Phys 211 - Intermediate Mechanics

Textbook - *Analytical Mechanics*, 6th Edition, by G.R. Fowles and G. R. Cassiday (Saunders, 1999)

Outline - An intermediate mechanics course covering kinematics, dynamics, energy, momentum, oscillations, rigid-body motion, gravitation.

Lecs. 1-4	Newtonian mechanics; rectilinear motion
Lecs. 5-7	Harmonic oscillator; free, damped and forced harmonic motion
Lecs. 8-9	General motion of a particle in 3 dimensions
Lecs. 10-13	Non-inertial reference frames
Lecs. 14-17	Central forces and celestial mechanics
Lecs. 18-19	Dynamics of systems of particles
Lecs. 20-22	Mechanics of rigid bodies
Lecs. 23-29	Rigid bodies in three dimensions
Lecs. 30-33	Oscillating systems

The following topics are included in supplementary lectures:

Supp. 1	Lorentz transformation
Supp. 2	Rutherford scattering
Supp. 3-5	Lagrangian and Hamiltonian mechanics

The lectures are available on-line at http://www.sfu.ca/~boal.

Grading - 10% weekly homework
25% midterm
65% final

Lecture 1: Kinematics (Fowles and Cassiday, Chap. 1)

We begin this course with a very rapid review of Newtonian mechanics, including kinematics in non-Cartesian coordinate systems and illustrations of velocity-dependent forces. Students are encouraged to review their Phys 120 notes as well as Chap. 1 of the text.

Kinematics and coordinate systems

Most of the kinematics problems of interest in this course involve objects moving in three dimensions. Inevitably, we are dealing with position, velocity and acceleration vectors **r**, **v**, **a** that are related by the usual vector equations

$$\mathbf{a} = d\mathbf{v}/dt \qquad \mathbf{v} = d\mathbf{r}/dt$$

In Cartesian coordinates, these equations clearly break up into independent equations

$$a_x = dv_x/dt \qquad a_y = dv_y/dt \qquad a_z = dv_z/dt$$

But not all situations have their simplest description in terms of Cartesian coordinates, circular motion being one example. So, we take a few minutes to look at kinematics in two other coordinate systems.

Plane polar coordinates (2D)

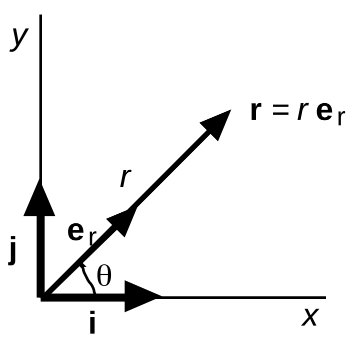
Standard Cartesian coordinates use a set of orthogonal (often space-fixed) unit vectors to represent the position **r** of an object

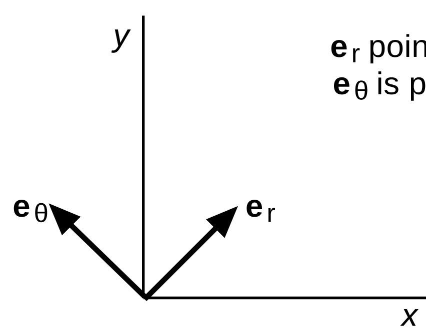
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

where **i** and **j** are unit vectors pointing along the x and y axes, respectively. As an alternative to Cartesians, polar coordinates (r, θ) may be more appropriate in some situations. For polars in two dimensions, we use two basis vectors : $\mathbf{e}_r$ points along **r** and $\mathbf{e}_\theta$ is perpendicular to $\mathbf{e}_r$. Thus, the position vector **r** has the very simple form

$$\mathbf{r} = r\,\mathbf{e}_r$$

although we pay the price that $\mathbf{e}_r$ may be time-dependent if the object moves.



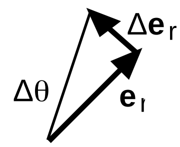


$\mathbf{e}_r$ points along **r**
$\mathbf{e}_\theta$ is perpendicular to $\mathbf{e}_r$

Recognizing that $\mathbf{e}_r$ and $\mathbf{e}_\theta$ may depend on time, the velocity equation becomes

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d(r\mathbf{e}_r)}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}$$

This expression is incomplete, because the velocity should be expressed in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Unless the object moves only radially, θ changes with time, as must the directions of $\mathbf{e}_r$ and $\mathbf{e}_\theta$. In time Δt, the angle θ changes by $\Delta\theta$, and $\mathbf{e}_r$ changes as



i) direction of $\Delta\mathbf{e}_r$ must be $\mathbf{e}_\theta$, perpendicular to $\mathbf{e}_r$ (since $\mathbf{e}_r$ is a unit vector of fixed magnitude that can only rotate).

ii) using the arc length equation, [*arc length*] = [*radius*]•[*angle in radians*], the magnitude of $\Delta\mathbf{e}_r$ must be $|\mathbf{e}_r|\,\Delta\theta = 1 \bullet \Delta\theta = \Delta\theta$. Thus

$$\Delta\mathbf{e}_r = \mathbf{e}_\theta\,\Delta\theta \qquad \text{so that} \qquad \frac{d\mathbf{e}_r}{dt} = \mathbf{e}_\theta\frac{d\theta}{dt}$$

or

$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$$

The acceleration vector **a** can be found in a similar way

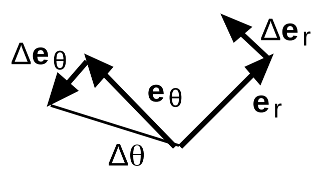
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left[\frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta\right]$$

$$= \frac{d^2r}{dt^2}\mathbf{e}_r + \frac{dr}{dt}\frac{d\mathbf{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{e}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{e}_\theta + r\frac{d\theta}{dt}\frac{d\mathbf{e}_\theta}{dt}$$

