Lecture 23 - Photon gas

What's Important:
- radiative intensity and pressure
- stellar opacity

Text: Carroll and Ostlie, Secs. 9.1 and 9.2

The temperature required to drive the Sun's nuclear furnace at its observed rate of energy production is about 15 million K. Yet the surface temperature, like that of all stars, is far less - just 5000 K in the case of the Sun. How can the surface temperature be so much less than the interior temperature, and what are the properties of the Sun's interior? To answer these questions, we first return to the description of photon wavelengths presented in Lec. 11. There, we found that a gas of photons is described by:

$$n_\gamma(E)\,dE = \frac{8\pi}{(hc)^3}\frac{E^2 dE}{e^{E/kt}-1} \tag{23.1}$$

where $n_\gamma(E)$ is the photon number density, the number per unit volume per unit energy between E and $E + dE$, and

$$u_\gamma(E)dE = En_\gamma(E)\,dE = \frac{8\pi}{(hc)^3}\frac{E^3 dE}{e^{E/kt}-1} \tag{23.2}$$

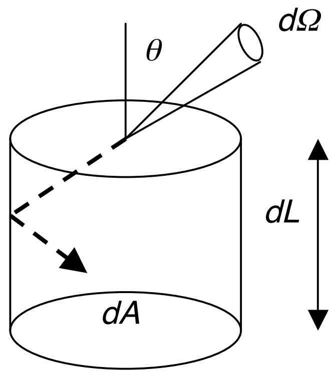
where $u_\gamma(E)$ is the energy per unit volume per unit energy between E and $E + dE$. In terms of wavelengths λ, the energy density is

$$u_\gamma(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{E/kt}-1}. \tag{23.3}$$

These expressions can be used to determine the flux and pressure of a photon gas.

Radiative intensity and flux

The first quantity that we derive from Eq. (23.3) is the intensity $I_\gamma(\lambda)$, which is the power per unit wavelength per unit area per unit solid angle, where the meaning of all the "pers" will become apparent momentarily (strictly speaking, this is the *specific intensity*). Let's consider the energy of the light contained in a cylinder of length dL and cross sectional area dA. We calculate this by making a mathematical cylinder with perfectly reflecting vertical walls, but open ends. The cylinder resides in an equilibrated photon gas and is initially empty at t = 0. Then, the ends are opened and light from the surrounding gas enters the cylinder from all directions. Note that it will take longer for photons entering the cylinder at shallow angles to traverse the cylinder than photons striking the end head-on.



Assuming the gas to be isotropic, then the amount of light entering is independent of the azimuthal angle ϕ. Further, light enters the top over polar angles θ between 0 and π/2, and through the bottom over angles between π/2 and π. Thus, the intensity must be integrated over

$$\int_0^{2\pi} d\phi \left(\int_0^{\pi/2} \ldots \sin\theta \, d\theta + \int_{\pi/2}^{\pi} \ldots \sin\theta \, d\theta \right) = 2\pi \int_0^{\pi} \ldots \sin\theta \, d\theta \,. \qquad (23.4)$$

Now, the θ integral is a little subtle, because both the time integral dt and the perpendicular area dA depend on θ.

Time dt: The time taken for the photons to fill the cylinder depends on their angle of entry. The vertical component of the photons velocity is $c \cos\theta$, so the time required for the vertical travel dL is

$$dt = dL \,/\, c \cos\theta.$$

Area dA: The intensity is per unit area facing the photons. But the area element dA in the diagram only faces photons with $\theta = 0$; at a general angle, the perpendicular area *decreases* to

$$\cos\theta \; dA$$

Taking these two effects into account, the energy filling the volume is

$$[\textit{energy per unit } \lambda] = 2\pi \int I_\gamma(\lambda) \; (dL \,/\, c \cos\theta) \; (\cos\theta \; dA) \; \sin\theta \; d\theta$$
$$= (2\pi/c) \int I_\gamma(\lambda) \; dA \; dL \; \sin\theta \; d\theta \qquad (23.5)$$

But $dA \, dL$ is the volume of the cylinder, so Eq. (23.5) can be rewritten as

$$[\textit{energy density per unit } \lambda] = (2\pi/c) \int I_\gamma(\lambda) \; \sin\theta \; d\theta. \qquad (23.6)$$

The left hand side is just $u_\gamma(\lambda)$, and the angle on the right hand side can be integrated away if $I_\gamma(\lambda)$ is isotropic to give

$$u_\gamma(\lambda) = (4\pi/c) \; I_\gamma(\lambda), \qquad \text{(isotropic)} \qquad (23.7)$$

where we have used $\int \sin\theta \; d\theta = 2$ over $0 \leq \theta \leq \pi$.

Lastly, the radiative flux $F_\gamma(\lambda)$ is the NET energy per unit wavelength passing through a unit area per unit time. This is written as

$$F_\gamma(\lambda) = \int I_\gamma(\lambda) \cos\theta \, d\Omega, \qquad (23.8)$$

where $\cos\theta$ arises from the orientation of the area element. If $I_\gamma(\lambda)$ is isotropic, as it would be for a photon gas in equilibrium, then the odd-integral in $\cos\theta$ vanishes; *i.e.*, $F_\gamma(\lambda) = 0$ and there is no net energy transport.

