Lecture 35 - Universal helium abundance

What's important:
- proton/neutron ratio
- helium production in the early universe

Text: Peebles, Chap. VIII

Universal helium abundance

We have described two characteristics of the Universe as a whole:
- expansion according to Hubble's Law
- 2.7 K microwave radiation background.

There is another characteristic as well, which is the chemical composition of the Universe as a whole:

Sun	78% H, 22% He
massive young stars	72% H, 28% He
ionized interstellar gas [emitting]	71% H, 29% He
dilute interstellar gas [absorbing]	74% H, 26% He

If we take into account the fact that helium has been produced in stars since the Big Bang, we find an average abundance of 24±1% He, the rest is hydrogen and a small amount of heavier elements.

In this lecture, we calculate the helium abundance arising from the Big Bang, and show that it is consistent with these measured values. The calculation is somewhat detailed, and involves several steps:

n / p ratio
- find the neutron to proton ratio at equilibrium as a function of temperature
- determine up to what time equilibrium holds
- follow the ratio when equilibrium is no longer supported

deuterium production
- find the deuterium abundance in equilibrium as a function of temperature and the n/p ratio
- determine the reaction pathways from deuterium.

Neutron to proton ratio

Equilibrium abundances

In our discussion of the distribution of particle energies, we described how the probability of a particle being in a given state of energy E is given by the Boltzmann factor $\exp(-\beta E)$, where $\beta = 1 / k_B T$. Previously, we took E to be the kinetic energy of a particle ($mv^2/2$ or pc), but it can be generalized to include the mass energy mc^2. Applying this idea to the probability of a nucleon being in a proton or neutron state, we would write

$$[n] \sim \exp(-m_n c^2 / k_B T)$$
$$[p] \sim \exp(-m_p c^2 / k_B T)$$

where [...] represents a number density. The prefactors in the proportionality are essentially the same, and can be eliminated by taking the ratio, leaving

$$\frac{[n]}{[p]} = \exp\left(-\frac{\Delta mc^2}{k_B T}\right) \qquad \text{(equilibrium)} \qquad (35.1)$$

where the proton-neutron mass difference Δm is

$$\Delta m = m_n - m_p$$

and

$$\Delta mc^2 = 1.293 \text{ MeV}.$$

Substituting numerical values,

$$\frac{\Delta mc^2}{k_B} = \frac{1.293 \times 10^6 \bullet 1.6 \times 10^{-19}}{1.38 \times 10^{-23}} = 1.5 \times 10^{10} (\text{K})$$

From the exponential in Eq. (13.1), one can see that the neutron abundance begins to drop as the temperature falls below 10^{10} K. For example, at $T = 10^{11}$ K

$$\frac{[n]}{[p]} = \exp\left(-\frac{1.5 \times 10^{10}}{10^{11}}\right) = 0.86.$$

A sample of the equilibrium values as a function of temperature (and time through $\Delta t = \Delta(2/n) H^{-1}$) are given below (from Peebles):

$T(K)$	t (s)	(n / n+p) % equilibrium only	(n / n+p) % with reactions
10^{12}	0.0001	49.6	49.6
10^{11}	0.011	46.3	46.3
$2\text{x}10^{10}$	0.27	32.0	33.0
10^{10}	1.1	18.2	23.8
10^9	182	$3\text{x}10^{-5}$ %	13.0

Notice how the neutron abundance has fallen off the face of the earth by 10^9 K.

Maintenance of equilibrium

Protons and neutrons are maintained in equilibrium through reactions such as

$$e^+ + n \leftrightarrow p + \text{anti-}\nu_e$$
$$\nu_e + n \leftrightarrow p + e^-$$

How rapidly these reactions occur depends on the number density of particles and their reaction cross section.

Now, at some point, the reactions will be so infrequent that the particles go out of equilibrium (as these reactions are today). This occurs when the reaction time is longer than the characteristic expansion time of the system:

reactions $t_{rxn} = 1 / (N\sigma v)$

expansion from $V = HR \rightarrow \quad \dfrac{dR(t)}{dt} = HR(t)$

which has the solution $R(t) = R_o \exp(-Ht)$

so the characteristic expansion time is $t_{exp} = H^{-1}$.

equilibrium requires $t_{rxn} < t_{expansion}$

Consider the neutrino-induced reaction

$\nu_e + n \leftrightarrow p + e^- \qquad \sigma \sim 4 \times 10^{-47}$ m^2.

