Vector Addition of Angular Momentum

We have already seen that spin (S) and orbital (L) angular momentum combine together to form total angular momentum (J). Each of the operators has the same set of commutation rules internally. What happens when we combine eigenstates of these operators together? There are several possibilities, of which examples are:

1 particle: $\vec{S} + \vec{L} \to \vec{J}$.

2 particles $\vec{S}_1 + \vec{S}_2 \to \vec{S}_{TOT}$

or $\vec{J}_1 + \vec{J}_2 \to \vec{J}_{TOT}$.

Let's just use the notation $\hat{J}$ and $|j\, m\rangle$ to represent any of the S, L, J combinations. We take two eigenstates $|j_1\, m_1\rangle^{(1)}$ and $|j_2\, m_2\rangle^{(2)}$ and form a combined state via:

$$|j_1\, j_2\, m_1\, m_2\rangle = |j_1\, m_1\rangle^{(1)}\, |j_2\, m_2\rangle^{(2)}.$$

The operators for which these states (rhs) have eigenvalues are

$$\vec{J}^{(1)} \cdot \vec{J}^{(1)} \qquad \vec{J}^{(2)} \cdot \vec{J}^{(2)} \qquad J_z^{(1)} \qquad J_z^{(2)}$$

Now, we use $J^{(1)}$ and $J^{(2)}$ to form states of total $\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$.
(or, more properly speaking $\vec{J} = \vec{J}^{(1)} \otimes \vec{1} + \vec{1} \otimes \vec{J}^{(2)}$)

operate on $|j_1\, m_1\rangle^{(1)}$

operate on $|j_2\, m_2\rangle^{(2)}$

It may be that $\vec{J}_z$ satisfies an eigenvalue equation while $\vec{J}_z^{(1)}$ and $\vec{J}_z^{(2)}$ do not. Now, let's denote the eigenvectors of $\hat{J}$, $\hat{J}_z$ by $|\alpha, J, M\rangle$

α is introduced in case we need other quantum numbers to describe the state.

$$\hat{\vec{J}} \cdot \hat{\vec{J}}\, |\alpha, J, M\rangle = J(J+1)\hbar^2\, |\alpha, J, M\rangle$$

$$\hat{J}_z\, |\alpha, J, M\rangle = M\hbar\, |\alpha, J, M\rangle .$$

As far as commutation relations are concerned,

$$\vec{J} \cdot \vec{J}\ ,\quad \vec{J}^{(1)} \cdot \vec{J}^{(1)}\ ,\quad \vec{J}^{(2)} \cdot \vec{J}^{(2)} \quad \text{and} \quad \hat{J}_z$$

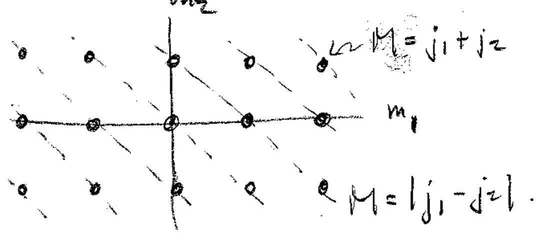
all commute and therefore have a common set of eigenvectors. Since $J^{(1)} \cdot J^{(1)}$ has $(2j_1+1)$ eigenvectors for a given j_1, and $J^{(2)} \cdot J^{(2)}$ has $(2j_2+1)$ " " " " j_2, then $\vec{J} \cdot \vec{J}$ must have $(2j_1+1)(2j_2+1)$ eigenvectors for fixed j_1, j_2. We note that

$$\begin{aligned}(2j_1+1)(2j_2+1) &= 4j_1 j_2 + 2j_1 + 2j_2 + 1 \\ &= 4j_1 j_2 + 2(j_1+j_2) + 1 \\ &> 2(j_1+j_2)+1\end{aligned}$$

In other words, there are more states in $(2j_1+1)(2j_2+1)$ than what is accounted for in $2(j_1+j_2)+1$. What do the sets of $|J, M\rangle$ states look like in terms of $|j_1 m_1\rangle$ $|j_2 m_2\rangle$.

First, we must determine how many sets of state J there are. Let's proceed by example:

Suppose that we have $j_1 = 2$, $j_2 = 1$. Then the combinations of m_1 and m_2 are



States of total M = 15.

$M = m_1 + m_2$	# states with M
3	1
2	2
1	3
0	3
⋮	

It turns out that there is a unique set of J's for each $|j_1 m_1\rangle|j_2 m_2\rangle$ combination. Hence we can label the eigenvectors by $|j_1\, j_2\, J\, M\rangle$. The coefficients which allow us to express $|j_1\, j_2\, J\, M\rangle$ in terms of the states $|j_1\, m_1\rangle|j_2\, m_2\rangle$ are called Clebsch - Gordan coefficients:

$$|j_1\, j_2\, J\, M\rangle = \sum_{m_1, m_2} |j_1\, j_2\, m_1\, m_2\rangle\langle j_1\, j_2\, m_1\, m_2 | j_1\, j_2\, J\, M\rangle$$

note that there is no sum over $j_1\, j_2$, since they commute with J^2 (and are fixed in any event).

