Lecture 9 - Examples of 3D motion

Text: Fowles and Cassiday, Chap. 4

In one dimension, the equations of motion to be solved are functions of only one position variable, namely \( x \). In three dimensions, there are three position variables, \( x, y, z \), and the possibility exists that the equations describing the motion are coupled: motion in the \( x \) direction is linked to motion in the \( y \) direction, for example. In our study of motion in three dimensions, we begin with so-called separable forces, in which the equations of motion are *not* coupled.

**Projectile motion in 3D with linear drag**

Consider the general projectile problem in which there is a constant force arising from gravity

\[-mgk,\]

where

- \( m \) = particle mass
- \( g \) = gravitational constant
- \( k \) = unit vector pointing in the +ve \( z \) direction

Newton’s second law for this situation reads

\[ md^2r / dt^2 = -mgk \]

which separates into 3 equations using Cartesian coordinates.

Now consider the slightly more difficult example of constant gravity plus linear drag:

\[ md^2r / dt^2 = -mgk - cv \]

which we recast into a differential equation for \( v \)

\[ dv / dt = -gk - \Gamma v \]

where \( \Gamma = c / m \) (=2\( \gamma \) in damped oscillations)

(note: different notation from Fowles, who uses \( \gamma \) for a variety of combinations)

This is a vector equation, including the \( v \), in three dimensions, but it nevertheless separates in 3:

\[
\begin{align*}
v_x / dt &= -\Gamma v_x \\
v_y / dt &= -\Gamma v_y \\
v_z / dt &= -\Gamma v_z - g
\end{align*}
\]

We have treated all of these equations before in our studies of drag in one dimension. The differential equations for the velocities yield:

\[
\begin{align*}
v_x &= v_{o,x} \exp(-\Gamma t) \\
v_y &= v_{o,y} \exp(-\Gamma t) \\
v_z &= v_{o,z} \exp(-\Gamma t) - (g / \Gamma)[1 - \exp(-\Gamma t)],
\end{align*}
\]

where \( v_o \) is the initial velocity.

Note the behavior of \( v_z \): if \( g = 0 \), then \( v_z \) just decays to zero with time, but if \( g > 0 \), \( v_z \) approaches \(-g / \Gamma\), as expected from the diff. equ. for \( v_z \) when \( dv_z / dt \) vanishes.

Proof of the last equation (dropping the \( z \)-subscripts):

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\[ \frac{dv}{dt} = -v_o e^{-\Gamma t} - g \frac{[0 - (-\Gamma)e^{-\Gamma t}]}{\Gamma} \]
\[ = -v_o e^{-\Gamma t} - g e^{-\Gamma t} \]
\[ = -\Gamma \left( v_o e^{-\Gamma t} - g \frac{1 - e^{-\Gamma t}}{\Gamma} \right) \]
\[ = -\Gamma \left( v_o e^{-\Gamma t} - g \frac{1 - e^{-\Gamma t}}{\Gamma} \right) + \Gamma \frac{g}{\Gamma} = -\Gamma v - g \]

These in turn can be integrated to give
\[ x(t) = (v_{ox} / \Gamma)[1 - e^{\Gamma t}] \]
\[ y(t) = (v_{oy} / \Gamma)[1 - e^{\Gamma t}] \]
\[ z(t) = [v_{oz} + g / \Gamma^2] \cdot [1 - e^{\Gamma t}] - gt / \Gamma \]

Proof of the last equation (dropping the \( z \)-subscripts):
Rewrite \( v_z \) as \((v_o + g / \Gamma) e^{\Gamma t} - (g / \Gamma)\) and integrate \( z = \int v dt = (1/\Gamma) \int v d(\Gamma t)\),
\[ z = (v_o + g / \Gamma)(1/\Gamma)[1 - e^{\Gamma t}] - gt / \Gamma. \]

Taking \( v_o \) to lie in the \( xz \) plane, there is no \( y \)-component to the motion and the projectile path looks like:

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parabola if \( \Gamma = 0 \)
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Note that although the solutions for \( r(\Gamma t) \) look pathological in the \( \Gamma \to 0 \) limit, in fact they are well-behaved, as can be seen by expanding the exponential functions
\[ e^x \sim 1 + x + x^2 / 2 + ... \]