The number density of neutrinos in thermal equilibrium at temperature T is given by

$N_\nu = (3/4)\ N_{boson} = 22.6\ (k_B T / hc)^3 = 22.6 \cdot (69.4 \cdot T)^3 \quad$ (T in K)

For example, at $T = 10^{10}$ K,

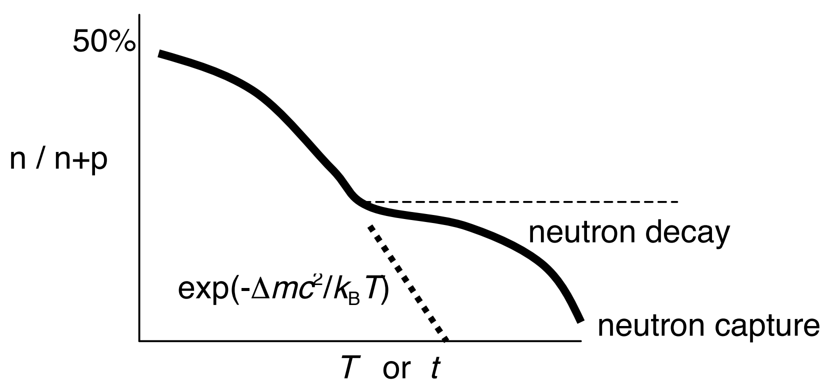
$N_\nu = 22.6\ (69.4 \times 10^{10})^3 = 7.5 \times 10^{36}$ m^{-3}.

so that

$t_{av} = (\sigma N_\nu c)^{-1} = (4\mathrm{x}10^{-47} \cdot 7.5\mathrm{x}10^{36} \cdot 3.0\mathrm{x}10^{8})^{-1} = 11$ seconds.

From the previous page, the characteristic expansion time is ~ 1 s at $T = 10^{10}$ K. Thus, the reactions will not be able to sustain an equilibrium below 10^{10} K.

The complete data display the behaviour (from Peebles)



The "dip-bump" structure arises when the p $\leftrightarrow$ n interconversion reactions go out of equilibrium and the neutron abundance no longer follows the equilibrium distribution. This occurs at $T \sim 10^{10}$ K, when n / n+p = 24%. Amusingly, the equilibrium reactions would *lower* the neutron abundance if the reactions persisted. Were the neutrons stable, their abundance would then be fixed at 24%. However, neutrons decay with a lifetime of the order ten minutes, so they still disappear, but at a slower rate.

Deuterium production

The formation of nuclei in the early universe goes through the sequential capture of protons and neutrons starting with

$$p + n \rightarrow d + \gamma \qquad B.E. = 2.225 \text{ MeV}$$

Note: p + p → no bound state n + n → no bound state.

The equilibrium abundance of deuterons (d or ^{2}H) at temperature T can be obtained from statistical mechanics, and has a form similar to the proton/neutron ratio:

$$\left(\frac{x_p x_n}{x_d}\right)_{\text{eq}} = \frac{4}{3}\frac{(2\pi k_B T)^{3/2}}{h^3 N}\left(\frac{m_p m_n}{m_d}\right)^{3/2} \exp(-B.E./k_B T) \qquad (35.2)$$

where

N = total baryon number density, $N_p + N_n + 2N_d + ...$

x_i = fractional abundances by number,

$x_p = N_p/N \qquad x_n = N_n/N \qquad x_d = N_d/N$

$x_p + x_n + 2x_d + ... = 1.$

To solve the equilibrium equation, we need an expression for N, the total baryon number density. Now, we know that in a matter-dominated universe, the number density is proportional to T^3, so we can use today's baryon number density to determine that in the early universe by

$N = (T / 3)^3 N_{\text{today}}$.