These are the Clebsch - Gordan coefficients and will be represented as

$$\langle j_1\, j_2\, m_1\, m_2 | j_1\, j_2\, J\, M\rangle = (j_1\, j_2\, m_1\, m_2 | J\, M)$$

There are a number of phase conventions used for the C-G coefficients. The one adopted here is $(j_1\, j_2\, j_1\; J - j_1 | J\, J)$ is real and positive.

With this convention, it can be shown that CG's are real, and hence

$$(j_1 j_2 m_1 m_2 | J M) = (J M | j_1 j_2 m_1 m_2).$$

so that we can express either basis set in terms of the other via

$$|j_1 j_2 J M\rangle = \sum_{m_1, m_2} |j_1 j_2 m_1 m_2\rangle (j_1 j_2 m_1 m_2 | J M) \quad \text{as before.}$$

$$|j_1 j_2 m_1 m_2\rangle = \sum_{J M} |j_1 j_2 J M\rangle (j_1 j_2 m_1 m_2 | J M)$$

The C-G's satisfy the orthonormality constraint that

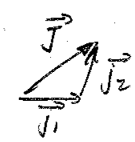
$$\sum_{m_1, m_2} (j_1 j_2 m_1 m_2 | J M)(j_1 j_2 m_1 m_2 | J' M') = \delta_{JJ'} \delta_{MM'}$$

$$\sum_{J M} (j_1 j_2 m_1 m_2 | J M)(j_1 j_2 m_1' m_2' | J M) = \delta_{m_1 m_1'} \delta_{m_2 m_2'}.$$

For the C-G's to be non-vanishing, at least the following situations must hold

i) $m_1 + m_2 = M$

ii) $|j_1 - j_2| \leq J \leq j_1 + j_2$ ← same as triangle condition



iii) $j_1 + j_2 + J$ = an integer ←

This comes from the symmetry of C-G under rotations by 2π. It just says that there must be an even number of half-odd-integer j's.

<u>Determining C-G's</u>

The most commonly used method for obtaining the j's is via raising and lowering operators ⊕ orthogonality:

$$J_\pm = J_x \pm i J_y = (j_{1x} + j_{2x}) \pm i (j_{1y} + j_{2y}) = j_{1+} + j_{2+}$$

Start with the stretched configuration $J = j_1 + j_2$ $M = j_1 + j_2$ and apply $\hat{J}_-$ successively. For example,

$j_1 = j_2 = \frac{1}{2}$: $$|\tfrac{1}{2}\ \tfrac{1}{2}\ 1\ 1\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle^1\ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle^2$$

There is only one term with $J=1$, $M=1$. So $(\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}|\ 1\ 1) = 1$
similarly $(\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2}|\ 1\ -1) = 1$

Then apply J_- to this, recalling

$$\hat{J}_\pm\ |j\ m\rangle = (j(j+1) - m(m\pm 1))^{1/2}\hbar\ |j\ m\pm 1\rangle$$

$$\Rightarrow \hat{J}_-\ |\tfrac{1}{2}\ \tfrac{1}{2}\ 1\ 1\rangle = (1(2) - 1(0))^{1/2}\hbar\ |\tfrac{1}{2}\ \tfrac{1}{2}\ 1\ 0\rangle = \sqrt{2}\hbar\ |\tfrac{1}{2}\ \tfrac{1}{2}\ 1\ 0\rangle$$

$$\hat{J}_-\ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle^1\ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle^2 = \underbrace{(\tfrac{1}{2}(\tfrac{3}{2}) - \tfrac{1}{2}(-\tfrac{1}{2}))^{1/2}}_{=(\frac{3}{4}+\frac{1}{4})^{1/2} = 1}\hbar\ [\ |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle^1\ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle^2 + |\tfrac{1}{2}\ \tfrac{1}{2}\rangle^1\ |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle^2]$$

$$\Rightarrow \quad (\tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}\ \tfrac{1}{2}\ |\ 1\ 0) = (\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}\ |\ 1\ 0) = \frac{1}{\sqrt{2}}$$

This completes the C-G's with $J=1$. There is one other state present, namely $J = |j_1 - j_2| = |\frac{1}{2} - \frac{1}{2}| = 0$ (and so $M=0$). It must be orthogonal to $|J=1\ M=0\rangle$, so

$$(\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}\ |\ 0\ 0) = -(\tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}\ \tfrac{1}{2}\ |\ 0\ 0) = \frac{1}{\sqrt{2}}.$$

choose this to be positive from convention above

Example : Spin-orbit coupling

For atomic systems, the spin of the atomic electrons couples to their orbital angular momentum. Since one frequently makes single-particle approximations for polyelectronic systems, then one often couples $\vec{L}$ to $\vec{S} = \frac{1}{2}$ to form states of total $\vec{J}$. Clearly, $J = \ell \pm 1/2$ (unless $\ell = 0$ in which case J is simply $1/2$). Using the raising and lowering operators, one can find in analogy with the $j_1 = 1/2$, $j_2 = 1/2$ situation:

$$(\ell \; 1/2, M-m_s, m_s \,|\, J\, M) =$$

	$J = \ell + 1/2$	$J = \ell - 1/2$
$m_s = 1/2$	$\left(\dfrac{\ell + M + 1/2}{2\ell + 1}\right)^{1/2}$	$-\left(\dfrac{\ell - M + 1/2}{2\ell + 1}\right)^{1/2}$
$m_s = -1/2$	$\left(\dfrac{\ell - M + \frac{1}{2}}{2\ell + 1}\right)^{1/2}$	$\left(\dfrac{\ell + M + 1/2}{2\ell + 1}\right)^{1/2}$

