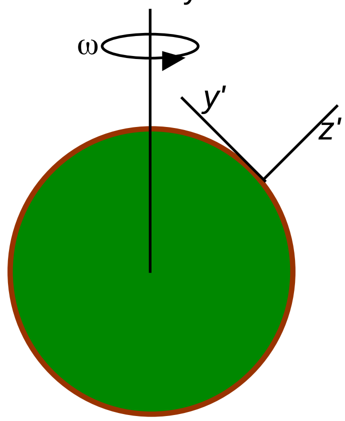
Lecture 13 - Projectile motion on the Earth's surface

Text: Fowles and Cassiday, Chap. 5

As one last example of motion in a rotating reference frame, we consider projectile motion, including components perpendicular to the Earth's surface. This is slightly more complicated than the Foucault pendulum, in which vertical motion can be safely ignored.

We start with the usual rotating coordinate system as with the Foucault pendulum,



and apply the following conditions:

- $d\omega / dt = 0$
- $\omega\mathbf{x}(\omega\mathbf{x}\mathbf{r}')$ is negligibly small compared to $\omega\mathbf{x}\mathbf{v}'$
- all physical forces other than gravity can be neglected
- the effective acceleration due to gravity is $\mathbf{g} = \mathbf{g}_o - \mathbf{A}_o$

This leaves us with the same equation as the Foucault pendulum but without the tension S :

$$m\mathbf{a}' = m\mathbf{g} - 2m\omega\mathbf{x}\mathbf{v}'$$

We determined $\omega\mathbf{x}(d\mathbf{r}'/dt)$ in the previous lecture to be

$$\omega\mathbf{x}\mathbf{v}' = [\omega \cos\lambda \, (dz' / dt) - \omega \sin\lambda \, (dy' / dt), \quad \omega \sin\lambda \, (dx' / dt), \quad -\omega \cos\lambda \, (dx' / dt)]$$

Hence, the acceleration equation reads, component by component

$$d^2x' / dt^2 = -2\omega \, [(dz' / dt) \cos\lambda - (dy' / dt) \sin\lambda] \qquad (1)$$

$$d^2y' / dt^2 = -2\omega \, (dx' / dt) \sin\lambda \qquad (2)$$

$$d^2z' / dt^2 = -g + 2\omega \, (dx' / dt) \cos\lambda \qquad (3)$$

Once again, these equations are not separable. We integrate with respect to time to obtain, trivially

$$dx' / dt = -2\omega \, (z' \cos\lambda - y' \sin\lambda) + v'_{o,x} \qquad (4)$$

$$dy' / dt = -2\omega x' \sin\lambda + v'_{o,y} \quad (5)$$
$$dz' / dt = -gt + 2\omega x' \cos\lambda + v'_{o,z} \quad (6)$$

The integration constants in Eqs. (4) - (6) are taken to be the Cartesian coordinates of the initial velocity vector $\mathbf{v}'_o$ because (4)-(6) must give $\mathbf{v}' = \mathbf{v}'_o$ at $t = 0$ and $\omega = 0$.

Substituting (5) and (6) into (1) allows us to generate an equation that depends only on x and some constants:

$$d^2x' / dt^2 = -2\omega\,(-gt + 2\omega x' \cos\lambda + v'_{o,z}) \cos\lambda + 2\omega\,(-2\omega x' \sin\lambda + v'_{o,y}) \sin\lambda \quad (7)$$

which becomes, after dropping terms of order ω^2

$$d^2x' / dt^2 = +2\omega gt \cos\lambda - 2\omega v'_{o,z} \cos\lambda + 2\omega v'_{o,y} \sin\lambda$$

Note that on the rhs, terms in $x(t)$ are missing, so that there is no restoring force leading to oscillatory behaviour. Integrating with respect to time leads to

$$dx' / dt = \omega gt^2 \cos\lambda - 2\omega t\,(v'_{o,z} \cos\lambda - v'_{o,y} \sin\lambda) + v'_{o,x} \quad (8)$$

(Taking the t --> 0 limit convinces one that the integration constant here is the same as in Eq. 4)

Integrate equation (8) with respect to time to obtain

$$x'(t) = (1/3)\omega gt^3 \cos\lambda - \omega t^2\,(v'_{o,z} \cos\lambda - v'_{o,y} \sin\lambda) + v'_{o,x}\, t + x_o \quad (9)$$

Now that we have found $x'(t)$, it can be substituted into Eqs. (5) and (6) to yield $y'(t)$ and $z'(t)$:

(9) --> (5) $$dy'/dt = -2\omega \sin\lambda\, [(1/3)\omega gt^3 \cos\lambda - \omega t^2(v'_{o,z} \cos\lambda - v'_{o,y} \sin\lambda) + v'_{o,x}\, t] + v'_{o,y}$$

Drop terms in ω^2 and integrate (note that x'_o doesn’t appear in Eq. (2); the remaining integration constants are y-dependent):

$$y'(t) = -2\omega \sin\lambda\, v'_{o,x}\, (1/2)t^2 + v'_{o,y}\, t + y'_o$$

or

$$y'(t) = v'_{o,y}\, t - \omega \sin\lambda\, v'_{o,x}\, t^2 + y'_o \quad (10)$$

(9) --> (6) $$dz'/dt = -gt + 2\omega\, [(1/3)\omega gt^3 \cos\lambda - \omega t^2(v'_{o,z}\cos\lambda - v'_{o,y} \sin\lambda) + v'_{o,x}\, t]\cos\lambda + v'_{o,z}$$

Dropping ω^2 terms and integrating

$$z'(t) = -(1/2)gt^2 + v'_{o,z}\, t + \omega v'_{o,x}\, t^2 \cos\lambda + z'_o \quad (11)$$

Eq's (9) to (11) contain the leading order time-dependence, as well as initial conditions $\mathbf{r}'_o$ and $\mathbf{v}'_o$. As expected, the equations of motion do *not* depend on the mass *m.*

Example An object is dropped at rest from a height h. Find $x'(t)$, $y'(t)$, $z'(t)$.

The initial conditions are $\mathbf{v}'_o = 0$; $x'_o = y'_o = 0$; $z'_o = h$
The vertical motion $z'(t)$ then reduces to

$$z'(t) = -(1/2)gt^2 + h \qquad (12)$$

which is just the usual expression for free-fall under gravity. The time taken for the object to reach the ground at $z' = 0$ is

$$t = (2h / g)^{1/2} \qquad (13)$$

In the x - direction, we have from Eq. (9)

$$x'(t) = (1/3)\,\omega g t^3 \cos\lambda$$

When the object hits the ground, the drift to the east is, from Eq. (13)

$$x' = (1/3)\omega g\,(2h / g)^{3/2} \cos\lambda$$

Lastly, from Eq. (11): $y'(t) = 0$ (no drift toward pole).

Typical magnitude:
Suppose $h = 10$ m, $\omega = 2\pi/(24 \bullet 3600) = 7.3 \times 10^{-5}$ s^{-1}, and $\lambda = \pi/4$. We then have

$$x' = 0.333 \bullet 7.3 \times 10^{-5} \bullet 9.81 \bullet (20/9.81)^{3/2} / \sqrt{2} = 4.5 \times 10^{-4} = 0.45 \text{ mm.}$$