Substituting $d\mathbf{e}_r/dt = (d\theta/dt)\,\mathbf{e}_\theta$, and collecting terms, we have

$$\mathbf{a} = \frac{d^2r}{dt^2}\mathbf{e}_r + r\frac{d\theta}{dt}\frac{d\mathbf{e}_\theta}{dt} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\mathbf{e}_\theta$$

To complete the evaluation of the acceleration, we need to find $d\mathbf{e}_\theta/dt$. As before



i) $\Delta\mathbf{e}_\theta$ must be perpendicular to $\mathbf{e}_\theta$, since $\mathbf{e}_\theta$ is a unit vector and can only rotate, not elongate.
ii) $\Delta\mathbf{e}_\theta$ is in the *opposite* direction to $\mathbf{e}_r$

By the usual arc length construction

$$\Delta\mathbf{e}_\theta = -\,\mathbf{e}_r\,\Delta\theta \Rightarrow d\mathbf{e}_\theta/dt = -\,\mathbf{e}_r\,d\theta/dt$$

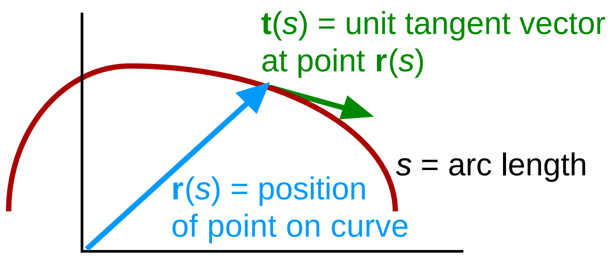
So

$$\mathbf{a} = \left(\frac{d^2r}{dt^2} - r\left[\frac{d\theta}{dt}\right]^2\right)\mathbf{e}_r + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\mathbf{e}_\theta$$

The text contains other examples of non-Cartesian coordinate systems, including cylindrical coordinates and spherical polar coordinates.

Curvature and arc lengths

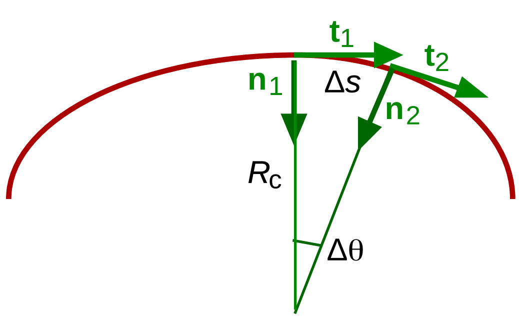
In polymer physics and many problems in engineering dealing with flexible beams, it is often useful to use a coordinate variable that follows the shape of the object. For instance, it may be convenient to use the arc length s as a parameter running along the curve, and then describe the curve by the set of points $\mathbf{r}(s)$.



The changing orientation of the curve is described by the unit tangent vector $\mathbf{t}(s)$ at position $\mathbf{r}(s)$, which is given by

$$\mathbf{t}(s) = \partial\mathbf{r}(s)/\partial s. \qquad (1.1)$$

This is like the unit vector for the velocity: $\mathbf{v} / |v| = [d\mathbf{r}/dt] / [ds / dt] = d\mathbf{r}/ds$. As expected for a unit vector, $\mathbf{t}$ is dimensioness.



Now consider two nearby positions 1 and 2 on a line as above. The rate of change of $\mathbf{t}$ with s provides a measure of the curvature at any given position. Now, the vector $\Delta\mathbf{t} = \mathbf{t}_2 - \mathbf{t}_1$ is perpendicular to the curve in the limit where positions 1 and 2 are infinitesimally close. Thus, the rate of change of $\mathbf{t}$ with s is proportional to the unit normal to the curve $\mathbf{n}$, and we define the proportionality constant to be the curvature C:

$$\partial\mathbf{t}/\partial s = C\mathbf{n}, \qquad (1.2)$$

where the curvature has units of inverse length.

The geometrical interpretation of C^{-1} as the radius of curvature can be seen by extrapolating the unit normals $\mathbf{n}_1$ and $\mathbf{n}_2$ to their point of intersection. If positions 1 and 2 are close by on the contour, then the arc is approximately a section of a circle with radius R_c. The arc length Δs of this section is just

$$\Delta s = R_c \Delta\theta, \qquad (1.3)$$

where $\Delta\theta$ is the angle subtending the arc. But $\Delta\theta$ is also the angle between $\mathbf{t}_1$ and $\mathbf{t}_2$; that is, $\Delta\theta = |\Delta\mathbf{t}|$ so that Eq. (1.3) can be rewritten as $|\Delta\boldsymbol{t}| / \Delta s = 1/R_c$. Comparing this last expression with Eq. (1.2) shows

$$C = 1/R_c. \qquad (1.4)$$