**PHYS 4xx net1 - Soft networks and their deformation**

Examples of two-dimensional networks in the cell:

(i) *membrane-associated cytoskeleton of the human erythrocyte*

(from Byers and Branton, 1985)

- a network of spectrin tetramers attached to cytoplasmic side of plasma membrane about midway along their length by the protein ankyrin
- each spectrin tetramer has a 200 nm contour length, but $r_{ee} \approx 70$ nm *in vivo*

(ii) *auditory outer hair cell*

(from Tolomeo, Steele, and Holley, 1996; bar is 200 nm long)

- lateral cortex lies on cytoplasmic side of the plasma membrane
- principal filaments are about 5-7 nm thick, spaced $\sim 60$ nm apart and form hoops around the axis of the cylinder; probably actin
- cross-linkers at intervals of $\sim 30$ nm (with a range of 10-50 nm) by thinner filaments just 2-3 nm thick, denoted by "cl" above; probably spectrin
(iii) **nuclear lamina**

(electron micrograph of a region about 2.5 µm in length from the nuclear lamina in a *Xenopus* oocyte; from Aebi *et al.*, 1986)

- four-fold connectivity; network is 10-20 nm thick
- filaments of the protein laminin are 10.5 ± 1.5 nm in diameter and are typically separated by about 50 nm

(iv) **peptidoglycan**

- bacterial cell wall is made from peptidoglycan network
- stiff chains of sugar molecules (cylinders) are cross-linked both in- and out-of-plane by flexible chains of amino acids (out-of-plane links are indicated by arrows)

*Strain tensor*

To describe the deformation of an object, we introduce a set of vectors \( \mathbf{u} \), such that a point moves from its original position \( \mathbf{x} \) to a new position \( \mathbf{x} + \mathbf{u} \).
• \( \mathbf{u} \) varies in magnitude and direction across the object (\( \mathbf{u} = \) constant corresponds to translation)
• \( \mathbf{u} \) may have non-zero partial derivatives in any Cartesian direction
• the strain tensor \( u_{ij} \), is related to the rate of change of \( \mathbf{u} \) with position \( \mathbf{x} \) by
  \[
  u_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_k (\frac{\partial u_k}{\partial x_i})(\frac{\partial u_k}{\partial x_j}) \right], \quad (i,j,k \text{ are Cartesian indices})
  \]  
  (1)
• \( u_{ij} \) has \( 2^2 = 4 \) components in 2D and \( 3^2 = 9 \) components in 3D
• \( u_{ij} \) is unitless and is symmetric in indices \( i \) and \( j \).

For small deformations, the last term in Eq. (1) may be neglected:
  \[
  u_{ij} \approx \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]. \quad \text{(small deformations)}
  \]  
  (2)

Example: Uniform scaling as in the diagram above:
  all \( x \) go to \( 1.1x \) \( \Rightarrow \ u_{xx} = (1.1 - 1)x / x = 0.1 \) everywhere
  all \( y \) go to \( 0.9y \) \( \Rightarrow \ u_{yy} = (0.9 - 1)y / y = -0.1 \) everywhere
  change in \( x \) does not depend on \( y \) \( \Rightarrow \ u_{xy} = u_{yx} = 0. \)

Stress tensor
• stress tensor \( \sigma_{ij} = \) force per unit area, taking into account the direction of force \( \mathbf{F} \)
  \[
  F_i = \sum_j \sigma_{ij} a_j.
  \]  
  (3)
• component in the \( i \)-direction of the net force, \( F_i \), is given by
• surface area vector \( \mathbf{a} \) is perpendicular to the surface
• \( \sigma_{ij} \) has units of energy density and is symmetric in indices \( i \) and \( j \).
• generally, the diagonal elements of the stress tensor correspond to compression and the off-diagonal elements to shear.
Example: An object under hydrostatic pressure $P$.

- $\mathbf{F}$ on a surface is in the opposite direction to the vector $\mathbf{a}$ describing the surface
  
  $$F_i = -P a_i = -P \sum_j \delta_{ij} a_j.$$  
  \hfill{(4)}

- comparing Eqs. (3) and (4):
  
  $$\sigma_{ij} = -P \delta_{ij}.$$  
  \hfill{(5)}

Elastic moduli

- for ideal springs in one dimension, the restoring force $f$ is proportional to the displacement from equilibrium $x$: $f = -k_{sp} x$, where $k_{sp}$ = spring constant.

- corresponding relationship for continuous materials reads $[\text{stress}] \propto [\text{strain}]$, or
  
  $$\sigma_{ij} = \sum_{k,l} C_{ijkl} u_{kl}.$$  
  \hfill{(6)}

- material-specific constants $C_{ijkl}$ are the elastic stiffness constants or elastic moduli; units of energy per unit volume for 3D materials, or energy per unit area for 2D

- the elastic moduli of two- or three-dimensional materials form a tensor, as opposed to the single $k_{sp}$ of an isolated spring

Just as the potential energy of a Hooke's law spring is quadratic in the square of the displacement, the change in the free energy density $\Delta F$ of a continuous object under deformation is quadratic in the strain tensor $u_{ij}$:

$$\Delta F = \frac{1}{2} \sum_{i,j,k,l} C_{ijkl} u_{ij} u_{kl}.$$  
  \hfill{(7)}

- symmetry considerations greatly reduce the number of independent components of $C_{ijkl}$ from $3^4 = 81$ terms in three dimensions, or $2^4 = 16$ in two dimensions.