Lecture 28 - Degenerate matter

*What's Important:*

- particle in a box
- Fermi energy

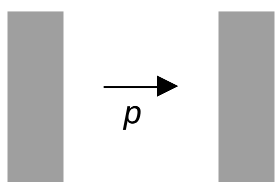
*Text*:

The protons and electrons (but not $^2$H and $^4$He) comprising a star are fermions, which collectively obey the Pauli exclusion principle that no two fermions can have a completely identical set of quantum numbers. We are already familiar with the consequences of this behavior through the Aufbau Principle that specifies how electrons are added to atomic orbitals. There, only two electrons, with opposing spin, can be accommodated by each 1s, 2s, $2p_x$, $2p_y$, $2p_z$...energy state.

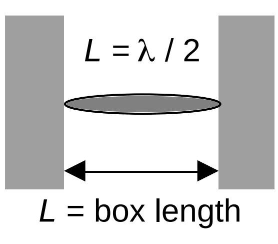
In this lecture, we address the situation in which localized interactions are unimportant, so that particle wavefunctions span an entire system, perhaps even as large as a star. We start by considering the motion of a particle between two reflective walls in one dimension, and then generalize the result to three dimensions. Application of our findings to white dwarfs and neutron stars are made in the next lecture.

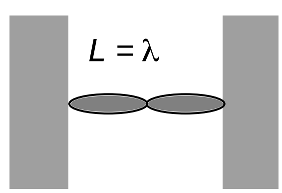
**Particle in a one-dimensional box**

This material should be familiar from an introductory course on quantum concepts like PHYS 285. Let's consider first the motion of a particle in one dimension between two perfectly reflective walls:




As there is no potential energy gradient in the region between the walls, the magnitude of the particle's energy and momentum must be constant. We quantize this system by demanding that the particle's de Broglie wave form a standing pattern:




The general relation between the wavelength $\lambda$ and the box size $L$ is

$$\lambda = 2L / n \qquad n = 1, 2, 3, ...$$

where $n$ is called the quantum number of the system. This means that the allowed values of the particle's momentum $p$ obey

$$p = h / \lambda = h / [2L / n] = nh / 2L. \qquad (28.1)$$

The corresponding kinetic energy is

$$E = p^2 / 2m = (nh / 2L)^2 / 2m$$

or

$$E_n = n^2 (h^2 / 8mL^2) \qquad n = 1, 2, 3, ... \qquad (28.2)$$

The little subscript $n$ is attached to $E$ to specify that it is the energy of the $n^{th}$ state.

## Particle in a three-dimensional cube

Next we place the particle in a three-dimensional cube. Because the directions are orthogonal, the solutions add independently in three dimensions. The corresponding kinetic energy is then

$$E = p^2 / 2m = (h^2 / 8mL^2)\ (n_x^2 + n_y^2 + n_z^2) \qquad n_i = 1, 2, 3, ... \qquad (28.3)$$

As we have set up this problem, each $n$ runs from 1 to ∞: clearly, $n = 0$ is not a wave ($\lambda = \infty$) and $n < 0$ makes no sense in our context.

## $T$=0 Fermi gas

So far, we have determined the states allowed for a single particle without interactions and without reference to its spin. Now, let's add fermions to these states. Physically, this applies to systems of electrons, protons, neutrons... individually or as multi-component systems. In three dimensions, the lowest lying states are

| states | number | $n_x$ | $n_y$ | $n_z$ | $n_x^2+n_y^2+n_z^2$ |
|---|---|---|---|---|---|
| ___ ___ ___ | 3 | 1 | 2 | 2 | 9 |
| | | 2 | 1 | 2 | 9 |
| | | 2 | 2 | 1 | 9 |
| ___ ___ ___ | 3 | 1 | 1 | 2 | 6 |
| | | 1 | 2 | 1 | 6 |
| | | 2 | 1 | 1 | 6 |
| ___ | 1 | 1 | 1 | 1 | 3 |

Placing fermions into these levels at $T = 0$ fills up the levels to some maximum value of the energy $E_{max}$

- all states with $E \leq E_{max}$ are occupied
- all states with $E \geq E_{max}$ are empty

The value of $E_{max}$ depends on the number of fermions $N$, and the value of their spin: each energy state can accommodate $2S + 1$ fermions of a given species, where $S$ is the spin of the particle. For protons, neutrons and electrons, $S = 1/2$, so that each energy state can hold two particles.

