Rutherford scattering

Text: similar to Fowles and Cassiday, Chap. 6

Most of the properties of nuclei and their constituents have been determined by scattering experiments in which energetic beams of particles are directed towards a target. Frequently, the interactions are repulsive (*e.g.*, between like charges) although that is not an important consideration. What allows one to obtain information about the nucleus is that the scattering is dominated by the Coulomb or strong interaction. Here we consider just Coulomb scattering, whose force is

$$f = kQq / r^2,$$

where k is a constant (much like gravity's G) , and Q and q are the charges.

Trajectories for Coulomb scattering

We consider now the motion of a charged particle in interaction with another charged particle whose position is fixed. The equation of motion to be solved is

$$d^2u / d\theta^2 + u = -f / m\ell^2 u^2$$

with $f = kQqu^2$, so

$$d^2u / d\theta^2 + u = - kQq / m\ell^2.$$

The attractive or repulsive nature of the interaction is borne by the product of the signs of the charges. Defining $\Gamma = -kQq$, then

$$d^2u / d\theta^2 + u = +\Gamma / m\ell^2$$

To within the sign on the rhs, this equation has the same functional form as the gravitational force problem, $d^2u / d\theta^2 + u = +\gamma/\ell^2$, except Γ may be:

$\Gamma > 0$ attractive $\qquad$ $\Gamma < 0$ repulsive.

The solution is the same as before, with $\gamma \to \Gamma / m$.

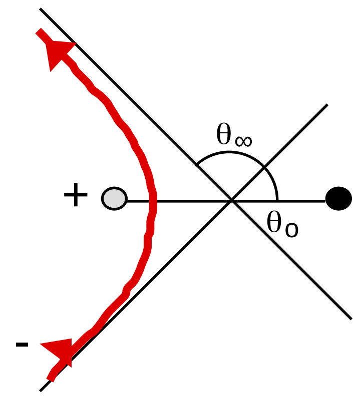
$$r = (m\ell^2/\Gamma) / (1 + e \cos\theta) \qquad (\Gamma \text{ is also hidden in the definition of } e) \qquad (1)$$

$$E = (\Gamma^2 / 2m\ell^2) \cdot (e^2 - 1). \qquad (2)$$

The sign of Γ does not affect the energy.

$e > 1$, attraction

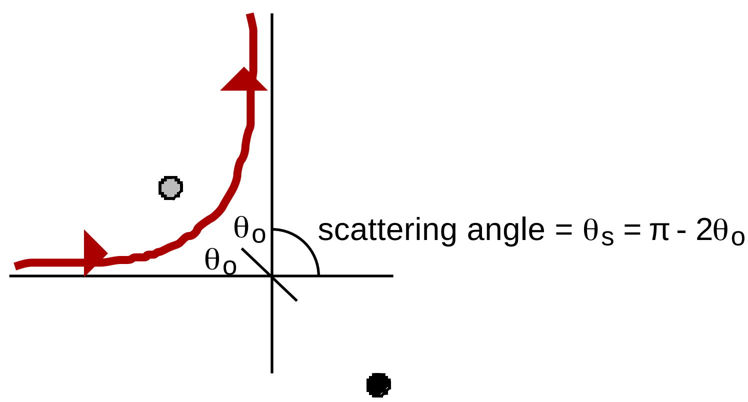
We now examine hyperbolic orbits with attractive interactions, as a prelude to obtaining scattering cross sections. Often, scattering experiments involve repulsive interactions, but the same formulae are found for attractive interactions. With our definition for Γ, ($\Gamma > 0$ for attraction) we can use our results from Lecs. 15 and 16 without change.



The asymptotes of the equation $r = (m\ell^2/\Gamma) / (1 + e\cos\theta)$ are at $\cos\theta_\infty = -1/e$. But the angle θ_o in the above diagram is given by $\theta_o = \pi - \theta_\infty$. Hence,

$\cos\theta_o = -\cos\theta_\infty$ so that $\cos\theta_o = 1/e$.

At its smallest (when e is large and the asymptotes are almost vertical), θ_∞ is just above $\pi/2$ and θ_o is just below $\pi/2$. It is usually more convenient to make one of the asymptotes lie parallel to the x - axis:

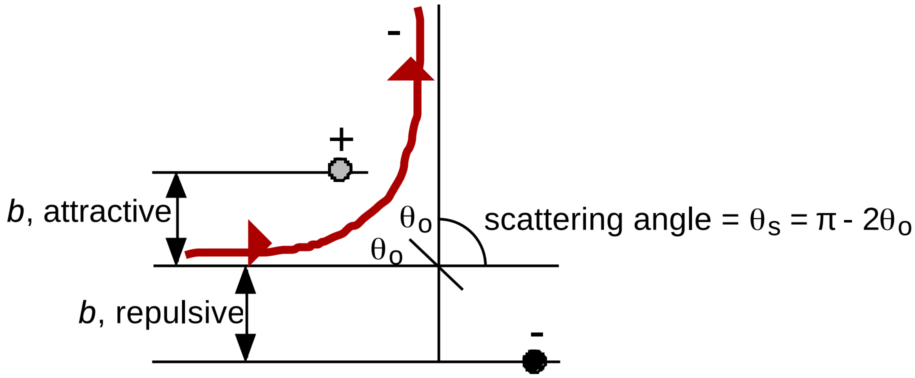


Now, the asymptote is displaced from passing through the charge because of the angular momentum l. In fact, for a specific energy E, the value of θ_o is a function of l:

$$E = (\Gamma^2 / 2m\ell^2)\bullet(e^2 - 1)$$
$$= (\Gamma^2 / 2m\ell^2)\bullet(1/\cos^2\theta_o - 1) \quad \text{(since } \cos\theta_o = 1/e \text{ is the asymptote)}$$
$$= (\Gamma^2 / 2m\ell^2)\bullet(\sin^2\theta_o / \cos^2\theta_o)$$

Hence
$$\tan\theta_o = (2m\ell^2 E / \Gamma^2)^{1/2}$$
$$= (2mE)^{1/2}(\ell/\Gamma) \qquad (4)$$

The displacement of the asymptote is called the impact parameter b:



We can relate b to the angular momentum per unit mass by

$$\ell = L/m = bv_o \qquad (5)$$

where v_o is the velocity as $r \rightarrow \infty$. But the velocity at infinite separation gives the total energy E, because the potential energy $V(r = \infty) = 0$:

$$E = (1/2)mv_o^2 \qquad (6)$$

Thus, Eq. (4) gives, upon substituting (5) for ℓ

$$\tan\theta_o = (2mE)^{1/2}\, bv_o /\Gamma$$
$$= (2mv_o^2E)^{1/2} b /\Gamma$$

Substituting (6) gives

$$\tan\theta_o = (4E^2)^{1/2} b /\Gamma$$

or $$\tan\theta_o = 2Eb /\Gamma. \qquad (7)$$

This relationship tells us how the scattering angle is related to the impact parameter. For example, in a head-on collision, $b \rightarrow 0$ so that $\theta_o \rightarrow 0$ from Eq. (7) and $\theta_s \rightarrow \pi$; that is, the particle is backscattered to its incoming trajectory.

Cross sections (optional)

An experiment measures the probability P for a beam particle to be scattered from a target, by comparing the number of particles scattered from a beam to the number of particles incident upon the target (see PHYS120 lecture 2). Theoretically, the probability of scattering is equal to the ratio of the effective area of the target objects (as seen by the incident beam of particles) compared to the total area of the target region exposed to the beam:

$$P = [\text{number of target particles exposed to the beam}]\ \sigma / A_T.$$

But, the number of target particles exposed to the beam is just the product of A_T with the number of particles per unit area n_T:

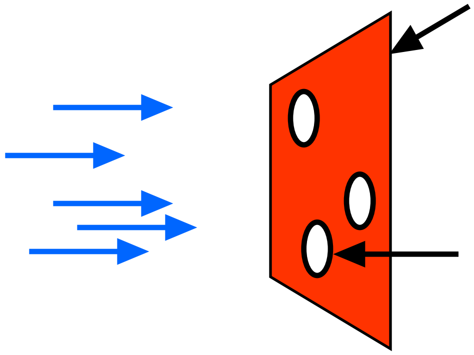
$$[\text{number of target particles exposed to the beam}] = n_T A_T.$$

Hence, the scattering probability is

$$P = n_T A_T \sigma / A_T$$

or

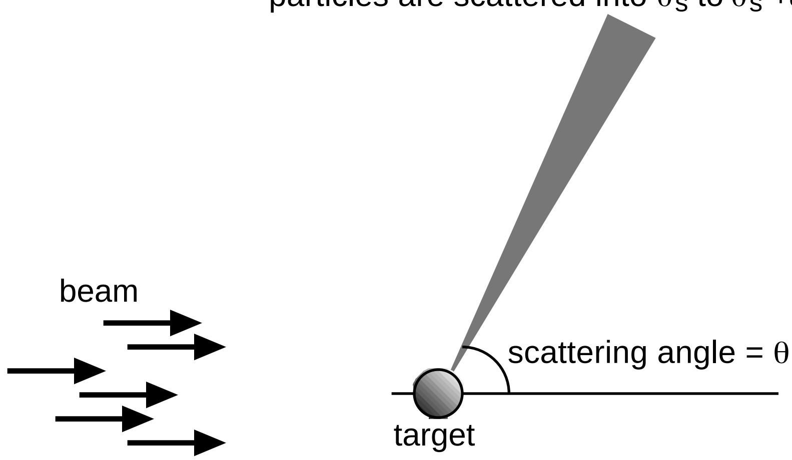
$$P = n_T \sigma. \qquad (8)$$



Here, σ is the total cross section, which is linked to the total probability that a particle is scattered in any direction.

