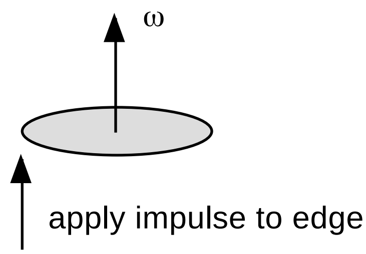
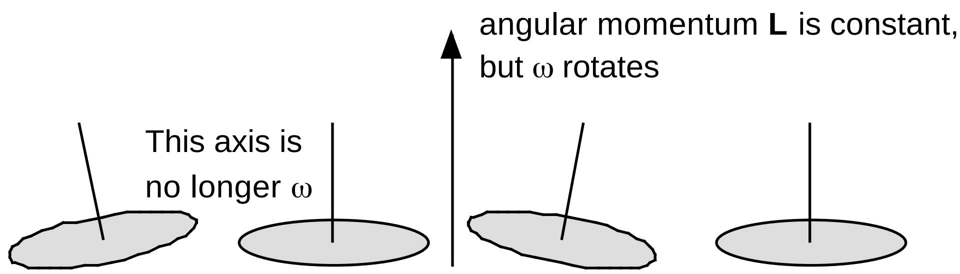
Lecture 26 - Torque-free rotation - body-fixed axes

Text:Fowles and Cassiday, Chap. 9
Demo: gyroscope, tennis racket, old CD

We now want to consider how the angular momentum **L** and the angular velocity ω behave when they are observed in a rotating coordinate system. We simplify matters for our first foray into this problem by considering a system *not* subject to an ongoing external torque. Consider, the motion of a spinning disk, like a gyroscope, whose angular velocity and angular momentum are perpendicular to the plane of the disk:



If we hit the disk on its side, generating a torque about the plane, then the plane of the disk will oscillate, as will the corresponding angular velocity vector ω (gyroscope demo):



After the rotating disk receives the impulse, it still has a component of ω perpendicular to the plane of the disk. But the disk rocks back and forth as well, meaning that there is an oscillating component of ω around the 1 and 2 body-fixed axes of the disk. That is, a rocking motion around the 1-axis corresponds to a non-vanishing ω_1, such that the vector describing ω may have several non-zero components (ω_1, ω_2, ω_3).

In this lecture, we describe the behavior of ω as seen by a rotating observer, which is not difficult to do mathematically, but is not what we see in the lab. In the following two lectures (27/28), we determine the motion according to a lab-based observer. Finally, we treat the gyroscope problem in detail in lecture 29.

Allowed motion of ω

Since we are considering only torque-free rotation, then according to a "stationary" observer, **L** is fixed in direction. But a frame rotating with respect to the stationary observer would observe **L** change direction, although the magnitude of **L** would not change. That is, according to a rotating observer,

L^2 = constant **L** changes.

The constant magnitude of the angular momentum implies that

$L_x^2 + L_y^2 + L_z^2$ = constant

--> $I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$ = constant (1)

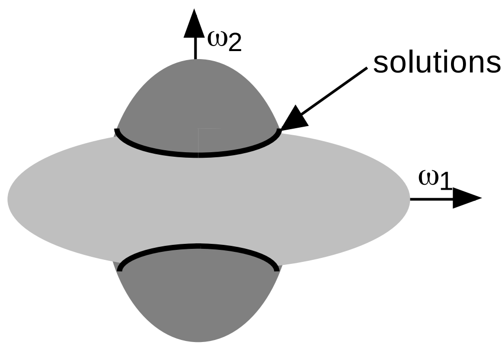
In other words, the constancy of L^2 is not enough to specify a value for ω: Eq. 1 is that of an ellipsoid in ω-space.

Under torque-free conditions, the rotational kinetic energy is also a constant, or

$K = \omega \bullet \mathbf{L}/2$ = constant

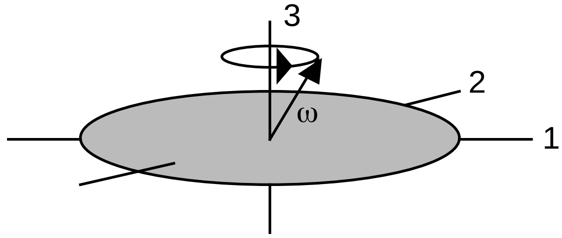
--> $I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2$ = constant (2)

where the constants in Eqs. (1) and (2) are obviously not equal. Eq. (2) is also that of an ellipsoid, just like Eq. (1). Now, ω must simultaneously satisfy Eqs. (1) and (2), although the solution is not necessarily unique; rather, the solution is the intersection of two ellispoids:



In some situations, the ellispoids may touch at a single point, giving a unique solution for ω. This arises if the inital motion corresponds to either the largest or smallest principal moment of inertia, in which case the body rotates steadily about the principal axis in question. Otherwise, ω precesses along the intersection locus of the two ellipsoids (*demo: tennis racket motion about principal axes*).

In lecture 25 we introduced a set of Euler's equations relating the torque exerted on an object to its angular velocity. Let's apply these equations to the situation in which there is a single axis of continuous symmetry (chosen as the 3-axis) with two equivalent axes perpendicular to it (the 1 and 2 axes).



