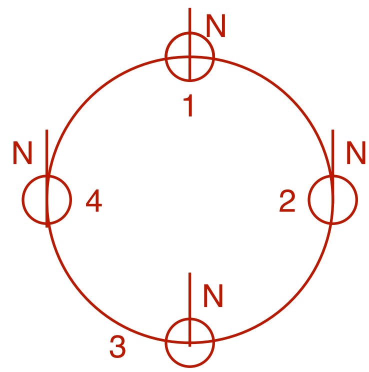
or, finally

$$x = L \,/\, [kT^2/(4\pi^2 m) - 1].$$

The planet Uranus has its axis of rotation in the same plane as its orbit about the Sun. Describe a uranian day (where is the Sun during the day) at its north pole and equator for the four positions shown in the diagram. How many different seasons would the pole and the equator experience (*i.e.* spring, summer, fall, winter).



Because of conservation of angular momentum, the axis of rotation always points in the same direction. Thus, the orientation of the axis is



At north pole:

Position	daylight
1	0%
2	on horizon
3	100%
4	on horizon

The north pole would experience 4 seasons, as on Earth.

At equator:

Position	daylight
1	on horizon
2	50%
3	on horizon
4	50%

The equator would experience 2 seasons.

The Earth rotates about its axis and also revolves around the Sun.
(a) What is the angular momentum of the Earth due to its rotation about its axis?
(b) What is the angular momentum of the Earth due to its motion around the Sun?
(c) If the Earth suddenly reversed its direction of rotation, what would be the fractional change in the length of an Earth year? Assume that the force causing this reversal acts only between the Earth and the Sun.
Need:
$M_{earth} = 5.98 \times 10^{24}$ kg $\quad R_{earth} = 6.37 \times 10^6$ m $\quad R_{e\text{-}s} = 1.50 \times 10^{11}$ m.

(a) The moment of inertia of the Earth is

$$I_{earth} = 2/5\ M_{earth}R_{earth}^2 = (2/5) \cdot 5.98 \times 10^{24} \cdot (6.37 \times 10^6)^2$$
$$= 9.71 \times 10^{37}\ \text{kg}\cdot\text{m}^2.$$

The angular frequency of rotation ω is

$$\omega_{earth} = 2\pi / 1\ \text{day} = 2\pi / (24 \times 3600) = 7.27 \times 10^{-5}\ \text{rad/sec}.$$

Thus, the angular momentum of the Earth due to its rotation is

$$L_{rot} = I_{earth}\omega_{earth} = 9.71 \times 10^{37} \cdot 7.27 \times 10^{-5} = 7.06 \times 10^{33}\ \text{kg}\cdot\text{m}^2/\text{s}.$$

(b) The angular momenum of the Earth-Sun system is found from

$$I_{e\text{-}s} = M_{earth}R_{e\text{-}s}^2 = 5.98 \times 10^{24} \cdot (1.50 \times 10^{11})^2$$
$$= 1.35 \times 10^{47}\ \text{kg}\cdot\text{m}^2.$$

The angular frequency of rotation ω is

$$\omega_{e\text{-}s} = 2\pi / 1\ \text{year} = 2\pi / (365 \times 24 \times 3600) = 1.99 \times 10^{-7}\ \text{rad/sec}.$$

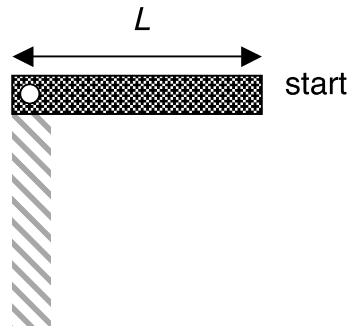
Thus, the angular momentum of the Earth due to its revolution is

$$L_{rev} = I_{e\text{-}s}\omega_{e\text{-}s} = 1.35 \times 10^{47} \cdot 1.99 \times 10^{-7} = 2.69 \times 10^{40}\ \text{kg}\cdot\text{m}^2/\text{s}.$$

(c) If the rotation reverses, then L_{rev} must increase by $2L_{rot}$ by conservation of angular momentum. Thus, the fractional change in the length of the Earth year is

$$2 \cdot 7.06 \times 10^{33} / 2.69 \times 10^{40} = 5.25 \times 10^{-7}.$$

A rod of uniform mass of mass M and length L is free to pivot about a hole drilled through one end, as in the diagram. Initially at rest, it is released from a horizontal position. What is its angular frequency as it swings through the vertical position, in terms of g and the characteristics of the rod? (11 marks)



