

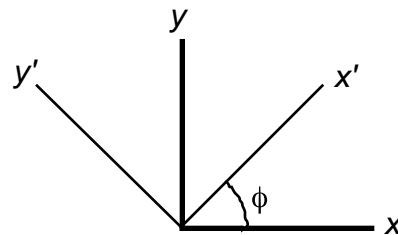
Lecs. 27/28 - Torque-free rotation - space-fixed axes

Text: Fowles and Cassiday, Chap. 9

Our calculation of Ω in the previous lecture is not the end of the story. The observed rotation as seen from a coordinate system fixed in *space* is *not* Ω . In the second example from Lecture 27, we saw that the precession of the rotational velocity ω as seen from a frame corotating with the Earth has a period of ~ 300 days. But the rotating frame itself has a period of ~ 1 day with respect to a "fixed" system, so the rotation of ω with respect to the fixed system is also on the order of 1 day. How do we relate these two observations?

Euler angles

We need a way of representing a general rotating coordinate system in three-dimensional space. Recall first what happens in two dimensions:



An arbitrary point (x, y) becomes (x', y') in the rotated system, where in matrix notation

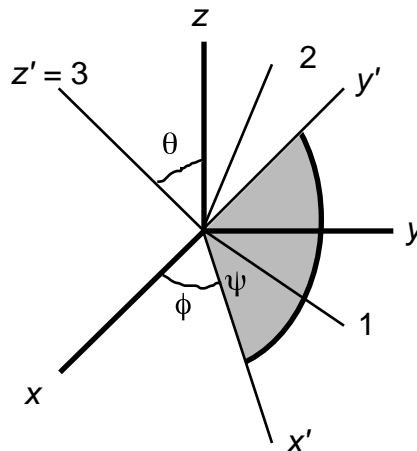
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus, one parameter (ϕ) is required to specify the orientation of the rotated axes.

It turns out that in three dimensions, three angles are required to specify the orientation of the rotated axes. A commonly chosen set of angles are the Euler angles, given by the following construction:

- rotate the xy axes by an angle ϕ about the z axis (in a counter-clockwise direction) leaving $x'y_{\text{tem}}$ in the same plane as xy . This is the same as the 2D rotation described above. $xyz \rightarrow x'y_{\text{tem}}z$
- rotate by θ about the new x' axis, which changes both y_{tem} and z' . The new z' -axis will be identified with the 3-axis of the body-fixed frame. $x'y_{\text{tem}}z \rightarrow x'y'z'$
- rotate by ψ about the new z' axis, which generates new 1,2 axes. $x'y'z' \rightarrow 12z' = 123$

First, in pictures we have



In matrix representation, the rotation is

$$(1, 2, 3) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$(1, 2, 3) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta \sin\phi & \cos\theta \cos\phi & 0 \\ \sin\theta \sin\phi & -\sin\theta \cos\phi & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$(1, 2, 3) = \begin{pmatrix} [\cos\psi \cos\phi - \sin\psi \cos\theta \sin\phi] & [\cos\psi \sin\phi + \sin\psi \cos\theta \cos\phi] & 0 \\ [-\sin\psi \cos\phi + \cos\psi \cos\theta \sin\phi] & [-\sin\psi \sin\phi + \cos\psi \cos\theta \cos\phi] & 0 \\ \sin\theta \sin\phi & -\sin\theta \cos\phi & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Now, we have three sets of axes with a common origin:

xyz is the space-fixed frame

$x'y'z'$ is the rotating frame, but not rotating around z' with the body

123 are the body-fixed principal axes.

- the z' axis is identified with the 3-axis
- the x' axis is the intersection of the 12 and xy planes; *i.e.*, x' lies in the xy plane
- the 12 plane is the same as the $x'y'$ plane, although the axes do not correspond; indeed, the 1 axis is rotated by an angle ψ about the 3 or z' axis.