Our next task is to find $E_{max}$. Corresponding to $E_{max}$ there is a maximal momentum $p_{max}$ given by

$$E_{max} = p_{max}^2 / 2m. \qquad (28.4)$$

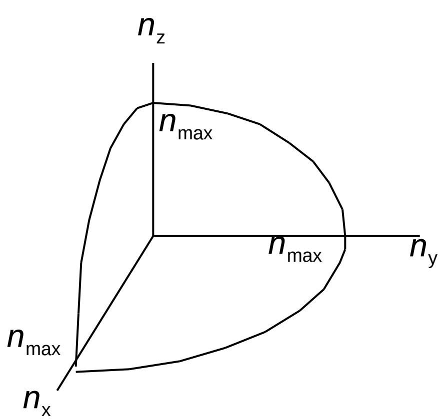
According to Eq. (28.1), the maximal value of $n$ is

$$n_{max} = 2Lp_{max} / h. \qquad (28.5)$$

The allowed states of the system correspond to any combination of $n_x$, $n_y$, or $n_z$ which satisfies

$$n_x^2 + n_y^2 + n_z^2 \leq n_{max}^2 \qquad (\text{so } E \leq E_{max}),$$

as a consequence of which no individual $n_x$, $n_y$, or $n_z$ is greater than $n_{max}$.




Each value of ($n_x$, $n_y$, $n_z$) inside the octant corresponds to one unique state. Thus, the number of states $N$ with $E \leq E_{max}$ is just the volume of the octant with positive $n_i$, or

$$N(E \leq E_{max}) = \frac{1}{8} \cdot \frac{4\pi}{3} n_{max}^3 \qquad (\text{times } 2S + 1) \qquad (28.6)$$

where the factor of 1/8 arises from the volume of the octant. We can work backwards from $n$'s to physical quantities as follows

$$\begin{aligned} N &= \frac{1}{8} \cdot \frac{4\pi}{3} n_{max}^3 = \frac{1}{8} \cdot \frac{4\pi}{3} \left( \frac{2Lp_{max}}{h} \right)^3 \\ &= \left( \frac{2^3 L^3}{8} \right) \left( \frac{4\pi}{3} p_{max}^3 \right) \frac{1}{h^3} \end{aligned} \qquad (\text{times } 2S + 1) \qquad (28.7)$$

The first factor on the RHS is just the volume of the box, $L^3$. It turns out that even if the shape of the boundary is a sphere instead of a cube, the same expression still applies, with $L^3$ replaced by the volume $V$ enclosed by the sphere. Thus, we can rewrite Eq. (28.7) as

$$\frac{N}{V} = \left(\frac{4\pi}{3} p_{\max}^3\right) \frac{1}{h^3} \qquad \text{(times } 2S + 1\text{)} \tag{28.8}$$

This expression can be inverted to give $p_{\max}$, which we now call the Fermi momentum $p_F$. And, of course, there is a Fermi energy

$$E_F = p_F^2 / 2m \tag{28.9}$$

corresponding to the maximum kinetic energy of the occupied states at $n_{\max}$.

### *S* = 1/2 gas

Our interest is primarily in electrons and protons, so let's explicitly write out the quantities of the $T = 0$ Fermi gas. Eq. (28.8) becomes

$$\frac{N}{V} = \frac{8\pi}{3} \left(\frac{p_{\max}}{h}\right)^3 \tag{28.10}$$

which yields

$$p_F = h \left(\frac{3N}{8\pi V}\right)^{1/3} . \tag{28.11}$$