The centre-of-mass position of the rod falls though a height difference $L/2$ as it falls from the horizontal to vertical position. Thus, the potential energy released is

$$\Delta P.E. = Mg\,(L/2).$$

The kinetic energy of the rod at any position is given by

$$K.E. = I\,\omega^2/2,$$

where I is the moment of inertia. For a rod pivoting about an axis through its end, the moment of inertia is

$$I = ML^2/3,$$

so the kinetic energy is

$$K.E. = (ML^2/3)\cdot(\omega^2/2) = ML^2\omega^2/6.$$

Equating this to the potential energy change, one has

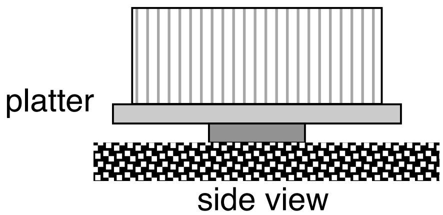
$$ML^2\omega^2 / 6 = Mg\,(L/2)$$

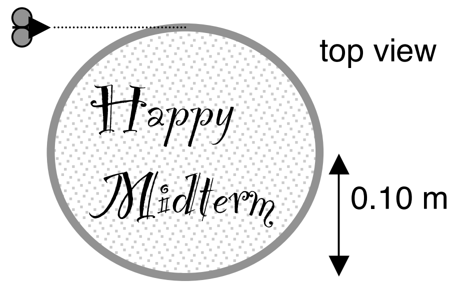
or

$$\omega^2 = 3g/L$$
$$\omega = (3g/L)^{1/2}.$$

If student used $I = ML^2/12$, then $K.E. = ML^2\omega^2 / 24$ and $\omega = (12g/L)^{1/2}$.

A cake rests on a platter, which can freely rotate without friction. A fly, travelling horizontally at 0.5 m/s, lands on the rim of the cake. What is the angular speed of the cake after the fly lands? The mass of the fly is 0.1 g, and that of the cake is 300 g, so that the mass of the fly is negligible compared to the cake once the fly lands. The radius of the cake is 0.10 m.





The initial angular momentum of the fly with respect to the axis of rotation of the cake is

$$L = Rp = Rmv,$$

where R is the radius of the cake (0.10 m) and m is the mass of the fly (0.0001 kg). Thus

$$L = 0.1 \cdot 0.0001 \cdot 0.5 = 5.0 \times 10^{-6} \text{ kg-m/s}.$$

By conservation of angular momentum, L of the cake after the fly lands must be 5.0×10^{-6} kg-m/s.

To find the angular frequency of the cake, we first need to evaluate its moment of inertia. For a cylinder,

$$I = mR^2/2 = 0.3 \cdot 0.1^2 / 2 = 1.5 \times 10^{-3} \text{ kg-m}^2.$$

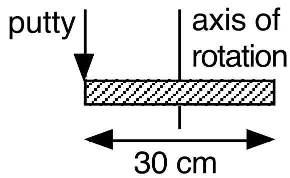
Then

$$L = I\omega$$

implies

$$\omega = L / I = 5.0 \times 10^{-6} / 1.5 \times 10^{-3} = 3.3 \times 10^{-3} \text{ rad/s}.$$

A record and turntable are rotating without friction at 0.6 revolutions per second. A piece of putty is dropped onto the edge of the record, where it sticks. What is the angular speed of the turntable after the putty sticks? The mass of the putty is 0.100 kg, and the mass of the record and turntable combined is 0.500 kg. Assume that there is no motor attached to the turntable.



We solve this problem using conservation of angular momentum.

The initial value of the angular frequency ω is

$$\omega_i = 2\pi \cdot 0.6 = 3.77 \text{ rad/sec,}$$

and the moment of inertia of the turntable is

$$I_{turn} = MR^2/2 = 0.500 \cdot 0.15^2 / 2 = 0.0056 \text{ kg-m}^2.$$

Thus, the initial angular momentum L is

$$L_i = I_{turn}\omega_i = 0.0056 \cdot 3.77 = 0.021 \text{ kg-m}^2\text{/s.}$$

The final moment of inertia is the sum of the turntable plus the putty:

$$I_f = I_{turn} + M_{putty}R^2$$
$$= 0.0056 + 0.1 \cdot 0.15^2$$
$$= 0.0056 + 0.0023 = 0.0079 \text{ kg-m}^2\text{/s.}$$

By conservation of angular momentum

$$L_f = L_i,$$

we have

$$I_f\omega_f = L_i,$$

or

$$\omega_f = L_i / I_f.$$

Numerically

$$\omega_f = 0.021 / 0.0079 = 2.68 \text{ rad/sec.}$$

An idealized simple pendulum consists of a mass m suspended from a massless string of length L, as in the diagram. The mass swings back an forth across its equilibrium position (where the string is vertical) and the angle θ varies with time as

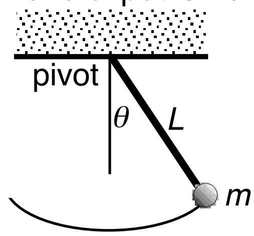
$$\theta(t) = A \sin(\beta t),$$

where A is the maximum value of θ and β is a constant.

(i) What is the angular frequency of rotation as a function of time?

(ii) What is the angular acceleration as a function of time?

(iii) At what position(s) is the magnitude of the angular acceleration a maximum? Are these positions consistent with what one expects from the forces on the mass?



