Lecture 12 - The Earth’s rotation

Text: Fowles and Cassiday, Chap. 5
Demo: basketball for the earth, cube for Cartesian frame

A coordinate system sitting with fixed orientation on the Earth’s surface (i.e., having the y-axis pointing towards the North Pole, the x-axis pointing east along a line of latitude and the z-axis perpendicular to the Earth’s surface) is both a rotating and an accelerating reference frame. Hence, fictitious forces are required in this frame to account for an object’s motion. The Earth rotates counter-clockwise as seen at North Pole, so $\omega$ is up.

The Plumb Line

Our first example is a simple plumb line: a point mass hung from a string. The coordinate system is illustrated below

\[ \lambda = \text{latitude (angle)} \]
\[ \rho = \text{distance of plumb line from axis of rotation} \]
\[ r_e = \text{distance of plumb line from the centre of the Earth} \]

We’ve drawn the Earth as if it has a slight bulge at the equator. Although we’ve drawn the tension in the plumb line $S$ as if it points along $z'$, in fact we don’t know that is the case yet (we use $S$ instead of $T$, which is reserved for period).

This is a particularly simple example because the plumb bob is stationary in the rotating frame. How do we describe the bob?
(i) it is stationary in the rotating frame: $a' = 0$ and $v' = 0$
(ii) it sits at the origin of the frame so $r' = 0$
(iii) the Earth’s rotational speed is constant, so $d\omega / dt = 0$

Thus, of the expression
\[ ma' = ma - 2m\omega x v' - m (d\omega / dt) x r' - m\omega x (\omega x r') - mA_o \]
we are left with

\[ 0 = ma - mA_o \]

or just \[ F = mA_o \]

Now the physical forces \( F \) acting on the bob are the true gravitational force \( mg_o \) toward the centre of the Earth and the tension \( S \) in the string.

The tension balances the locally measured acceleration due to gravity \( g \), which is different from the true acceleration \( g_o \). That is

\[ S = -mg \]

\[ F = -mg + mg_o = mA_o \quad \text{(vector equation)} \]

or \[ g = g_o - A_o \]

To evaluate the difference between \( g \) and \( g_o \), we use the law of sines

\[ \frac{\sin \epsilon}{mA_o} = \frac{\sin \lambda}{mg} \]

\( A_o \) is the centripetal acceleration (and \(-A_o\) is the centrifugal acceleration) given by

\[ A_o = \omega^2 r_e = \omega^2 r_o \cos \lambda \]

\[ \implies \sin \epsilon = \frac{\omega^2 r_o \sin \lambda \cos \lambda}{mg} \]

\[ = \frac{\omega^2 r_o \sin 2\lambda}{2g} \]

As expected, \( \epsilon \) vanishes at the equator (\( \lambda = 0 \)) and at poles (\( 2\lambda = 180 \) degrees). The largest value of \( \epsilon \) is at \( \lambda = 45 \) degrees:

\[ \omega = \frac{2\pi}{(24 \times 3600)} = 0.000073 = 7.3 \times 10^{-5} \text{ radians/sec} \]

\[ r_e = 6400 \text{ km} = 6.4 \times 10^6 \text{ m} \]

\[ \sin \epsilon \sim \epsilon = (7.3 \times 10^{-5})^2 \cdot 6.4 \times 10^6 \cdot 1 \div (2 \times 9.8) = 0.0017 \text{ radians} \]

or \( \epsilon \sim 0.0017 \times 57.3 = 0.1 \) of a degree.

Lastly, because the rotational motion affects both the surface of the Earth as well as the bob, then \( S \) remains perpendicular to the Earth’s surface. We use this fact in the
next example, and always subsume $A_o$ with $g_o$ to allow us to work with the local acceleration $g$. At the equator, $A_o = (7.3 \times 10^{-5})^2 \cdot 6.4 \times 10^6 = 0.034 \text{ m/s}^2$.

**Foucault Pendulum**

Now we take the plumb bob from the previous example and allow it to swing back and forth. The Foucault pendulum is simply a mass on a string, but the string is not restricted to move in a particular plane.

The forces acting on the pendulum bob are

$$m \left( \frac{d^2 r'}{dt^2} \right) = mg + S - 2m\omega x'v',$$

where we have rolled $g_o - A_o$ into the local acceleration $g$. But because the tension $S$ changes magnitude and direction as the bob swings, it is not always true that $S = -mg$.

The local centripetal acceleration term $\omega x'(\omega x'r')$ has been dropped in favour of the Coriolis force, which is much more important for this problem [The local force $\omega x'(\omega x'r')$ depends upon $r'$ which is with respect to the $x'y'z'$ origin on the surface of the Earth, whereas $A_o$ depends on $r_e >> r'$].