Differential cross section

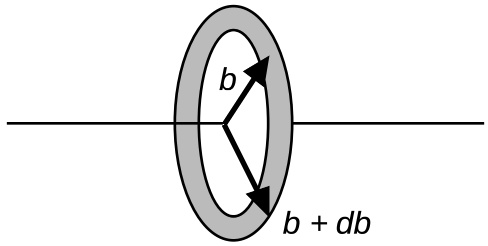
We know from our study of trajectories that some collisions at small *b* lead to large scattering angles, while others at large *b* lead only to small scattering angles. We define $\sigma(\theta_s)$ as that part of the cross section associated with a specific scattering angle θ_s. Now $\sigma(\theta_s)$ is a differential cross section in the sense that it is an area per unit solid angle, where solid angles have units of square radians or steradians. In the diagram below, the solid angle subtended by a polar angle $d\theta_s$ and azimuthal angle $d\phi$ is $\sin\theta_s \, d\theta_s d\phi$. The solid angle covered by the cone between θ_s and $\theta_s + d\theta_s$, which includes all values of ϕ, is thus $2\pi \sin\theta_s \, d\theta_s$.



Thus, that part of the effective area of the target particle associated with the solid angle $2\pi \sin\theta_s \, d\theta_s$ is

$$[\text{effective area for solid angle}] = \sigma(\theta_s) \bullet 2\pi \sin\theta_s \, d\theta_s. \qquad (9)$$

But we can calculate the effective area knowing the impact parameter range that corresponds to θ_s. From the scattering geometry



$$[\text{area of target from } b \text{ to } b + db] = \pi(b + db)^2 - \pi b^2$$
$$= 2\pi b db \qquad (10)$$

Comparing equations (9) and (10),

$$2\pi b db = 2\pi\sigma(\theta_s)\sin\theta_s \, d\theta_s$$

$$\Rightarrow \quad \sigma(\theta_s) = (b / \sin\theta_s)|db / d\theta_s|. \qquad (11)$$

The absolute value around $db / d\theta_s$ is needed because the derivative is negative: θ_s increases as b decreases.

Simplifying Eq. (11) involves a lot of trig. We know that

$$\tan\theta_o = 2Eb / \Gamma$$

or

$$b = (\Gamma / 2E)\tan\theta_o. \qquad (12)$$

Thus

$$db / d\theta_s = (d\theta_o / d\theta_s)(db / d\theta_o)$$
$$= [d(\pi/2 - \theta_s/2) / d\theta_s][db / d\theta_o]$$
$$= -(1/2)(\Gamma / 2E)(d \tan\theta_o / d\theta_o) \qquad (13)$$

However, the derivative of the tangent is

$$d \tan\theta / d\theta = d(\sin\theta / \cos\theta) / d\theta$$
$$= (\sin^2\theta + \cos^2\theta) / \cos^2\theta$$
$$= 1 / \cos^2\theta$$

Thus, Eq. (13) becomes

$$|db / d\theta_s| = \Gamma / 4E\cos^2\theta_o. \qquad (14)$$

Collecting terms

$$\sigma(\theta_s) = (b / \sin\theta_s)\left| db / d\theta_s \right|$$
$$= (\Gamma / 2E) \tan\theta_o \, (\Gamma / 4E) / (\cos^2\theta_o \bullet \sin\theta_s)$$

At this point, we have to convert θ_o to $\pi/2 - \theta_s/2$; further, we need

$$\sin(\pi/2 - \theta_s/2) = \cos(\theta_s/2) \quad \text{and} \quad \cos(\pi/2 - \theta_s/2) = \sin(\theta_s/2)$$

The trig functions then become

$$\tan\theta_o / (\cos^2\theta_o \sin\theta_s) = [\cos(\theta_s/2) / \sin(\theta_s/2)] / [\sin^2(\theta_s/2) \sin\theta_s]$$
$$= \cos(\theta_s/2) / [\sin^3(\theta_s/2) \sin\theta_s]$$
$$= 1 / [2\sin^4(\theta_s / 2)]$$

where we have used

$$\sin\theta_s = 2 \sin(\theta_s/2) \cos(\theta_s/2)$$

Hence, our expression for the cross section becomes

$$\sigma(\theta_s) = \Gamma^2 / 16E^2 \sin^4(\theta_s/2) \qquad (15)$$

This is the Rutherford cross section and has been derived from purely classical considerations. It was used by Rutherford to deduce that the nucleus acted like a point scattering centre in deflecting a beam of the nuclei (produced in the decay of other radioactive nuclei). Note that $\sigma(\theta_s)$ is infinite at $\theta_s = 0$, and finite at $\theta = \pi$:

$$\sigma(\pi) = \Gamma^2 / 16E^2.$$