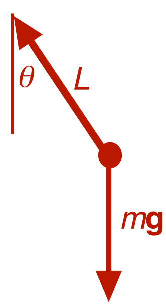
(i) The angular frequency of rotation ω is given by

$$\begin{aligned}\omega(t) &= \mathrm{d}\theta(t) / \mathrm{d}t \\ &= \mathrm{d}/\mathrm{d}t\,[A\sin(\beta t)] \\ &= A\beta\ \cos(\beta t)].\end{aligned}$$

(ii) The angular acceleration α is

$$\begin{aligned}\alpha &= \mathrm{d}\omega/\mathrm{d}t \\ &= \mathrm{d}/\mathrm{d}t\,[A\beta\ \cos(\beta t)] \\ &= -A\beta^2 \sin(\beta t)\end{aligned}$$

(iii) Since both α and θ have the same functional form, then α is a maximum when θ is a maximum. At the extreme positions of the swing, the torque τ also is a maximum from the behaviour of **R** x **F**:



$\boldsymbol{\tau} = \mathbf{R} \times \mathbf{F} = Lmg \sin\theta$

Thus, τ is a maximum when θ is a maximum.

A mouse of mass 20 grams walks around the edge of a horizontal turntable, which may be viewed as uniform disk of mass 200 grams. If both the turntable and mouse are initially at rest, how much does the turntable rotate relative to the ground while the mouse makes one complete circle relative to the turntable?

•The initial angular momentum is zero.
•By conservation of angular momentum the angular momenta of the mouse and turntable must be equal in magnitude.
---> $I_{mouse}\,\omega_{mouse} = I_{t\text{-}table}\,\omega_{t\text{-}table}$
•Since the angular displacement of the objects is $\Delta\theta = \omega t$,
then for any given time $I_{mouse}\,\omega_{mouse} t = I_{t\text{-}table}\,\omega_{t\text{-}table} t$
or $I_{mouse}\,\theta_{mouse} = I_{t\text{-}table}\,\theta_{t\text{-}table}$
or $\theta_{mouse} = (I_{t\text{-}table} / I_{mouse})\theta_{t\text{-}table}$.

The moments of the objects are $I_{mouse} = 20R^2$

$I_{t\text{-}table} = (1/2)\,200R^2$

so that $(I_{t\text{-}table} / I_{mouse}) = 200 / (2 \cdot 20) = 5$

Hence $\theta_{mouse} = 5\,\theta_{t\text{-}table}$.

The mouse completes one circle around the outside

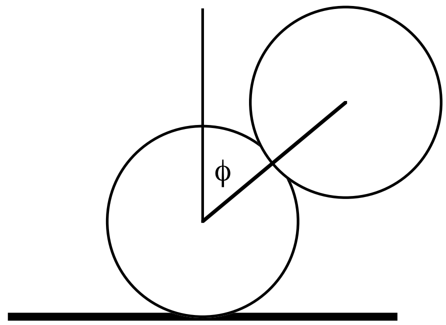
$\theta_{mouse} + \theta_{t\text{-}table} = 2\pi$

or $5\theta_{t\text{-}table} + \theta_{t\text{-}table} = 2\pi$

or $6\theta_{t\text{-}table} = 2\pi$

or $\theta_{t\text{-}table} = \pi/3 = 60^{o}$

Two identical balls (of mass M and radius R) sit on top of one another. The bottom ball is glued to a table. The top ball rolls without slipping down the bottom ball, and finally falls off. At what angle θ do the balls loose contact?

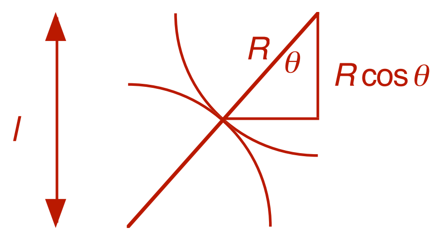


Need $I = 2/5\ MR^2$.

Consider the upper ball dropping through a height h while still in contact with the lower ball's surface. The change in gravitational potential energy is Mgh and this gives the ball a velocity v and rotational speed ω,

$$Mgh = I\omega^2/2 + Mv^2/2. \qquad (1)$$

To proceed further, we need to relate the change in height to θ.



The change in height $h = 2R - l$,

but $l = 2R \cos\theta$.

Hence

$$h = 2R\,(1 - \cos\theta).$$

We also need to relate v to ω. Since the top ball moves in a circular path of radius $2R$ around the bottom ball, then

$$v = 2R \cdot \omega.$$

Returning to Eq. (1),

$$Mg \cdot 2R\,(1 - \cos\theta) = [2/5\ MR^2]\cdot(v\,/\,2R)^2/2 + Mv^2/2$$

$$2gR\,(1 - \cos\theta) = (2/5)\cdot(1/2)\cdot(1/4)\ v^2 + v^2/2 \qquad \text{(cancelling } M\text{)}$$

$$2gR\,(1 - \cos\theta) = v^2/20 + v^2/2$$

$$= 11v^2/20. \qquad (2)$$

Now, as long as the centripetal acceleration

$a_c = v^2/2R$ (note the $2R$)

is less than the radial component of **g**, the ball will stay in contact. The radial component of **g** is

$g\cos\theta$,

so the condition for loss of contact is

$v^2/2R = g\cos\theta$. (3)

Substituting (3) into (2) gives

$2gR(1 - \cos\theta) = 11\,[2Rg\cos\theta]/20$

$1 - \cos\theta = 11\cos\theta/20$

$1 = 31\cos\theta/20$

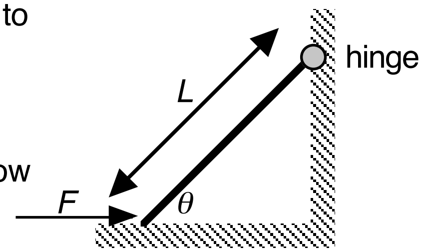
or

$\cos\theta = 20/31$.

