Lecture 19 - Degenerate stars

What's Important:

- pressure of a degenerate Fermi gas
- equilibrium of degenerate stars
- neutron stars

Text: Gasirowicz, Chap. 9; Carroll and Ostlie, Chap. 15

Pressure of a degenerate Fermi gas

From the previous lecture, the total energy of a degenerate Fermi gas of spin 1/2 particles is

$$E_{TOT} = \frac{\pi^3\hbar^2}{10m}\left(\frac{3N}{\pi}\right)^{5/3} V^{-2/3}. \qquad (1)$$

The pressure of the gas can be obtained from this by differentiation

$$P_{\text{deg}} = -\partial E_{\text{TOT}} / \partial V \qquad \text{at fixed } N, \qquad (2)$$

so

$$\begin{aligned} P_{\text{deg}} &= -\left(-\frac{2}{3}\right)\frac{\pi^3\hbar^2}{10m}\left(\frac{3N}{\pi}\right)^{5/3} V^{-5/3} \\ &= \frac{\pi^3\hbar^2}{15m}\left(\frac{3N}{\pi V}\right)^{5/3} \end{aligned} \qquad (3)$$

The density ρ can replace N/V in the last expression.

Equilibrium of degenerate stars

In normal stars, the nuclear fires generate energy which saves the star from gravitational collapse. But even in the absence of an energy source, the pressure from degenerate electrons (or neutrons, for dense stars) may balance the gravitational pressure and maintain mechanical equilibrium. To find this condition, we first must determine the pressure from gravity. First, recall that the total gravitational potential energy is given by

$$V_{\text{grav}} = -(3/5)\, GM^2/R. \qquad (4)$$

Students may have seen this derived in earlier courses; the calculation involves the following steps:

1. the mass of a thin shell of material at radius r is $4\pi r^2 \rho \, dr$
2. the gravitational potential of the shell arising from its attraction to the mass within r is
   $-G\,(4\pi r^2 \rho\, dr)\,(4\pi r^3 \rho /3) / r = 3G\,(4\pi/3)^2\rho^2\, r^4 dr$
3. the integral of this potential over all concentric shells of the star is
   $-3G\,(4\pi/3)^2\rho^2 \int_o^R r^4 dr = -3G\,(4\pi R^3 \rho /3)^2 / 5R = -(3/5)\, GM^2/R.$

Knowing the potential, one can find the pressure by differentiation

$P_{\text{grav}} = -\partial V_{\text{grav}}/\partial V$ (sorry for the notation)

with the substitution $R = (3V / 4\pi)^{1/3}$. Thus,

$$P_{grav} = -\frac{3}{5}\left(\frac{4\pi}{3}\right)^{1/3} GM^2 \frac{dV^{-1/3}}{dV}$$

$$= -\frac{3}{5}\left(-\frac{1}{3}\right)\left(\frac{4\pi}{3}\right)^{1/3} GM^2 V^{-4/3} \qquad (5)$$

$$= \frac{1}{5}\left(\frac{4\pi}{3}\right)^{1/3} GM^2 V^{-4/3}$$

The equilibrium point is determined by equating Eqs. (3) and (5). Let's find it for a normal star.

- In Eq. (3), the mass is that of an electron m_e, and the number of electrons is $N_{\text{nuc}}/2$, one electron for every proton. This is the left-hand side of Eq. (6).
- In Eq. (5), the mass of the star is the number of nucleons N_{nuc} times the nucleon mass m_n. This is the right hand side of Eq. (6).

$$\frac{\pi^3 \hbar^2}{15 m_e}\left(\frac{3N_{nuc}}{2\pi}\right)^{5/3} V^{-5/3} = \frac{1}{5}\left(\frac{4\pi}{3}\right)^{1/3} G N_{nuc}^2 m_{nuc}^2 V^{-4/3} \qquad (6)$$

or

$$\left(\frac{3}{4\pi}\right)^{1/3} \frac{5}{G N_{nuc}^2 m_{nuc}^2} \frac{\pi^3 \hbar^2}{15 m_e}\left(\frac{3N_{nuc}}{2\pi}\right)^{5/3} = V^{1/3}$$

$$V^{1/3} = \left(\frac{3}{4\pi}\right)^{1/3} \frac{\pi^3 \hbar^2}{3 G N_{nuc}^{1/3} m_{nuc}^2 m_e}\left(\frac{3}{2\pi}\right)^{5/3}$$

Multiplying the LHS by $(3 / 4\pi)^{1/3}$ to obtain the radius of the star, we have

$$R_{\text{lim}} = \left(\frac{3}{4\pi}\right)^{1/3} V^{1/3} = \left(\frac{3^4 \pi^2}{2^9}\right)^{1/3} \frac{\hbar^2}{G N_{nuc}^{1/3} m_{nuc}^2 m_e} \qquad (7)$$

(Gas 9-34 incorrect: 128 -> 512)

As an example, a one solar mass star has 10^{57} nucleons, from which the limiting radius can be calculated to be

$R_{\text{lim}} = 7620$ km disagrees with Gas

In other words, the pressure arising from the electron degeneracy is sufficient to support a one solar mass star with a radius of 7600 km.

