

Hamilton's equations

Text: Fowles and Cassiday, Chap. 10

We finish our discussion of Lagrangian mechanics with Hamilton's equations. In the Lagrangian approach, the principal kinematic variables are the generalized coordinates q_i and their corresponding velocities. In Hamilton's approach, the velocities are replaced by momenta p_i conjugate to the coordinates.

How do we obtain p_i ? For the moment, assume that the potential energy $V(r)$ is independent of velocity:

$$\partial V / \partial \dot{q}_i = 0$$

Then

$$\partial L / \partial \dot{q}_i = \partial K / \partial \dot{q}_i$$

Now, in Cartesian coordinates, the kinetic energy is

$$K = (m/2) \sum_i \dot{q}_i^2$$

$$\partial K / \partial \dot{q}_i = m \dot{q}_i$$

But the right hand side is just the momentum $p = m(dq/dt)$. Thus

$$p_i = \partial L / \partial \dot{q}_i \quad (\text{Cartesians, } V \text{ independent of } \dot{q}_i)$$

We propose that this relationship is true in general; that is, for a generalized coordinate q_i there exists a *conjugate momentum* p_i defined by

$$p_i = \partial L / \partial \dot{q}_i \tag{1}$$

We can find the time derivative of the momentum by substituting (1) into Lagrange's equation

$$(\partial L / \partial q_i) - d/dt(\partial L / \partial \dot{q}_i) = 0$$

to yield

$$\dot{p}_i = \partial L / \partial q_i \tag{2}$$

In other words, we can obtain a generalized momentum and its time derivative from the Lagrangian using Eqs. (1) and (2). So what do we do with p_i ? The first thing to do is check under what conditions it is conserved. Conservation of p_i implies that Eq. (2) vanishes, or

$$\text{if } L \text{ is independent of } q_i, \quad \dot{p}_i = 0$$

More importantly, we can construct an object called the Hamiltonian H with p_i :

$$H = \sum_i p_i \dot{q}_i - L \tag{3}$$

This can be recast into a more familiar form, first by substituting (1)

$$\sum_i p_i \dot{q}_i = \sum_i (\partial L / \partial \dot{q}_i) \dot{q}_i \tag{4}$$

If the potential energy depends only on coordinates

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial K}{\partial \dot{q}_i} + 0 \quad (5)$$

so (4) is

$$p_i \dot{q}_i = \frac{\partial K}{\partial \dot{q}_i} \dot{q}_i \quad (6)$$

But for homogeneous functions (*i.e.*, same exponent for the variables), Euler showed that

$$x_i \frac{\partial f}{\partial x_i} = kf \quad (7)$$

where k is the exponent of the variable in the function (*i.e.*, $k = 2$ for x^2 *etc.*).

Applying (7) to (6) gives

$$p_i \dot{q}_i = 2K \quad (8)$$

Lastly, substituting (8) into (3)

$$H = p_i \dot{q}_i - L = 2K - (K - V) = K + V$$

or

$$H = K + V \quad (9)$$

so that H is the total energy of the system.

Equations of motion

Lagrange's equation provided a relationship between q and, after some work, its *second* time derivative (recall example of simple harmonic motion). We now find an equation of motion based on the Hamiltonian, rather than the Lagrangian. The situation is different:

Lagrangian: $2 \times 3N$ variables q_i, \dot{q}_i

Hamiltonian: $3 \times 3N$ variables q_i, \dot{q}_i, p_i

To obtain the equations of motion, we perform a variation on H , just as we obtained Lagrange's equations from a variation on L . Starting with

$$H = p_i \dot{q}_i - L$$

we find

$$H = p_i [\dot{q}_i \dot{q}_i + \dot{q}_i p_i - (\partial L / \partial \dot{q}_i) \dot{q}_i - (\partial L / \partial q_i) q_i] \quad (10)$$

Now the first and third terms cancel by virtue of the definition

$$p_i = \partial L / \partial \dot{q}_i$$

so we are left with:

$$H = p_i [\dot{q}_i p_i - (\partial L / \partial q_i) q_i] \quad (11)$$

But if we look at H simply as a function of q_i and p_i , then in general

$$H = p_i [(\partial H / \partial p_i) p_i + (\partial H / \partial q_i) q_i] \quad (12)$$

Comparing Eqs. (11) and (12)

$$\partial H / \partial p_i = \dot{q}_i \quad (13a)$$

$$\partial H / \partial q_i = -\dot{p}_i \quad (13b)$$

after substituting Eq. (2)

$$\dot{p}_i = \partial L / \partial q_i$$

Eqs. (13) are Hamilton's equations of motion. Just to contrast the difference:

Lagrange: N equations, each second order in time (d^2/dt^2)

Hamilton: $2N$ equations, each first order in time (d/dt)

Example: 1D harmonic oscillator

This problem involves only x and p . Although some of the steps are a little obvious, this is the solution:

Start with x and its derivative:

$$K = m\dot{x}^2/2 \quad V = kx^2/2$$

Then

$$L = K - V = m\dot{x}^2/2 - kx^2/2$$

$$p = \partial L / \partial \dot{x}$$

$$= \partial (m\dot{x}^2/2) / \partial \dot{x}$$

$$= m\dot{x}$$

Thus, in terms of x and p , we have

$$H = \dot{x}p - L = \dot{x}p - (m\dot{x}^2/2 - kx^2/2)$$

$$= p^2/m - p^2/2m + kx^2/2$$

$$= p^2/2m + kx^2/2$$

So far, so good. We started with L and now have H , although we could have written down the last equation by inspection. Next, the equations of motion:

$$\partial H / \partial p = \dot{x} \quad \dot{x} = \partial (p^2/2m) / \partial p = p/m$$

$$\partial H / \partial x = -\dot{p} \quad -\dot{p} = \partial (kx^2/2) / \partial x = kx$$

These two equations could be integrated as they stand. If we want to recover our usual expression for simple harmonic motion, we substitute one equation into the other:

$$\begin{aligned} -\dot{p} &= kx & -d(m\dot{x})/dt &= kx \\ \text{or} & & m\ddot{x} &= -kx \end{aligned}$$