Now, $x'y'z'$ is rotating with respect to the xyz system, but does not completely follow the body (123). We call the angular velocity of the rotating frame ω' . Note that ω and ω' are different, and may have different components in each frame: only two rates of angular change, $d\theta/dt$ and $d\phi/dt$, contribute to ω' . In the $x'y'z'$ frame, ω' has the components:

(i) $\omega'_{x'}$ is the rate of change of angle θ about the x' axis, so that

$$\omega'_{x'} = d\theta/dt \quad (1a)$$

The other rate of change is that of ϕ about the fixed z -axis, which is $d\phi/dt$. In terms of the $y'z'$ components:

(ii) $\omega'_{y'}$ is the rate of change of angle about the y' axis, so that

$$\omega'_{y'} = (d\phi/dt)_{y'} = (d\phi/dt)\sin\theta \quad (1b)$$

(ii) $\omega'_{z'}$ is the rate of change of angle about the z' axis, so that

$$\omega'_{z'} = (d\phi/dt)_{z'} = (d\phi/dt)\cos\theta \quad (1c)$$

Clearly, when xy and $x'y'$ planes coincide (that is, the z and z' axes coincide so that $\theta = 0$), we find (as expected)

$$\omega'_{x'} = 0 \quad \omega'_{y'} = 0 \quad \omega'_{z'} = (d\phi/dt).$$

The only difference between ω and ω' is that ω includes the rotation of the 12 axes about z' or 3 -axis, which adds $d\psi/dt$ to $\omega_{z'}$. Thus, ω and ω' in the primed frame are

$$\omega_{x'} = \omega'_{x'} = d\theta/dt \quad (2a)$$

$$\omega_{y'} = \omega'_{y'} = (d\phi/dt)\sin\theta \quad (2b)$$

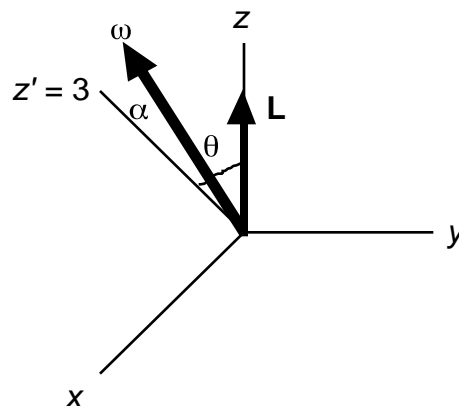
$$\omega_{z'} = \omega'_{z'} + d\psi/dt = (d\phi/dt)\cos\theta + d\psi/dt \quad (2c)$$

Wobble (axial symmetry)

Our objective is to determine the rate at which ω wobbles around \mathbf{L} . We choose the angular momentum vector \mathbf{L} to lie along the z -axis, so that

$$L_x = L_y = 0 \quad L_z = L$$

The direction of \mathbf{L} is called the invariable line, since \mathbf{L} is conserved. As is clear from the figure



L lies in the $y'z'$ plane, so that in the $x'y'z'$ frame,

$$L_{x'} = 0 \quad (3a)$$

$$L_{y'} = L \sin \theta \quad (3b)$$

$$L_{z'} = L \cos \theta \quad (3c)$$

Special case - axial symmetry

Again, we look at an object with axial symmetry (2 inequivalent axes) and take the 3-axis to be the symmetry axis. Then the moments about the principal axes are labelled

$$I_{x'x'} = I_{y'y'} = I_1 = I_2 = I$$

$$I_{z'z'} = I_s$$

Now, the inertia tensor is diagonal in frame $x'y'z'$, (which looks like 1-2-3 frame, but omits the motion about the 3-axis) so $L_{x'} = I \omega_{x'}$. Hence, Eq. (3a) with $L_{x'} = 0$ tells us that

$$\omega_{x'} = 0.$$

In other words, ω must lie in the $y'z'$ plane, and its rate of rotation is therefore $d\phi/dt$ (i.e., the same rate as y' -axis has about z' -axis)

As in the previous lecture, we define α to be the angle between z' and ω . Because ω lies in the $y'z'$ plane, then

$$\omega_{y'} = \omega \sin \alpha \quad \omega_{z'} = \omega \cos \alpha \quad (4)$$

$$L_{y'} = I \omega \sin \alpha \quad L_{z'} = I_s \omega \cos \alpha \quad (5)$$

From (3b) and (3c), we see that

$$L_{y'}/L_{z'} = \tan \theta \quad (6)$$

But from Eqs. (5), we find

$$L_{y'}/L_{z'} = (I/I_s) \tan \alpha \quad (7)$$

Hence

$$\tan \theta = (I/I_s) \tan \alpha \quad (8)$$

This gives the relationship between the cones that ω sweeps out in two different frames: ω sweeps out the body cone in the rotating system, and the space cone in the fixed system.

