PHYS 4xx Net 3 - Properties of two-dimensional networks

Six-fold spring networks under stress

When the network is placed under a two-dimensional tension $\tau$, $s$ changes from its unstressed value $s_o$ to a new value $s_\tau$. We evaluate this behavior for a spring network.

- calculate $s_\tau$ by minimizing the enthalpy $H$
  \[ H = E - \tau A \]  
  (1)

- at 3 springs per vertex, the energy per vertex is $(3/2)k_{sp}(s - s_o)^2$

- the area per vertex is
  \[ A_v = \sqrt{3} \frac{s^2}{2} \]  
  (2)

- hence, the enthalpy per vertex $H_v$ is
  \[ H_v = (3/2)k_{sp}(s - s_o)^2 - \sqrt{3} \tau s^2/2 \]  
  (3)

- take the derivative of $H_v$ to find $s_\tau$:
  \[ 0 = \frac{\partial H_v}{\partial s} = \frac{\partial}{\partial s} \left[ (3/2)k_{sp}(s - s_o)^2 - \sqrt{3} \tau s^2/2 \right] \]
  \[ = (3/2)2k_{sp}(s - s_o) - \sqrt{3} \cdot 2\tau s/2 \]
  \[ = \sqrt{3} \cdot [\sqrt{3} k_{sp}(s - s_o) - \tau s] \]

  then
  \[ [\sqrt{3} k_{sp} - \tau ]s = \sqrt{3} k_{sp}s_o \]
  \[ s = \sqrt{3} k_{sp}s_o / [\sqrt{3} k_{sp} - \tau ] \]

  or
  \[ s_\tau = s_o / (1 - \tau / [\sqrt{3} k_{sp}]) \]  
  (six-fold symmetry)  
  (4)

- (4) shows that the network expands without bound as the tension approaches a critical value $\tau_{exp}$
  \[ \tau_{exp} = \sqrt{3} k_{sp} \]  
  (six-fold symmetry),  
  (5)
expansion at large tension because both the energy of the springs and the pressure term $\tau A_v$ scale like $s^2$ at large extensions; of course, physical networks could reach a maximum bond length

the minimum value of the enthalpy per vertex is

$$H_{v,\text{min}} = (3/2) k_{sp}(s_i - s_o)^2 - \sqrt{3} \, \tau s_i^2/2$$

$$= (3/2) k_{sp}s_o^2 \left[ 1/(1 - \tau / \sqrt{3} \, k_{sp}) - 1 \right] - \sqrt{3} \, \tau s_o^2 / 2(1 - \tau / \sqrt{3} \, k_{sp})^2$$

$$= s_o^2(k_{sp} \left[ \tau / k_{sp} \right]^2 - \sqrt{3} \, \tau) / 2(1 - \tau / \sqrt{3} \, k_{sp})^2$$

$$= \tau s_o^2 \left[ (\tau / k_{sp} - \sqrt{3}) / 2(1 - \tau / \sqrt{3} \, k_{sp}) \right]^2$$

$$= \sqrt{3} \, \tau s_o^2 \left[ (\tau / \sqrt{3} \, k_{sp}) - 1 \right] / 2(1 - \tau / \sqrt{3} \, k_{sp})^2$$

or

$$H_{v,\text{min}} = - (\sqrt{3} / 2) \tau s_o^2 / (1 - \tau / \sqrt{3} \, k_{sp}) \quad \text{(equilateral plaquettes)} \quad (6)$$

Eq. (6) is not the global minimum of $H$ under compression ($\tau < 0$) if shapes other than equilateral plaquettes are considered

the smallest $\tau A$ contribution at $\tau < 0$ is given by plaquettes with zero area

isosceles triangles with two short sides of length $s_I$ and a long side of length $2s_I$

have the lowest spring energy at zero area (3 springs per vertex):

$$H_{v,\text{iso}} = (k_{sp}/2) \cdot [(2s_I - s_o)^2 + 2(s_I - s_o)^2]$$

minimum value of $H_{v,\text{iso}}$ as a function of $s_I$ is at $\partial H_{v,\text{iso}} / \partial s_I = 0$, or