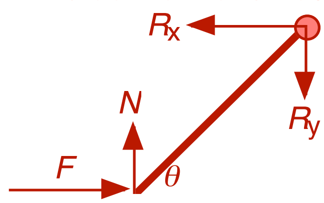
This gives

$\theta = 49.8^{\circ}$.

A thin massless rod is attached to a wall by a hinge and rests on a frictionless floor at an angle θ with respect to the horizontal. A force *F* is applied horizontally to the rod at the point where it touches the floor.
(a) What is the magnitude of the total reaction force on the rod at the hinge as a function of *F* and θ only (show a free-body diagram; 11 marks).
(b) For a given *F*, what is the minimum reaction force and for what angle θ does it occur? (2 marks)



(a) First, set up a free-body diagram to define the forces.



From the conditions of static equilibrium:

in *x*-direction $R_x = F$ (1)

in *y*-direction $R_y = N$ (2)

no rotation around bottom of rod $FL\sin\theta = NL\cos\theta$. (3)

Solve Eq. (3) for *N*:

$N = F\sin\theta / \cos\theta$. (4)

Substitute (1) - (4) into the expression for the reaction force:

$R^2 = R_x^2 + R_y^2 = F^2 + N^2 = F^2(1 + [\sin\theta / \cos\theta]^2)$

Simplifying

$R^2 = F^2(\cos^2\theta + \sin^2\theta) / \cos^2\theta$

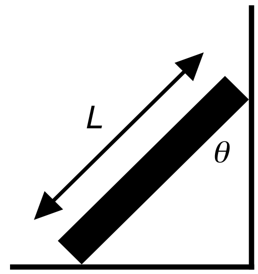
or

$R = F/\cos\theta$.

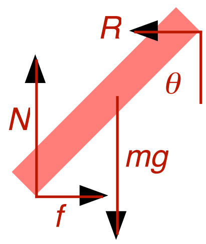
(b) The minimum value of the reaction force occurs when $\cos\theta = 1$, or $\theta = 0$, at which position

$R = F$.

A thin book of mass M and length L sits on a shelf and leans against a wall at an angle θ with respect to the vertical. The coefficient of friction between the book and the shelf is equal to μ. What is the minimum value of μ required to stop the book from sliding? (Include a free-body diagram and neglect the thickness of the book compared to its length)



First, construct a free-body diagram of the book and determine the relevant forces:



There are 4 forces on the book, but only one force is known, namely mg. Hence, three equations are needed to solve for the forces (2 translations and one rotation). In addition, θ is not known, and so the maximal friction equation $f = \mu N$ must also be used.

no net force in x-direction $R = f$ (1)
no net force in y-direction $N = mg$ (2)
no rotation around bottom of book $mg \cdot (L/2) \cdot \sin\theta = RL \cos\theta$. (3)

Substitute (1) into (3), cancelling L

$(1/2)\ mg\ \sin\theta = f \cos\theta$,

or

$\tan\theta = 2f / mg$ (5)

But the maximal friction force is

$f = \mu N$,

or, from Eq. (2)

$f = \mu mg$. (6)

Substituting (6) into (5) yields

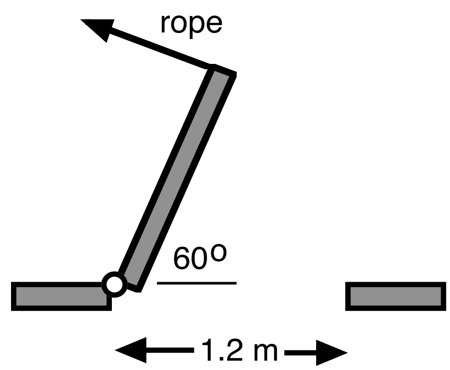
$\tan\theta = 2\mu mg / mg = 2\mu$. (7)

A trap door which is 1.2 m to the side has a mass of 25 kg. The door is attached to the floor by a hinge, and is pulled up by a rope perpendicular to the surface of the door. If the door is held at an angle of 60 ^{o} with respect to the horizontal:

(a) what is the tension *S* in the rope?

(b) what is the force on the hinge from the door and rope?

In (a) and (b), quote the magnitude of the force and direction with respect to the floor.



(a) From the equilibrium condition that there is no rotational motion:

Apply around hinge

total torque = 0 = + $S \cdot 1.2$ - $mg \cdot (1.2 / 2) \cdot \cos 60^o$

$= 1.2S - 0.3 \cdot 25 \cdot 9.81$

--> $1.2S = 73.6$

--> $S = 61.3$ N

The direction of *S* is along the rope, 30^o with respect to the horizontal.

