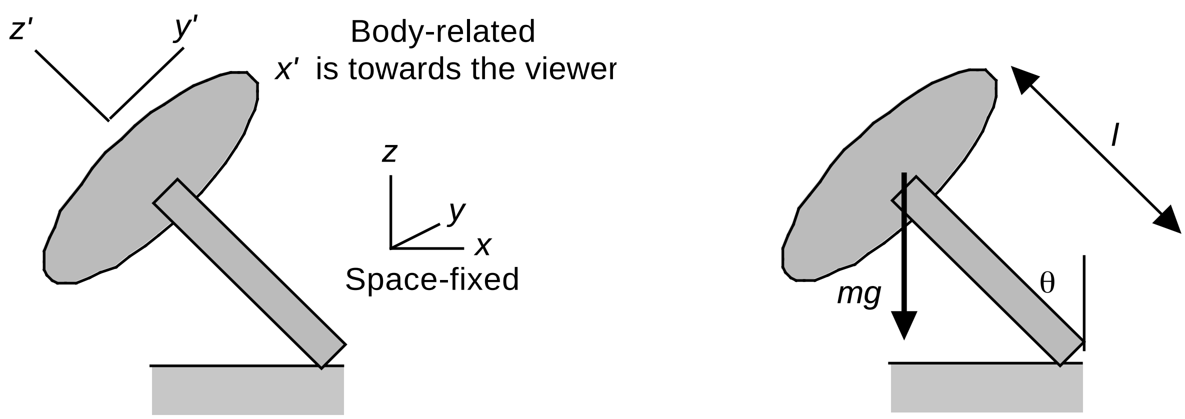
Lecture 29 - Rotation with torque - gyroscope

Text: Fowles and Cassiday, Chap. 9

We now generalize our treatment of rotational motion by including a torque. We consider the specific problem of gyroscopic precession, and leave the interested student to read about related problems such as rotating wheels.

The notation is

- xyz are space-fixed axes
- $x'\,y'\,z'$ are body-related axes with z' along the axis of the gyroscope (z' lies along a principal axis, but $x'y'$ do not rotate with the body).



The torque acts about the x' axis, which is made to lie in the xy plane (see diagram).

From the free-body diagram above, the components of the torque in the rotating system are

$$\tau_{x'} = mg\ell \sin\theta$$
$$\tau_{y'} = \tau_{z'} = 0.$$

From a few lectures ago, the relationship between torque and angular momentum in the rotating frame is:

$$\tau = (d\mathbf{L} / dt)_{rot} + \omega' \mathbf{x} \mathbf{L}$$

As it should, this expression involves ω' of the rotating frame, not ω of the body. Since torque is measured with respect to a physical position, it makes sense to locate the coordinate origin at the pivot point. Further, we need $I_{,PIV}$ not just I of the body-fixed frame, where $I_{,PIV} = I + m\ell^2$ from the parallel axis theorem.

We now write the torque equations in component form. Since $x'y'z'$ correspond to the principal axes of the gyroscope, then **L** is diagonal in ω:

$$L_{x'} = I_{x'x'}\,\omega_{x'} = I_{,PIV}\; d\theta/dt$$
$$L_{y'} = I_{y'y'}\,\omega_{y'} = I_{,PIV}\; (d\phi/dt) \sin\theta$$
$$L_{z'} = I_{z'z'}\,\omega_{z'} = I_s\; (d\phi/dt \cos\theta + d\psi/dt) = I_s S$$

In this last equation, the spin S is introduced to replace $\omega_{z'}$ and simplify the equations. The components of ω' (the angular velocity of the rotating frame) are, in the rotating system:

$$\omega'_{x'} = (d\theta/dt)$$
$$\omega'_{y'} = (d\phi/dt)\sin\theta$$
$$\omega'_{z'} = (d\phi/dt)\cos\theta$$

Now we can substitute into the torque equations.