The solution for \( z \) must be taken to second order:
\[ z = (v_o + g / \Gamma)^2 (1 / \Gamma) (1 - 1 + \Gamma t - (\Gamma t)^2 / 2 ...) - gt / \Gamma \]
\[ = v_o t + gt / \Gamma - gt^2 / 2 - gt / \Gamma \left( -v_o \Gamma t^2 / 2, \right. \text{which vanishes with} \Gamma) = v_o t - gt^2 / 2. \]

**Isotropic oscillator in 2D**

The one-dimensional oscillator can be easily generalized to higher dimensions by adding more equations, each of which contains only a given coordinate. In 2D, for example,
\[ d^2x / dt^2 = -(k / m)x \]
\[ \frac{d^2y}{dt^2} = -(k/m)y \]

etc. This oscillator is said to be isotropic, in that the spring constant \( k \) is the same in both directions (see textbook for anisotropic oscillators).

The solutions to these equations have the familiar form

\[ x = A \cos(\omega t + \alpha) \quad y = B \cos(\omega t + \beta) \]

where

- \( A, B \) are amplitudes
- \( \alpha, \beta \) are phase angles
- \( \omega = (k/m)^{1/2} \)

These two solutions give us a parametric equation (in time) for the motion in the \( xy \) plane. It is straightforward to solve one equation for the time, then substitute into the other to find \( y \) as a function of \( x \).

If \( \alpha = \beta \), then the path is

If \( |\alpha - \beta| = \pi / 2 \), so the amplitudes are out of phase, then the path is an ellipse

Lastly, if \( \alpha - \beta \) is just arbitrary (but fixed), then the path is a rotated ellipse:

This line of reasoning can be generalized to 3D as shown in the text.
Charged particle in a magnetic field

Lastly, we consider the motion of a particle with charge $q$ in a magnetic field $B$. For simplicity, we consider just motion of the particle in the plane perpendicular to $B$. The force experienced by the particle is $F = qv \times B$, which gives the particle an acceleration perpendicular to $v$ and $B$ of

$$ F = ma = qvB \quad \Rightarrow \quad a = \left(\frac{qB}{m}\right)v $$

But $a = v^2 / R$ for centripetal acceleration, so

$$ v^2 / R = \left(\frac{qB}{m}\right)v \quad \text{or} \quad v / R = \frac{qB}{m} $$

Since the period of the motion is $T = 2\pi R / v$, then

$$ T = 2\pi \left(\frac{m}{qB}\right) $$

Lastly, the angular frequency $\omega = 2\pi / T$

$$ \Rightarrow \omega = \frac{qB}{m} $$

We repeat all of this in Cartesian coordinates. Starting with the cross product ($B$ points along $z$)

$$ v \times B = (v_y B, -v_x B, 0), $$

which says that $F_y$ is negative if the initial velocity lies along the positive $x$-axis (agrees with the right-hand rule for cross products). Newton’s law $F = ma$ becomes

$$ m \left(\frac{d^2x}{dt^2}\right) = qB \left(\frac{dy}{dt}\right) \quad (1) $$

$$ m \left(\frac{d^2y}{dt^2}\right) = -qB \left(\frac{dx}{dt}\right) \quad (2) $$

Now, these equations are coupled: the force is not separable. We can trivially integrate once with respect to $t$ to obtain

$$ \frac{dx}{dt} = \left(\frac{qB}{m}\right)y + c_y \quad (3) $$

$$ \frac{dy}{dt} = -\left(\frac{qB}{m}\right)x + c_x \quad (4) $$

and substitute (4) into (1)

$$ \frac{d^2x}{dt^2} = \omega\left[-\omega x + c_x\right] $$

or

$$ \frac{d^2x}{dt^2} = -\omega^2 (x - a) \quad \text{with} \quad a = c_x / \omega $$

This expression is just a variant of simple harmonic motion with $x$ shifted to $x - a$. The solution is

$$ x - a = A \cos(\omega t + \varphi) \quad (5) $$

Substituting $dx / dt$ from (5) into (3) also gives

$$ y - b = -A \sin(\omega t + \varphi) \quad \text{with} \quad b = -c_y / \omega $$

Thus, the motion is a circle centered on $(a, b)$, and the particle travels with an angular frequency of $\omega = qB / m$ as expected. Adding a $z$-component to $v_o$ turns the path into a helix.
B towards viewer