(b) [11 marks] The net force from the door and rope is the vector sum $\mathbf{F}_{net} = \mathbf{F}_{door} + \mathbf{F}_{rope}$:

In y-direction: net force on hinge $F_y = - mg + S\sin30^o$

$= -25 \cdot 9.8 + 61.3 \cdot \sin30^o$

$= -245 + 30.7$

$= -214.4$ N

In x-direction: net force on hinge $F_x = - S\cos30^o$

$= - 61.3 \cos30^o$

$= - 53.1$ N

The total force on the hinge is then

$(F_x^2 + F_y^2)^{1/2} = (214.4^2 + 53.1^2)^{1/2}$

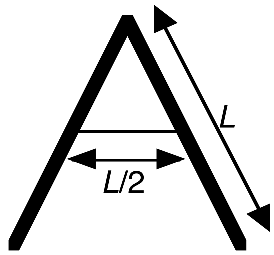
$= 220.9$ N

The force makes an angle of

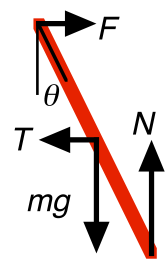
$\tan\theta = F_y / F_x = (-214.4) / (-53.1) = 4.04$

or = 76^o below the horizontal, pointing to the left.

A step ladder consisting of two identical ladders, each of mass *M* and length *L*, is held together by a massless crossbar attached to the midpoints of the ladders. What force is exerted on the crossbar by each of the ladders, if the length of the crossbar is *L*/2? Assume that the forces at the top of the ladders are strictly horizontal, and ignore friction between the ladders and the floor. Solve for all angles, and express your answer in terms of *mg* only. Show a free-body diagram for one of the ladders.



The tension *T* and the normal force *N* are defined in the free-body diagram through



In the vertical direction: $N = mg$

Torque around the top of the ladder:

clockwise: $mgL / 4 + T(L/2)\cos\theta$

counterclockwise: $NL\sin\theta = mgL\sin\theta$

Equating torques

$mgL / 4 + T(L/2)\cos\theta = mgL\sin\theta$

$(T/2)\cos\theta = mg\sin\theta - mg / 4$

$T = 2mg\,[\sin\theta - 1/4] / \cos\theta.$

To proceed further, we need to eliminate the angles. Clearly

$\sin\theta = (L/4) / (L/2) = 1/2$

from which

$\cos\theta = [1 - \sin^2\theta]^{1/2} = [1 - 1/4]^{1/2} = \sqrt{3}/2.$

This gives

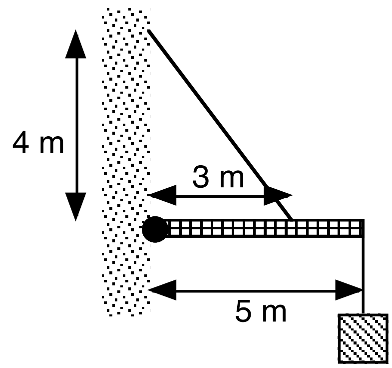
$T = 2mg\,[1/2 - 1/4] / (\sqrt{3}/2)$

$= 4mg\,[1/4] / \sqrt{3}$

or

$T = mg / \sqrt{3}.$

One end of a uniform 50 kg beam 5 m in length is attached to a vertical wall by a hinge. It is held horizontally by a flexible cable which connects to the beam 3 m from the wall. A 200 kg weight is attached to the beam.
(a) What is the tension *T* in the cable (7 marks)?
(b) What force *F* does the hinge exert on the beam (8 marks)?



(a) To find the tension *T* in the cable, consider the torques around the hinge:

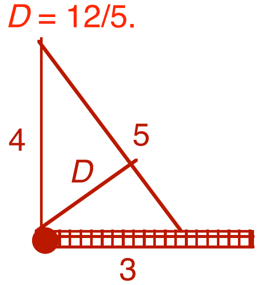
$$DT = 5 \cdot 200g + 2.5 \cdot 50g.$$

where *D* is the perpendicular distance from the hinge to the cable.
What is *D*? From Pythagoras' theorem, the length of the cable is $\sqrt{(3^2 + 4^2)} = 5$. By similar triangles in the diagram,

$$D / 4 = 3/5$$

or

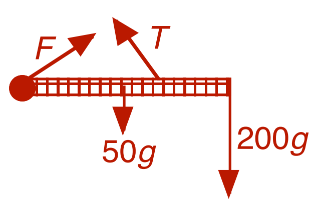
$$D = 12/5.$$



Thus,

$$T = (1000 + 125)g / D = 1125 \cdot 9.8 / (12/5) = 4590 \text{ N}.$$

(b) Now that we know *T*, we can construct a free-body diagram for the beam:



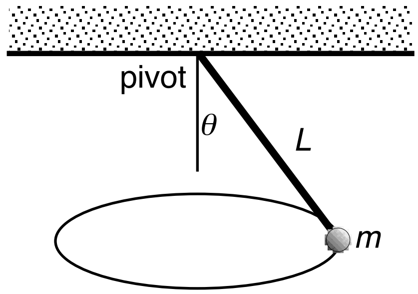
x-component of tension = -(3/5) 4590 = 2750 N
y-component of tension = (4/5) 4590 = 3670 N

Hence,

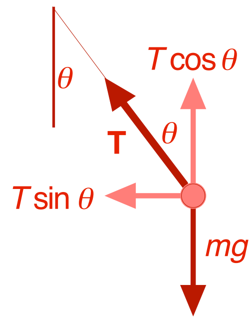
x-component of F = 2750 N
y-component of $F = -3670 + 50g + 200g = -1220$ N

Total force from hinge = $(2750^2 + 1220^2)^{1/2} = 3010$ N.