$$0 = \partial / \partial s_I [(2s_I - s_o)^2 + 2(s_I - s_o)^2]$$

$$= 2 \cdot 2 \cdot (2s_I - s_o) + 2 \cdot 2 \cdot (s_I - s_o)$$

$$= 8s_I - 4s_o + 4s_I - 4s_o = 12s_I - 8s_o$$

Hence

$$s_I = 2s_o / 3$$
Thus, the enthalpy per vertex of the equilateral network rises with pressure ($\tau < 0$) according to Eq. (6) until it exceeds Eq. (7) at a collapse tension $\tau_{\text{coll}}$

$$H_{V,\text{iso}} = \frac{k_{sp}}{2} \cdot \left[ \frac{(4s_o/3 - s_o)^2 + 2(2s_o/3 - s_o)^2}{(1/3)^2 + 2(1/3)^2} \right] = k_{sp}s_o^2/6$$

(7)

Thus, $H_{V,\text{min}} = k_{sp}s_o^2/6$

that is

- $(\sqrt{3}/2)\tau \simeq (1 - \tau / [\sqrt{3} k_{sp}] ) = k_{sp}s_o^2/6$
- $(\sqrt{3}/2)\tau = (k_{sp}/6)(1 - \tau / [\sqrt{3} k_{sp}])$
- $\tau = (k_{sp}/3\sqrt{3})(1 - \tau / [\sqrt{3} k_{sp}])$
- $\tau = k_{sp}/3\sqrt{3} - \tau /9$
- $(8/9)\tau = k_{sp}/3\sqrt{3}$

or

$$\tau_{\text{coll}} = - (\sqrt{3}/8) k_{sp} \quad \text{(six-fold symmetry)}, \quad \text{(8)}$$

or, equivalently, at an equilaterial spring length $s_t/s_o = 8/9$.

**Elastic moduli for networks under stress**

Elastic moduli can be obtained by the same method as in Net 2 for springs at zero temperature and no stress. For variety, we take a different approach for the compression modulus, going back to its definition

$$K_A^{-1} = A^{-1} \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau} \left[ \frac{1}{s_t^2} \right] = (2/s_t) \frac{\partial}{\partial \tau} \left[ s_o / (1 - \tau / [\sqrt{3} k_{sp}] ) \right]$$

$$= (2/s_o) \frac{\partial}{\partial \sigma} \left[ s_o / (1 - \tau / [\sqrt{3} k_{sp}] ) \right]$$

$$= (2s_o/s_t) \frac{\partial}{\partial \tau} \left[ s_o / (1 - \tau / [\sqrt{3} k_{sp}] ) \right]$$

$$= (2/\sqrt{3} k_{sp})(1 - \tau / [\sqrt{3} k_{sp}])^{-1}$$

thus

$$K_A = (\sqrt{3} k_{sp}/2)(1 - \tau / [\sqrt{3} k_{sp}]) \quad \text{(this returns our previous expression when } \tau = 0) \quad \text{(9)}$$

The shear modulus can be obtained following the same route as before.

© 2010 by David Boal, Simon Fraser University. All rights reserved; further resale or copying is strictly prohibited.
\[ \mu = (\sqrt{3} \ k_{sp} / 4) \cdot (1 + \sqrt{3} \ \tau / k_{sp}), \]  
(10)

- the Poisson ratio is a measure of how a material contracts in a transverse direction when stretched longitudinally; in two dimensions (stress along the \( x \)-axis)
\[ \alpha_p = - \frac{u_{yy}}{u_{xx}}, \]  
(11)

- (the negative sign gives \( \alpha_p > 0 \) for conventional materials)

- we can show, from the 3D result in Appendix D, that in 2D:
\[ \alpha_p = (K_A - \mu) / (K_A + \mu). \]  
(12)

- a triangular network at zero temperature and stress has \( K_A/\mu = 2 \rightarrow \alpha_p = 1/3 \)

- (9) and (10) give
\[ \alpha_p = \frac{[1 - 5\tau / (\sqrt{3} \ k_{sp})]}{[3 + \tau / (\sqrt{3} \ k_{sp})]} \]  
(six-fold symmetry).  
(13)

- thus, \( \alpha_p \) becomes negative over the range \( \sqrt{3} / 5 < \tau / k_{sp} < \sqrt{3} \)