1. *x*-component

$$\tau_{x'} = dL_{x'}/dt + \omega'_{y'} L_{z'} - \omega'_{z'} L_{y'}$$

which becomes upon substitution

$$mg\ell \sin\theta = I_{,PIV} d^2\theta/dt^2 + (d\phi/dt)\sin\theta \bullet I_s S - (d\phi/dt)\cos\theta \bullet I_{,PIV}(d\phi/dt)\sin\theta$$

or

$$mgl \sin\theta = I_{,PIV} d^2\theta/dt^2 + I_s S (d\phi/dt)\sin\theta - I_{,PIV} (d\phi/dt)^2 \sin\theta\cos\theta \qquad (1)$$

2. *y*-component

$$\tau_{y'} = dL_{y'}/dt + \omega'_{z'} L_{x'} - \omega'_{x'} L_{z'}$$

or

$$0 = I_{,PIV} d[(d\phi/dt)\sin\theta]/dt + (d\phi/dt)\cos\theta \bullet I_{,PIV} (d\theta/dt) - (d\theta/dt) I_s S$$
$$0 = I_{,PIV} d[(d\phi/dt)\sin\theta]/dt + I_{,PIV} (d\phi/dt)(d\theta/dt)\cos\theta - I_s S (d\theta/dt) \qquad (2)$$

3. *z*-component

$$\tau_{z'} = dL_{z'}/dt + \omega'_{x'} L_{y'} - \omega'_{y'} L_{x'}$$

or

$$0 = I_s (dS/dt) + (d\theta/dt) \bullet I_{,PIV} (d\phi/dt)\sin\theta - (d\phi/dt) \bullet \sin\theta\, I_{,PIV} (d\theta/dt)$$
$$0 = I_s (dS/dt) \qquad (3)$$

Now, Eq. (3) for the *z*-component tells us that $dS/dt = 0$, or equivalently, $L_{z'}$ is a constant:

$$I_s S = L_{z'} = \text{constant} \qquad (4)$$

In other words, the component of the angular momentum **L** along *z'* is a constant.

Steady precession

Consider now the case where the precession of the gyroscope is steady, *i.e.* there is no change in the angle between *z* and *z'*

$$d\theta/dt = d^2\theta/dt^2 = 0,$$

so that Eq. (1) gives, after canceling a $\sin\theta$:

$$mg\ell = 0 + I_s S (d\phi/dt) - I_{,PIV} (d\phi/dt)^2 \cos\theta \qquad (5)$$

This equation is quadratic in $d\phi/dt$, and has the following solution (by the usual quadratic formula):

$$d\phi/dt = [I_s S \pm (I_s^2 S^2 - 4I_{,PIV}\, mg\ell \cos\theta)^{1/2}] / (2I_{,PIV} \cos\theta) \qquad (6)$$

This expression gives two possible solutions for the precession rate; usually a simple top assumes the slower rate. Further, it gives a condition for there to be any physical solution at all, namely

$$I_s^2 S^2 > 4I_{,PIV}\, mg\ell \cos\theta \qquad (7)$$

We are familiar with the consequences of this equation in part from the behaviour of a spinning top. If the top is spinning very fast, it will not fall over. The vertical stability condition is found by putting $\theta = 0$ in Eq. (7), to yield

$$I_s^2 S^2 > 4I_{,PIV}\, mg\ell \qquad \text{(vertical stability)} \qquad (8)$$

So long as $S^2 > 4I_{,PIV}\, mg\ell / I_s^2$ the top can maintain its vertical orientation. If the top slows down and S violates the inequality, then the orientation is no longer vertical and the top begins to precess. The same kind of condition on the minimum speed for stability can be found for rolling wheels.

How do we regain the precession formula from first year, where we considered a gyroscope with its axis in the horizontal plane ($\theta = \pi/2$)? Returning to Eq. (5), with $\theta = \pi/2$,

$$(5) \quad \text{-->} \quad mg\ell = I_s S\,(d\phi/dt) - 0$$

or

$$d\phi/dt = mg\ell / I_s S = mg\ell / L_{z'}$$