Conversion to neutrons

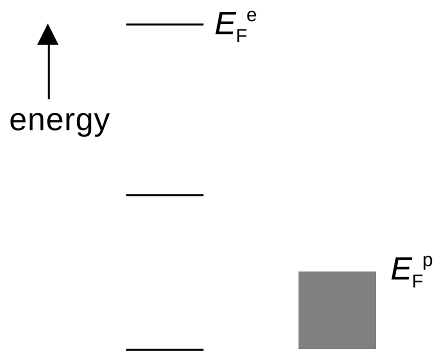
What happens when a star is sufficiently massive that its gravitational attraction overwhelms the electron pressure? Let's assume that the star is largely electrons, protons and neutrons. The Fermi *momentum* of all particle species with the same number density will be identical, given that

$$p_F = h\,(3N / 8\pi V)^{1/3} \qquad (8)$$

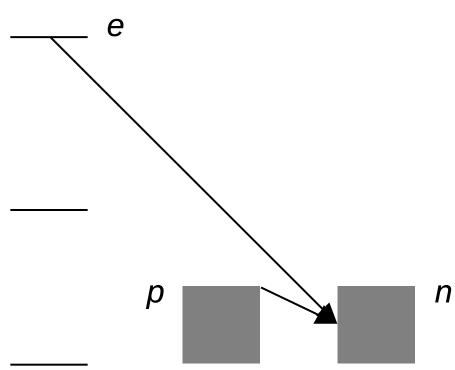
where N/V is the particle number density. However, the Fermi *energy* will vary according to particle mass, with *lighter* particles having *higher* energy:

$$E_F = p_F^2 / 2m \quad (9)$$

In the case of protons and electrons, the Fermi energy of the electrons will be 2000 times higher than that of the protons. Thus, the energy levels will look like (we can't do justice to a factor of 2000!)



When the difference between the Fermi energies exceeds the difference between the proton and neutron mass energies Δmc^2, an electron near E_F^e can be captured by a proton near E_F^p to create a neutron. That is



The amount of energy required for electron capture (with the release of a massless neutrino) is

$$\Delta mc^2 = m_n c^2 - m_p c^2 - m_e c^2$$

but

$$m_n - m_p - m_e = (1.6750 - 1.6726 - 0.00091) \times 10^{-27} = 1.5 \times 10^{-30} \text{ kg}$$

so

$$\Delta mc^2 = 1.5 \times 10^{-30} \bullet (3.0 \times 10^8)^2 = 1.34 \times 10^{-13} \text{ J.}$$

As $E_F^e \sim 2000 E_F^p$, let's just equate

$$E_F^e = \Delta mc^2 = 1.34 \times 10^{-13} \text{ J.}$$

The density corresponding to this threshold for capture can then be found from inverting Eq. (9) **which is a non-relativistic expression, although the**

electrons with E_F must be relativistic given the magnitude of Δmc^2:

$$E_F^e = \frac{1}{2m_e} h^2 \left(\frac{3N}{8\pi V}\right)^{2/3}$$

whence

$$\frac{(2m_e \Delta mc^2)^{3/2}}{h^3} \cdot \frac{8\pi}{3} = \frac{N}{V}$$

and

$$\frac{N}{V} = \frac{8\pi}{3} \frac{(2 \bullet 9.11 \times 10^{-31} \cdot 1.34 \times 10^{-13})^{3/2}}{(6.626 \times 10^{-34})^3} = 3.5 \times 10^{36} m^{-3} \qquad (10)$$

This number density (of protons or electrons) is much higher than the density of our Sun, which comes in at around 10^{30} m^{-3}. Thus, the conversion point of a one-solar-mass neutron star would have a radius about 1 / $(3 \times 10^6)^{1/3}$ = 0.007 that of the Sun, or

$R_{\text{threshold}} \sim 0.007 \bullet 7 \times 10^5 = 5000$ km.

Neutron stars

Once the star has converted to neutrons, it continues to collapse. At some point, the neutron degeneracy may balance the gravitational attraction, just as we calculated above for the electron gas. Changing the factors of two, and the electron mass, the limiting radius is

$$R_{\lim} = \left(\frac{3}{4\pi}\right)^{1/3} V^{1/3} = \left(\frac{3^4 \pi^2}{2^4}\right)^{1/3} \frac{\hbar^2}{G N_{nuc}^{1/3} m_{nuc}^3} \qquad (11)$$

From Eq. (11), a fully collapsed neutron star with a mass of **TWO** solar masses has:

- a nucleon number density in the region of 10^{39} m^{-3}
- a radius of 10.4 km.

This number density is still much less than the number density of nuclear matter at 1.4 x 10^{44} m^{-3}.