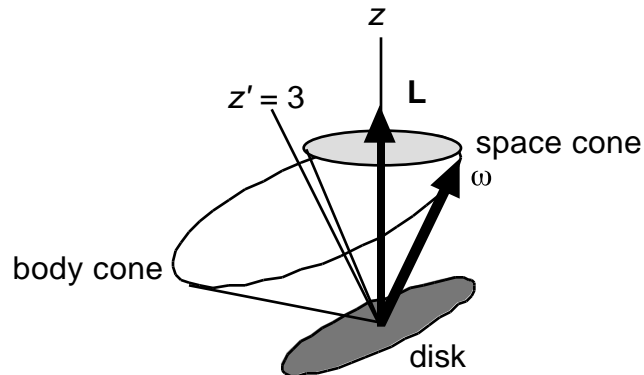
These cones intersect differently for a rod, compared to a disk.

For a pancake-shape (oblate):

$$I_s > I \quad (I_s = 2I \text{ for a lamina})$$

and

$$\tan \theta < \tan \alpha \quad \text{or} \quad \theta < \alpha$$



θ = angle between \mathbf{L} and z' α = body-cone angle between z' and ω
Hence, $\alpha - \theta$ = space cone angle (sign is reversed for a cylindrical object)

Back to the task of finding $d\phi/dt$. Comparing (2b) with (4) for ω_y , we have

$$(d\phi/dt)\sin\theta = \omega \sin\alpha$$

$$\text{---> } d\phi/dt = \omega(\sin\alpha/\sin\theta)$$

This last equation can be cleaned up by using Eq. (8) for the angles. It takes a bit of trigonometry to complete the task:

$$\begin{aligned} \text{---> } \tan\theta &= \sin\theta / \cos\theta = \sin\theta / (1 - \sin^2\theta)^{1/2} = 1 / [1/\sin^2\theta - 1]^{1/2} \\ \text{---> } 1/\sin^2\theta - 1 &= 1/\tan^2\theta \\ \text{---> } 1/\sin\theta &= (1/\tan^2\theta + 1)^{1/2} = (1 + \tan^2\theta)^{1/2} / \tan\theta \end{aligned}$$

Hence

$$d\phi/dt = \omega(\sin\alpha/\sin\theta) = \omega(\tan\alpha/\tan\theta)(1 + \tan^2\theta)^{1/2} / (1 + \tan^2\alpha)^{1/2}$$

Now we substitute Eq. (8) to get rid of $\tan\theta$:

$$d\phi/dt = \omega(I_s / I) \{ [1 + (I / I_s)^2 \tan^2\alpha] / (1 + \tan^2\alpha) \}^{1/2}$$

But

$$1 + \tan^2\theta = 1 + \sin^2\theta/\cos^2\theta = 1 / \cos^2\theta$$

so

$$\begin{aligned} d\phi/dt &= \omega (I_s / I) \{ [1 + (I / I_s)^2 \tan^2\alpha] \cos^2\alpha \}^{1/2} \\ &= \omega (I_s / I) \{ \cos^2\alpha + (I / I_s)^2 \sin^2\alpha \}^{1/2} \\ &= \omega (I_s / I) \{ \cos^2\alpha + (I / I_s)^2 (1 - \cos^2\alpha) \}^{1/2} \\ &= \omega (I_s / I) \{ \cos^2\alpha (1 - (I / I_s)^2) + (I / I_s)^2 \}^{1/2} \end{aligned}$$

or

$$d\phi/dt = \omega \{ \cos^2\alpha [(I_s / I)^2 - 1] + 1 \}^{1/2}$$

Whew!

Example

We return to the problem of the Earth's rotation, where we had

$$\cos\alpha = 1.000\dots$$

$$I_s/I = 1.00327$$

Then

$$\begin{aligned} d\phi/dt &= \omega \{ \cos^2\alpha [(I_s / I)^2 - 1] + 1 \}^{1/2} \\ &= \omega \{ 1 \cdot (I_s / I)^2 - 1 + 1 \}^{1/2} \\ &= \omega (I_s / I) = 1.00327 \omega \end{aligned}$$

Now, ω is the angular frequency of the Earth (2 π radians per day), so that

$$2\pi / (d\phi/dt) = 2\pi / (1.00327 \omega) = 0.997 \text{ days.}$$

That is, as seen by a "stationary" observer, ω wobbles at the rate of about a turn per day.