As we'll show in a moment, the equilibrium density shifts to deuterons over a very narrow temperature range, so that it is no great error to use the critical baryon number density (n_{crit} = 5 nucleons/m^3) instead of N_{today} (we'll return to this below). Further, we approximate $m_d \sim 2m_n$ or $2m_p$, so

$$\left(\frac{x_p x_n}{x_d}\right)_{\text{eq}} = \frac{4}{3}\frac{(2\pi k_B T)^{3/2}}{h^3}\left(\frac{2.7^3}{n_{crit}T^3}\right)\left(\frac{m_p}{2}\right)^{3/2} \exp(-B.E./k_B T) \qquad (35.3)$$

Explicitly substituting for all the numerical constants, we are left with

$(x_p x_n/x_d)_{\text{eq}} = 3.4 \times 10^{26}\ T^{-3/2} \exp(-B.E.\ /k_B T)$ (dimensionless)

Now, because nucleosynthesis occurs at such elevated temperatures, we use the simplifying notation

$T_6 = T / 10^6 \qquad T_9 = T / 10^9 \qquad T_{10} = T / 10^{10}$ *etc*

so that

$(x_p x_n/x_d)_{\text{eq}} = 1.09 \times 10^{13}\ T_9^{\ -3/2} \exp(-25.8\ /T_9)$.

The exponentials and factors of 10^{13} tell us that the abundances may change rapidly with temperature, so we take logarithms to write

$\ln(x_p x_n/x_d)_{\text{eq}} = 30.0 - 3/2 \cdot \ln(T_9) - 25.8\ /T_9$.

Here are a few sample values according to this equation

T_9	$\ln(x_p x_n / x_d)_{eq}$	
10	24.0	
5	22.4	
1	4.2	
0.8	-1.92	(equilibrium shifts to d)
0.1	-225	

So, the equilibrium shifts VERY SHARPLY at T just under 10^9 K, and the available neutrons are scooped up to make deuterons.

From earlier in the lecture, the neutron abundance at this temperature is 13%. As we confirm in a moment, the deuterons are rapidly converted to ^{4}He, whose abundance follows from

$$\%\text{He by weight} = 2N_n / (N_n+N_p) = 2 \bullet 13\% = 26\%,$$

where the factor of 2 arises because there are two nucleon masses (p+n) for every neutron in ^{4}He. Thus, we predict a universal abundance of 26% by weight ^{4}He arising from the early universe, very close to the observed 24±1%. The time when $T = 10^9$ K is given in the previous table at 180 seconds, or three minutes.

Why don't we just get d? Reactions like

$$d + d \rightarrow p + {}^3\text{He}$$
$$d + d \rightarrow n + {}^3\text{H}$$

are fast because they are strong or electromagnetic, and lead to product nuclei that are more deeply bound (per nucleon) than d. Same with ^{4}He, which is much more deeply bound (6 MeV per bound nucleon, compared to 1.1 MeV for deuterium). In other words, the available deuterium is rapidly converted to ^{4}He through the further addition of nucleons.

Why stop at helium, why not keep on making heavier elements? In nature, there are NO stable nuclei with $A = 5$: ^{4}He is a bottleneck, which can only be overcome through combinations of reactants facing a large coulomb barrier. This WILL happen in the early universe but the rate is reduced, so the production of nuclei like ^{6}Li is low.

Constraints on baryon density

The baryon density of the universe appears in both (13.2) and (13.3), the latter by extrapolating today's baryon density back to $T = 10^9$ K with the usual R^{-3} scaling. Having established that the overall picture of primordial nucleus production is very close to correct, one could invert the logic and use today's measured values to find how close ρ_{today} is to ρ_{crit}. That is

- predict ^{4}He, ^{6}Li ... in terms of $\rho_{early\ universe}$.
- scale $\rho_{early\ universe}$ to obtain $\rho_{today} = \rho_{early\ universe} (T_{today} / 10^9)^3$.

The results of this approach do change from year to year as measurements improve, but some recent values are:

from ^{2}H $\rho_{today} / \rho_{crit}$ = 1.3 - 2.8 %
from ^{4}He $\rho_{today} / \rho_{crit}$ = 1.0 - 3.3 %
from ^{6}Li $\rho_{today} / \rho_{crit}$ = 1.0 - 8.0 %.

ρ_{today} is baryon density

These figures exclude the uncertainties of the experimental measurements, which will broaden out the ranges even further.