A conical pendulum is different from a simple pendulum in that the mass *m* executes a circular motion about the pivot at a constant angle θ. That is, the velocity vector *v* of the mass lies in a horizontal plane. Find *v* as a function of *L* (the length of the string), *g* (the acceleration due to gravity), and θ. Show a complete solution, including a free-body diagram of the forces on the mass.



The free-body diagram of the mass involves the tension in the string *L* and force due to gravity.



x-direction: unbalanced force of $T \sin\theta$ y-direction: $T \cos\theta = mg$

In the x-direction, the tension is unbalanced and gives rise to a centripetal acceleration

$$a = v^2 / R$$

where *R* is the radius of the orbit in the plane. Thus,

$$T \sin\theta = ma = mv^2 / R$$

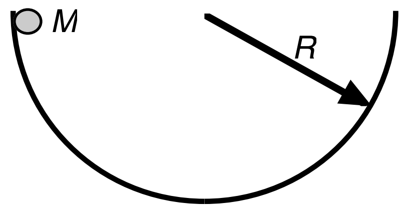
But $T = mg / \cos\theta$, so

$$mv^2 / R = \sin\theta \, mg \,/ \cos\theta \quad \text{->} \quad v^2 = Rg \sin\theta \,/ \cos\theta$$

But the orbital radius is given by $R = L \sin\theta$, so

$$v^2 = Lg \sin^2\theta \,/ \cos\theta \quad \text{->} \quad v = (Lg \,/ \cos\theta)^{1/2} \sin\theta$$

A point-like object of mass M slides down a frictionless surface in the shape of a hemisphere (of radius R). If the object starts from rest at the top rim of the surface, find the reaction force on the object when it reaches the bottom of the surface. Quote your answer in terms of M, R, and g, as needed.



The kinetic energy of the object when it reaches the bottom of the surface can be found from the change in the potential energy:

$$mgR = mv^2/2 \quad \text{or} \quad v^2/R = 2g.$$

Thus, the centripetal acceleration is $v^2/R = 2g$, and the net force is $2mg$.

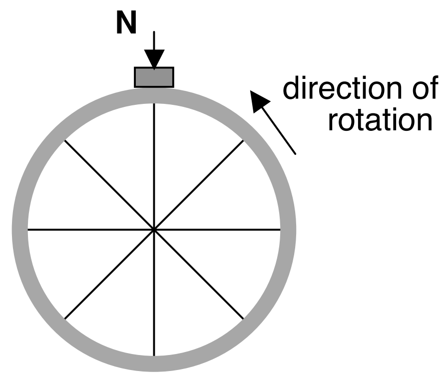
The net force on the object, $2mg$, results from the normal force acting upwards, and the weight mg acting downwards. Thus, the normal force must be $3mg$.

A thin bike wheel of radius 0.2 m and mass 2.0 kg is spinning at 5 revs/second. A block is pushed against the top of the wheel with a force **N** of 10 N. The coefficient of kinetic friction between the block and the wheel is 0.9.

(a) What is the angular momentum of the wheel before the block is applied (include the sign)?

(b) What torque does the block exert on the wheel (include the sign)?

(c) How long does it take for the wheel to stop?



(a) The angular frequency of rotation ω is given by $\omega = 2\pi \cdot 5\ s^{-1}$.
The moment of inertia for the bike wheel is $I = MR^2 = 2 \cdot (0.2)^2 = 0.080$ kg-m^2.
Hence, the angular momentum of the wheel is $L = I\omega = 0.08 \cdot (2\pi \cdot 5) = 2.51$ kg-m^2/s.
Since the motion is counter-clockwise, then the sign is positive.

(b) The frictional force on the block is $f = \mu N$, and this gives rise to a torque of

$$\boldsymbol{\tau} = \mathbf{R} \times \mathbf{F} = R\mu N = 0.2 \cdot 0.9 \cdot 10 = 1.8\ \text{kg-m}^2/\text{s}^2.$$

The torque opposes the motion of the wheel and is therefore negative.