Let the moments of inertia about the principal axes be

$I_s = I_3$

$I = I_1 = I_2$.

For torque-free rotation, $\tau_i = 0$ in Euler's equations, leaving

$$I (d\omega_1 / dt) + \omega_2\omega_3(I_s - I) = 0 \quad (3)$$

$$I (d\omega_2 / dt) + \omega_3\omega_1(I - I_s) = 0 \quad (4)$$

$$I_s (d\omega_3 / dt) = 0. \quad (5)$$

Eq. (5) can be integrated immediately, telling us that ω_3 is independent of time

ω_3 = constant.

To make life somewhat simpler, we define a new constant Ω in terms of the constant value of ω_3:

$$\Omega \equiv \omega_3(I_s - I)/I \quad (6)$$

Then, Eqs. (3) and (4) can be rewritten as

$$d\omega_1 / dt + \Omega\omega_2 = 0 \quad (7a)$$

$$d\omega_2 / dt - \Omega\omega_1 = 0. \quad (7b)$$

These two equations are coupled, but are straightforward to solve. Just differentiate Eq. (7a) with respect to time, then substitute (7a) for $d\omega_2 / dt$ into the result. That is, the derivative of (7a) gives

$$d^2\omega_1 / dt^2 = -\Omega(d\omega_2 / dt)$$

which becomes, from (7b)

$$d^2\omega_1 / dt^2 = -\Omega^2\omega_1 \quad (8)$$

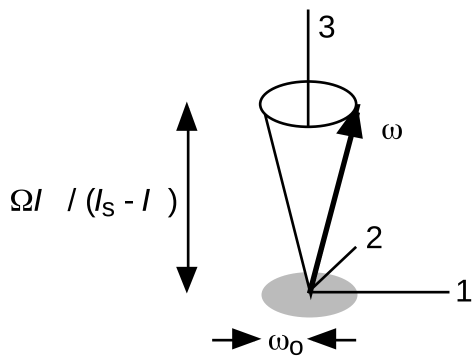
Now, Eq. (8) is just the expression for simple harmonic motion, so that ω_1 has the solution

$$\omega_1(t) = \omega_o \cos\Omega t. \quad (9)$$

A similar treatment for ω_2 shows that it too must obey simple harmonic motion. To find the correct amplitude and phase, we substitute Eq. (9) into (7b) to obtain

$$\omega_2(t) = \omega_o \sin\Omega t.$$

Clearly, the projection of ω in the 1,2 plane is just a circle of radius ω_o and angular frequency Ω (where $\omega_o^2 + \omega_3^2 = \omega^2$). In other words, *seen in the rotating frame*, ω precesses about the 3-axis with an angular frequency Ω:



If we let α be the angle between the 3-axis and ω, then from the definition of Eq. (6):

$$\omega_3 = \omega \cos\alpha = I \;\; \Omega / (I_s - I \;\;).$$

Hence,

$$\Omega = \omega \cos\alpha \; [(I_s / I \;\;) - 1], \qquad (10)$$

which gives the angular frequency of precession Ω in terms of the moments of inertia and the angle α between the symmetry axis and the rotational axis.

Examples

1. Rotation of a thin disk (*demo: rotating CD*).

For a thin disk, such as a frisbee or a china plate, the perpendicular axis theorem can be used to relate the moments of inertia:

$$I_s = I_1 + I_2 = 2I \;\; . \qquad (11)$$

Thus, Eq. (10) reads

$$\Omega = \omega \cos\alpha \; .$$

Consider the behaviour of Ω and ω_o at $\alpha = 0$ (to plane) and $\alpha = \pi/2$ (in plane).

2. Rotation of the Earth

The Earth is slightly flattened (oblate) so that it has inequivalent symmetry axes. Further, the rotational axis is very slightly off the North Pole, with an angle

$$\alpha = 0.2 \text{ arc seconds.}$$

To appreciate just how small this angle is, we convert to radians

3600 arc seconds = 1 degree = π / 180 radians

--> 1 arc second = π / (180 x 3600) = 1 / 206265 radians

Then $\alpha = 0.2 / 206265 = 9.7 \times 10^{-7}$ radians.

For small angles, $\cos\alpha = 1 - \alpha^2/2$, so that for our purposes here, $\cos\alpha = 1$.

Now, the measured oblateness of the Earth corresponds to $I_s / I = 1.00327$, so that

$$\Omega = \omega \cos\alpha \, [(I_s / I) - 1]$$
$$= \omega \bullet 1 \bullet (1.00327 - 1)$$
$$= 0.00327\omega.$$

Since the angular frequency of the Earth is $\omega = 2\pi$/day, then

$$\Omega = 0.00327 \bullet 2\pi \text{ / day}$$
$$= 0.0205 \text{ / day}.$$

Put another way, the period T of rotation is

$$T = 2\pi/\Omega = 2\pi / (0.00327 \bullet 2\pi) \text{ days}$$

or

$$T = 306 \text{ days}.$$

(This is the rotational period of the ω-axis as measured by an observer on the Earth).