Radiative pressure

Because photons carry momentum courtesy of $E = pc$, a gas of photons exerts a pressure on its surroundings. When a single photon bounces off a surface at an angle θ with respect to the vertical, its change of momentum is

$$\Delta p = 2p\cos\theta = 2E\cos\theta / c, \qquad (23.9)$$

as the incident and reflective angles are the same. An equal and opposite change in momentum is experienced by the surface. To find the force on the surface, we need to sum over all wavelengths λ and integrate over all angles, just as we did in deriving Eq. (23.5). We use the intensity $I_\gamma(\lambda)$, which is the energy per unit wavelength,

per unit time; here, the time interval is Δt
per unit area; here, the area element is $\cos\theta \, \Delta A$
per unit solid angle; here, the angle is $d\Omega$.

Hence, the energy of the photons is

$$I_\gamma(\lambda) \, \Delta t \cos\theta \, \Delta A \, d\Omega \, d\lambda,$$

and Eq. (23.9) becomes

$$\begin{aligned} \Delta p \, d\lambda &= (2/c) \cos\theta \, I_\gamma(\lambda) \, \Delta t \cos\theta \, \Delta A \, d\Omega \, d\lambda \\ &= (2/c) \, I_\gamma(\lambda) \, \Delta t \, \Delta A \cos^2\theta \, d\Omega \, d\lambda. \qquad (23.10) \end{aligned}$$

We rearrange terms to read

$$[(\Delta p / \Delta t) / \Delta A] \, d\lambda = (2/c) \, I_\gamma(\lambda) \cos^2\theta \, d\Omega \, d\lambda.$$

The term $\Delta p / \Delta t$ is the force experienced by the surface, which becomes the pressure $P_\gamma(\lambda)$ when divided by ΔA. After integrating over all angles

$$P_\gamma(\lambda) = (2/c) \int I_\gamma(\lambda) \cos^2\theta \, d\Omega. \qquad (23.11)$$

Grabbing the factor of 2 from the front, we note that the angular integral immediately yields, for an isotropic $I_\gamma(\lambda)$,

$$\int_0^{2\pi} d\phi = 2\pi$$

$$2\int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta = \int_{-1}^{1} \cos^2\theta \, d\cos\theta = 2/3$$

Thus,

$$P_\gamma(\lambda) = (4\pi/ 3c)\, I_\gamma(\lambda). \qquad (23.12)$$

For black-body radiation, we know from Eq. (23.7) that

$$u_\gamma(\lambda) = (4\pi/c)\, I_\gamma(\lambda),$$

so

$$P_\gamma(\lambda) = u_\gamma(\lambda)/3$$

or

$$P_\gamma = U_\gamma/3, \qquad (23.13)$$

after integrating over all wavelengths. Recall that $U_\gamma = 7.565 \times 10^{-16}\ T(\mathrm{K})^4\ \mathrm{J/m^3}$.

Stellar opacity

Let's now return to the question of stellar temperatures. How do we reconcile the observed surface temperature of the Sun (5000 K) with the calculated interior temperature (15 million K) indicated by the Sun's nuclear reactions? Conceptually, the question is what did a photon last scatter from as it left the Sun: did it come straight from the interior like a neutrino, or did it scatter repeatedly, sampling cooler environments on its way?

We established some time back that the mean time between reaction t_{rxn} is

$$t_{\mathrm{rxn}} = 1 / (N\sigma v),$$

where we neglect thermal averages over the cross section σ *etc*. Reworking this:

$$vt_{\mathrm{rxn}} = 1 / N\sigma,$$

where the product vt_{rxn} is a length scale called the mean free path l (it's the speed times the mean time between reactions). That is

$$l = 1 / N\sigma. \qquad (23.14)$$

What is the mean free path for photons scattering from electrons and protons at the surface of the Sun? We perform a crude calculation:

> The density of electrons, if they were spread uniformly across the Sun, would be similar to the density of baryons ($N_{\mathrm{baryons}} = 10^{30}\ \mathrm{m^{-3}}$ in Lec. 22); thus, for electrons and protons, we expect
>
> $$N = 2N_{\mathrm{baryons}} = 2 \times 10^{30}\ \mathrm{m^{-3}}.$$
>
> Use $10^{-36}\ \mathrm{m^2}$ for an electromagnetic cross section.
> Thus,
>
> $$l = 1 / 2\times10^{30} \cdot 10^{-36} \sim 5 \times 10^5\ \mathrm{m} \sim 500\ \mathrm{km}.$$

Corrections:

- the measured density in the photosphere is much lower at 10^{23} m^{-3}.
- the cross section is much higher
- the net result is still $l \sim 10^5$ m.

The main point is that the mean free path is about 0.1% of the solar radius, and photons scatter repeatedly as they work their way out of the core.

Aside:

Rather than quote elementary cross sections for the components of a system, one often quotes something called the opacity κ, such that the mean free path is

$$l = 1 / \rho\kappa, \tag{23.15}$$

where ρ is the mass density of the material. To account for the change from number density to mass density, the opacity has units of [*area*]/[*mass*].

Because of scattering and reaction processes, the intensity of a particle beam decreases with distance s as

$$dI = - l^{-1} I \, ds,$$

so that the intensity decays like

$$I = I_o \exp(-l / s).$$

One can choose either representation of l to evaluate this last expression.