(c) The torque generates an angular deceleration α from

$$\tau = I\alpha$$

from which

$$\alpha = \tau / I = 1.8 / 0.08 = 22.5\ s^{-2}.$$

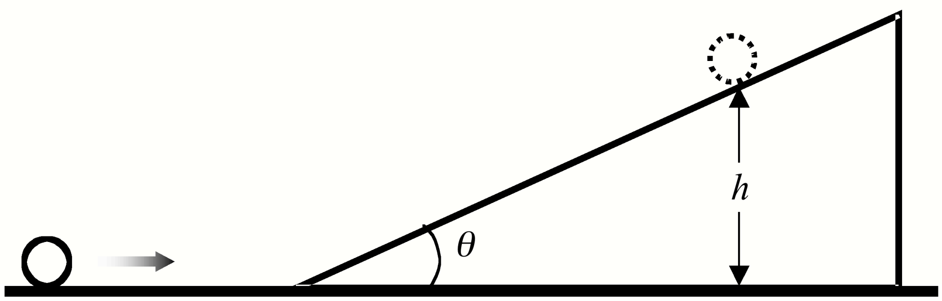
Using

$$\alpha = \Delta\omega / t = \omega_{final}/t,$$

the time taken for the wheel to stop must be

$$t = \omega_{final}/\alpha = 10\pi / 22.5 = 1.40 \text{ seconds}$$

As shown in the figure below, a small solid ball of radius R rolls without slipping on a horizontal surface at a linear velocity v_o = 20 m/s and then rolls up the incline. If friction losses are negligible, what is the linear speed of the ball when its height h = 21 m? For numerical convenience, use g = 10 m/s^2.



The kinetic energy of a rolling object has both translational and rotational contributions

$$K = mv^2/2 + I\omega^2/2.$$

The moment of inertia of a sphere is

$$I = 2mR^2/5,$$

so

$$K = mv^2/2 + mR^2\omega^2/5.$$

But $v = \omega R$ here, so

$$= mv^2/2 + mv^2/5$$
$$= mv^2 (1/2 + 1/5)$$
$$= mv^2 (7/10).$$

By conservation of energy, as the ball rolls up the hill

$$\Delta K = -\Delta U,$$

so

$$(7/10) \bullet (mv_o^2 - mv^2) = mgh$$

or

$$v_o^2 - v^2 = (10/7)\, gh$$
$$v^2 = v_o^2 - (10/7)\, gh.$$

Numerically, this becomes

$$v^2 = 20^2 - (10/7) \bullet 10 \bullet 21$$
$$= 100$$
$$v = 10 \text{ m/s}.$$

As a small solid sphere rolls without slipping over the top of a hill on a track, its speed v_{top} is 4.0 m/s. if friction losses are negligible, what is the speed v of the sphere when it is 1.4 m below the top? For numerical simplicity, use $g = 10$ m/s^2.

The kinetic energy of a rolling object has both translational and rotational contributions

$$K = mv^2/2 + I\omega^2/2.$$

The moment of inertia of a sphere is

$$I = 2mR^2/5,$$

so

$$K = mv^2/2 + mR^2\omega^2/5.$$

But $v = \omega R$ here, so

$$K = mv^2/2 + mv^2/5$$
$$= mv^2\,(1/2 + 1/5)$$
$$= mv^2\,(7/10).$$

By conservation of energy, as the ball rolls down the hill

$$\Delta K = -\Delta U,$$

so

$$(7/10) \cdot (mv^2 - mv_{top}{}^2) = mgh$$

or

$$v^2 - v_{top}{}^2 = (10/7)\, gh$$
$$v^2 = v_{top}{}^2 + (10/7)\, gh.$$

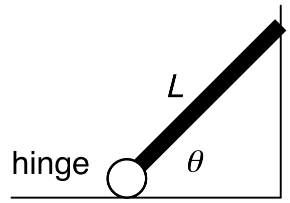
Numerically, we have

$$v^2 = (4.0)^2 + (10/7) \cdot 10 \cdot 1.4$$
$$= 16 + 20 = 36$$

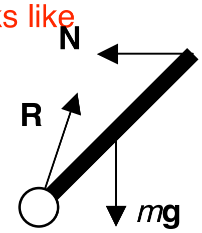
Hence

$$v = 6 \text{ m/s}.$$

A bar of mass *M* and length *L* leans up against a frictionless wall, as shown. The bottom of the bar is held against the floor by a hinge, around which the bar is free to rotate. The hinge cannot slide, so the bar makes a fixed angle θ with respect to the floor. What is the magnitude of the force that the hinge exerts on the bar?



A free-body diagram of the bar looks like



From the condition of zero torque:

clockwise torque = counterclockwise torque

$(mgL/2)\cos\theta = NL\sin\theta$

so

$N = (mg/2)\cos\theta / \sin\theta$.

Thus, the components of the reaction force exerted by the hinge must be

$R_x = (mg/2)\cos\theta / \sin\theta$

and

$R_y = mg$,

so

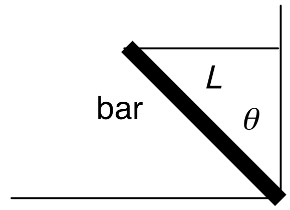
$R^2 = [(mg/2)\cos\theta / \sin\theta]^2 + (mg)^2$

$= (mg)^2 [1 + (\cos\theta / 2\sin\theta)^2]$

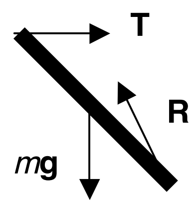
Lastly,

$R = mg [1 + (\cos\theta / 2\sin\theta)^2]^{1/2}$.

A bar of mass *M* and length *L* is held against a wall by a horizontal wire, as shown. The bottom of the bar is held wedged against the base of the wall, making an angle θ with respect to the wall. What is the magnitude of the force exerted on the bar at the corner?



