Lecture 25 - Principal axes

Text: Fowles and Cassiday, Chapter 9
Demo: tennis racket

We formally introduced the principal axes in Chap. 9 as the set of eigenvectors that diagonalize the inertia tensor. It is not necessary that the axes pass along some obvious symmetry axis of the object (although symmetry is often used to identify a likely choice of principal axes); that I_{ij} is symmetric guarantees that such axes exist. Let's run through the diagonalization procedure as an example.

Diagonalizing the inertia tensor is no different from any standard matrix. One finds the eigenvalues λ from

$$\begin{vmatrix} I_{xx} - \lambda & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} - \lambda & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} - \lambda \end{vmatrix} = 0$$

and then substitutes to find the three eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$

$$(\mathbf{I} - \lambda_i \mathbf{1})\mathbf{e}_i = 0$$

Example Let's evaluate the eigenvalues of the square plate problem again from lecture 25. Recalling for a square plate with sides of length a and the coordinate origin at one corner of the plate:

$$I_{xx} = I_{yy} = (1/3)\, ma^2 \qquad I_{zz} = (2/3)ma^2$$
$$I_{xy} = -ma^2/4 \qquad I_{xz} = I_{yz} = 0$$

The determinant to be evaluated is

$$\begin{vmatrix} (1/3 - \Lambda) & -1/4 & 0 \\ -1/4 & (1/3 - \Lambda) & 0 \\ 0 & 0 & (2/3 - \Lambda) \end{vmatrix} = 0$$

where a factor of ma^2 has been removed by writing $\Lambda = \lambda / ma^2$. Hence

$$(2/3 - \Lambda)[(1/3 - \Lambda)^2 - (-1/4)^2] = 0$$

which factors into

$$(2/3 - \Lambda) = 0 \quad \text{-->} \quad \Lambda = 2/3$$
$$(1/3 - \Lambda)^2 = (1/4)^2 \quad \text{-->} \quad \Lambda = 1/3 \pm 1/4$$
$$\text{-->} \quad \Lambda = 1/12$$
$$\text{or } \Lambda = 7/12$$

From the structure of the determinant, $\mathbf{e}_3$ lies along the z - axis, $\mathbf{e}_1$ and $\mathbf{e}_2$ lie in the xy plane. As expected from the perpendicular axis theorem:

$$\Lambda_1 + \Lambda_2 = 1/12 + 7/12 = 2/3 = \Lambda_3$$

Just to check the eigenvectors, we solve for $\mathbf{e}_1$ with $\Lambda_1 = 1/12$

$$I_{xx}/ma^2 - \Lambda_1 = I_{yy}/ma^2 - \Lambda_1 = 1/4$$
$$I_{zz}/ma^2 - \Lambda_1 = 2/3 - 1/12 = 7/12$$

=>

$$\begin{pmatrix} 1/4 & -1/4 & 0 \\ -1/4 & 1/4 & 0 \\ 0 & 0 & 7/12 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

Hence, the three equations arising from the matrix product are:

$$1/4\, c_1 - 1/4\, c_2 = 0$$
$$-1/4\, c_1 + 1/4\, c_2 = 0$$
$$7/12\, c_3 = 0$$

Immediately, $c_3 = 0$ and $c_1 = c_2$. Normalizing via

$$c_1^2 + c_2^2 + c_3^2 = 1$$

gives

$$\mathbf{e}_1 = (1/\surd 2,\ 1/\surd 2,\ 0).$$

By explicit substitution, or just using our experience to construct an orthogonal vector,

$$\mathbf{e}_2 = (1/\surd 2,\ -1/\surd 2,\ 0).$$

Now, we see that $\mathbf{e}_1$ and $\mathbf{e}_2$ lie along the diagonals of the square. This is as expected, since we saw in the example of lecture 25 that $\mathbf{L} \parallel \omega$ if ω lies along the diagonal.

Euler's equations for a rigid body

Demo: flip a tennis racket to show how ω rotates with respect to the body-fixed axes.

Some lectures back, we discussed the apparent behaviour of *linear* kinematic quantities such as velocity when viewed in a rotating reference frame; examples were

$$d\mathbf{x} / dt = d\mathbf{x}' / dt + \omega \mathbf{x} \mathbf{r} \qquad (1)$$

Now consider what happens to *angular* quantities such as the angular momentum, when viewed in a frame rotating with angular velocity ω. First, we just look upon **L** as a vector, unrelated to the angular velocity of the object, and we are simply observing **L** in the rotating frame.

We start with

$$\mathbf{L} = L_x\mathbf{i} + L_y\mathbf{j} + L_z\mathbf{k} = L_x'\mathbf{i}' + L_y'\mathbf{j}' + L_z'\mathbf{k}'$$

and find by the same arguments that led to Eq. (1):

$$d\mathbf{L} / dt = d\mathbf{L}' / dt + \omega \mathbf{x} \mathbf{L}$$

where the meaning of $d\mathbf{L}'/dt$ is

$$d\mathbf{L}'/dt = \mathbf{i}'(dL_x'/dt) + \mathbf{j}'(dL_y'/dt) + \mathbf{k}'(dL_z'/dt)$$

and the time-derivatives of ($\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$) have disappeared into $\omega\mathbf{x}\mathbf{L}$.

Now, the equivalent of Newton's second law for torques reads

$$\tau = d\mathbf{L}/dt,$$

so

$$\tau = d\mathbf{L}'/dt + \omega\mathbf{x}\mathbf{L} \qquad (2)$$

Now, we examine **L** in more detail by giving the reference frame the same ω as the object. Using the inertia tensor representation, the two terms on the rhs of Eq. (2) are:

$$\omega\mathbf{x}\mathbf{L} = \omega\mathbf{x}(\mathbf{I}\bullet\omega)$$
$$d\mathbf{L}'/dt = \mathbf{I}\bullet(d\omega/dt)$$

The magnitudes of the vectors in the two frames may be the same, but their components will be different. We choose the rotating frame ($\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$) to correspond to the principal axes of the object (1, 2, 3), so that

$$\mathbf{L} = \mathbf{I}\bullet\omega$$
$$= (I_1\omega_1, I_2\omega_2, I_3\omega_3),$$

and the cross product has components in the 1-2-3 frame of

$$\omega\mathbf{x}\mathbf{L} = \begin{matrix} \omega_2 I_3\omega_3 - \omega_3 I_2\omega_2 \\ \omega_3 I_1\omega_1 - \omega_1 I_3\omega_3 \\ \omega_1 I_2\omega_2 - \omega_2 I_1\omega_1 \end{matrix} = \begin{matrix} \omega_2\omega_3(I_3 - I_2) \\ \omega_1\omega_3(I_1 - I_3) \\ \omega_1\omega_2(I_2 - I_1). \end{matrix}$$

Substituting this representation into the the torque equation, we end up with Euler's equations (for components along the principal axes):

$$\tau_1 = I_1\, d\omega_1/dt + \omega_2\omega_3(I_3 - I_2)$$
$$\tau_2 = I_2\, d\omega_2/dt + \omega_1\omega_3(I_1 - I_3)$$
$$\tau_3 = I_3\, d\omega_3/dt + \omega_1\omega_2(I_2 - I_1).$$