Choose $x', y'$ to form a plane tangent to the Earth’s surface, and choose $z'$ to therefore lie along $g$:

![Diagram](image)

From $v' = v_x' i' + v_y' j' + v_z' k'$, the Cartesian components of $\omega x'v'$ are

$$\omega x'v' = [\omega \cos \lambda (dz'/dt) - \omega \sin \lambda (dy'/dt),$$

$$\omega \sin \lambda (dx'/dt),$$

$$-\omega \cos \lambda (dx'/dt)].$$

Now the components of the tension $S$ can be written in terms of the length of the string and the coordinates of the bob: comparing $S_x / S$ with $x' / L$, for example, gives $S_x = -(x'/L)S$. 

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\[ S_x' = -(x' / L)S \]
\[ S_y' = -(y' / L)S \]
\[ S_z' = +[(L - z') / L]S \]

Note that the - signs are required to obtain the correct orientations.

Returning to our equation for \( \mathbf{a}' \), we can write out the \( x' \) and \( y' \) components as
\[
m (d^2x' / dt^2) = -(x' S / L) - 2m \omega \left[(dz' / dt) \cos \lambda - (dy' / dt) \sin \lambda\right]
\]
\[
m (d^2y' / dt^2) = -(y' S / L) - 2m \omega (dx' / dt) \sin \lambda.
\]

We make two approximations in the small angle situation, where the motion is nearly horizontal:
• \( dz' / dt \sim 0 \) corresponding to no vertical motion
• \( S = mg \) (in magnitude) since the pendulum is almost vertical

\[
=> \quad d^2x' / dt^2 = -(g / L) x' + 2 \Omega (dy' / dt)
\]
\[
d^2y' / dt^2 = -(g / L) y' - 2 \Omega (dx' / dt)
\]

where \( \Omega = \omega \sin \lambda \) is the component of \( \omega \) in the \( z' \) direction, \( i.e., \) it's the local vertical component of \( \omega \).

Somewhat like the \( q \mathbf{v} \times \mathbf{B} \) problem of charged particles in a magnetic field, the equations of motion are now coupled in the \( x' \) and \( y' \) directions. They can be uncoupled by defining yet another rotating coordinate system \( X, Y \) which rotates \textbf{clockwise} around \( z' \) with an angular frequency of \( -\Omega = -\omega \sin \lambda \).

\[
x' = X \cos \Omega t + Y \sin \Omega t
\]
\[
y' = -X \sin \Omega t + Y \cos \Omega t
\]
Substituting this transformation into the first equation for \( d^2x'/dt^2 \):

\[
d^2x'/dt^2 = -(g/L)x' + 2\Omega dy'/dt
\]

becomes

\[
d^2(\cos\Omega t + Y \sin\Omega t)/dt^2 = -(g/L)(\cos\Omega t + Y \sin\Omega t) + 2\Omega d(-X \sin\Omega t + Y \cos\Omega t)/dt
\]

or

\[
d/dt( dX/dt \cos\Omega t - X\Omega \sin\Omega t + dY/dt \sin\Omega t + Y\Omega \cos\Omega t) = -(g/L)(\cos\Omega t + Y \sin\Omega t)
\]

\[
+ 2\Omega [-dX/dt \sin\Omega t - X\Omega \cos\Omega t + dY/dt \cos\Omega t - Y\Omega \sin\Omega t]
\]

Dropping terms of order \( \Omega^2 \)

\[
(d^2X / dt^2 + gX / L) \cos\Omega t + (d^2 Y / dt^2 + gY / L) \sin\Omega t = 0.
\]

Each of the coefficients of \( \cos\Omega t \) or \( \sin\Omega t \) must vanish for arbitrary time, leaving just the usual simple harmonic motion equations for \( X, Y \):

\[
\begin{align*}
\frac{d^2X}{dt^2} + (g / L)X &= 0 \\
\frac{d^2Y}{dt^2} + (g / L)Y &= 0
\end{align*}
\]

That these equations give the usual SHM (including the period \( 2\pi \sqrt{L/g} \)) is really not very surprising. What is important to note is that the \( XY \) coordinates rotate with an angular frequency \( \omega = \sin\lambda = \Omega \) with respect to \( x'y' \): that is, the plane of oscillation rotates with respect to a coordinate system on the Earth’s surface. Using

\[
\omega = 2\pi / T
\]

the period of rotation \( T_{\text{foucault}} \) is then

\[
T_{\text{foucault}} = T_{\text{earth}} / \sin\lambda, \quad \text{where } T_{\text{earth}} = 24 \text{ hours}.
\]

Examples:

- North pole, \( \lambda = \pi/2 \), \( T' = 24 \text{ hours} \)
- Equator, \( \lambda = 0 \), \( T_{\text{foucault}} = \infty \) (no rotation)
- \( \lambda = 45 \text{ degrees} \), \( T_{\text{foucault}} = 24 / (1/\sqrt{2}) = (\sqrt{2})(24) = 34 \text{ hours} \).

The effect was first demonstrated by French physicist Jean Foucault in 1851; pendulum rotates clockwise in northern hemisphere.
Effects on surface winds:

- Low pressure with Coriolis, in northern hemisphere
- CCW in middle
- Back and forth motion of pendulum
- Coriolis force make pendulum rotate clockwise in northern hemisphere