A free-body diagram of the bar looks like



From the condition of zero torque:

clockwise torque = counterclockwise torque

$(mgL/2)\sin\theta = TL\cos\theta$

so

$T = (mg/2)\sin\theta / \cos\theta$.

Thus, the components of the reaction force exerted by the hinge must be

$R_x = (mg/2)\sin\theta / \cos\theta = (mg/2)\tan\theta$

and

$R_y = mg$,

so

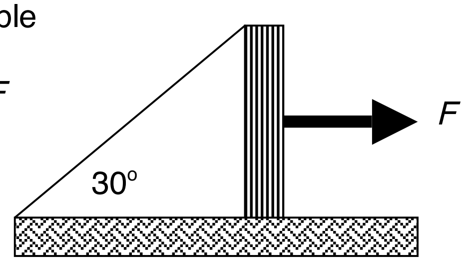
$R^2 = [(mg/2)\tan\theta]^2 + (mg)^2$

$= (mg)^2 [1 + (\tan\theta / 2)^2]$

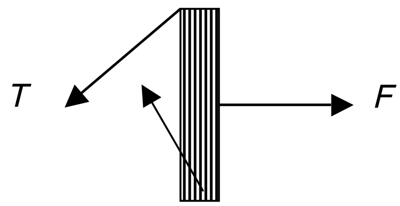
Lastly,

$R = mg [1 + (\tan\theta / 2)^2]^{1/2}$.

The base of a massless bar of length L is fixed to a table so that it cannot slip. The top of the bar is attached to the table by a thin wire, as shown. A horizontal force F is applied to the mid-point of the bar by a rope. If the thin wire snaps when it experiences a tension greater than 400 N, what is the maximum force that can be applied through the rope?
(Include a free-body diagram)



The free-body diagram of the bar looks like



From the condition of zero torque about the bottom of the bar:

clockwise torque = counterclockwise torque

$(FL/2) = TL \sin 60^{\circ}$

so

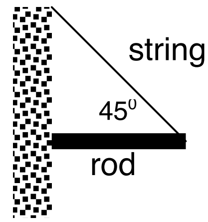
$F = 2T \sin 60^{\circ}$.

If the thin wire snaps at 400 N, then the maximum force that can be applied is

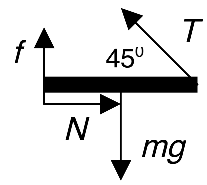
$F = 2 \cdot 400 \sin 60^{\circ}$

$= 693$ N.

A rod of mass m and length L is attached at one end to a massless string bearing a tension T. The other end rests against a wall, as shown, but does not slide because of the frictional force it experiences. What is the minimum coefficient of static friction for the rod to remain in place?



The free-body diagram of the rod looks like



Balancing forces in the horizontal direction gives

$$T\cos 45^{\circ} = N \quad (1)$$

Balancing forces in the vertical direction gives

$$mg = T\sin 45^{\circ} + f \quad (2)$$

Demanding zero torque around the far end of the rod gives

$$mg \cdot L/2 = fL$$

or

$$f = mg/2. \quad (3)$$

Substituting (1) into (2) gives

$$mg = N + f$$

and adding (3) yields

$$mg = N + mg/2.$$

Thus

$$N = mg/2. \quad (4)$$

Substituting (3) and (4) into the maximum static friction equation

$$f = \mu_S N$$

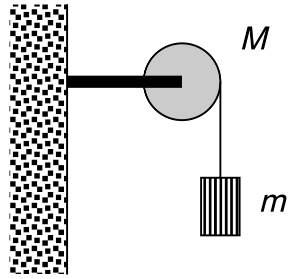
yields

$$mg/2 = \mu_S\, mg/2$$

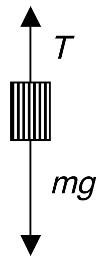
or

$$\mu_S = 1.$$

An object of mass m is tied to a light string wound around a uniform solid cylindrical wheel of mass M and radius R. The wheel is free to rotate. The object is released, making the wheel turn. Draw the applicable free-body diagrams and **derive** an algebraic expression for the angular acceleration α of the wheel as a function of m, M, R and g.



The free-body diagram of mass m is

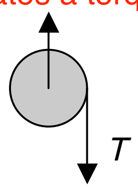


The tension is less than mg because the object is accelerating down. Thus

$$T = mg - ma.$$

At the wheel, the tension generates a torque τ:

$$\tau = TR$$
$$= (mg - ma)R$$



In turn, this torque produces an angular acceleration

$$\tau = I\alpha$$

or

$$(mg - ma)R = I\alpha.$$

The linear acceleration a is given by

$$a = \alpha R$$

so

$$(mg - m\alpha R)R = I\alpha.$$

Solving for α:

$$mgR - m\alpha R^2 = I\alpha$$
$$mR^2\alpha + I\alpha = mgR$$
$$\alpha = mgR / (mR^2 + I)$$

The moment of inertia of the wheel itself is

$$I = MR^2/2$$

so we are left with

$$\alpha = mgR / (mR^2 + MR^2/2)$$

or

$$\alpha = mg / (m + M/2)R.$$