Example: Hadronic physics of isospin

In elementary particle physics, one characterizes hadrons (strongly interacting particles) by their isospin. Isospin has nothing to do with spin, but it does have the same transformation properties. For example:

$$\left.\begin{array}{l}\text{proton}\\ \text{neutron}\end{array}\right\} \quad I = 1/2 \quad \begin{array}{l} I_3 = +1/2 \\ I_3 = -1/2 \end{array}$$

$$\text{pion} \quad \left.\begin{array}{l}\pi^+\\ \pi^0 \\ \pi^-\end{array}\right\} \quad I = 1 \quad \begin{array}{l} I_3 = +1 \\ I_3 = 0 \\ I_3 = -1. \end{array}$$

When these particles react, they form states of total isospin I_{TOT}. For example $\pi\,\pi$ $\quad i_1 = 1 \quad i_2 = 1$

The Clebsch - Gordan coefficients give:

I_{TOT} $I_{TOT,3}$

$(2i_1+1)(2i_2+1) = 9$

5 states	2,2	$\pi^+\pi^+$
	2,1	$\frac{1}{\sqrt{2}}(\pi^+\pi^0 + \pi^0\pi^+)$
	2,0	$\frac{1}{\sqrt{6}}(\pi^+\pi^- + 2\pi^0\pi^0 + \pi^-\pi^+)$
	etc.	
3 states	1,1	$\frac{1}{\sqrt{2}}(\pi^+\pi^0 - \pi^0\pi^+)$
	1,0	$\frac{1}{\sqrt{2}}(\pi^+\pi^- - \pi^-\pi^+)$ ← note: no $\pi^0\pi^0$.
	etc.	
1 state?	0,0	$\frac{1}{\sqrt{3}}(\pi^+\pi^- - \pi^0\pi^0 + \pi^-\pi^+)$

3 - j symbols

There are some inherent symmetries in the C - G coefficients which allow them to be expressed in an alternative form called 3 - j symbols

$$3j\text{'s} \rightarrow \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - m_3}}{(2j_3+1)^{1/2}} \left(j_1\, j_2\, m_1\, m_2 \middle| j_3\, -m_3\right)$$

note sign.

The 3 - j symbols have the symmetries:

i) even permutation $(1\,2\,3) = (2\,3\,1) = (3\,1\,2)$

ii) odd permutation $(3\,2\,1) = (2\,1\,3) = (1\,3\,2) = (-1)^{j_1+j_2+j_3} \times (1\,2\,3)$.

Irreducible Tensor Operators

We introduced the behaviour of $|j\,m\rangle$ under rotations by

$$\hat{R}(\alpha,\beta,\gamma)|j\,m\rangle = \sum_{m'} D^{(j)}_{m'm}(\alpha,\beta,\gamma)\,|j\,m'\rangle$$

What about the behaviour of operators: $\hat{R}\hat{T}\hat{R}^{-1}$? The simplest form is just

$$\hat{R}(\alpha,\beta,\gamma)\,\hat{S}\,R^{-1}(\alpha,\beta,\gamma) = \hat{S}.$$

The operator $\hat{S}$ transforms like a scalar number under rotations.
The next most complicated operator would be a vector operator $\vec{V}$, which transforms like a vector ($\vec{J}$) under rotations.
In analogy with the raising and lowering operators, we form

$$\hat{V}_{+1} = -(\hat{V}_x + i\hat{V}_y)/\sqrt{2} \qquad \hat{V}_{-1} = (\hat{V}_x - i\hat{V}_y)/\sqrt{2}$$
$$\hat{V}_0 = \hat{V}_z$$

Then: $$\hat{R}(\alpha,\beta,\gamma)\hat{V}_q\hat{R}^{-1}(\alpha,\beta,\gamma) = \sum_{q'} \hat{V}_{q'}\,\overline{D^{(1)}_{q'q}(\alpha,\beta,\gamma)}.$$

This rotates components of $\hat{V}$ around.

In general, we refer to sets of operators $T^{(k)}$ which transform like spherical harmonics as being an irreducible tensor operator of degree k.

irreducible: can't be decomposed further under angular momentum

tensor: usual scalar, vector, tensor...

with the transformation property

$$\hat{R}(\alpha,\beta,\gamma)\,\hat{T}^{(k)}_q\,R^{-1}(\alpha,\beta,\gamma) \equiv \sum_{q'} T^{(k)}_{q'}\,D^{(k)}_{q'q}(\alpha,\beta,\gamma)$$

To find the commutation properties of $T_q^{(k)}$, we proceed as follows:
Suppose we now replace $D_{q'q}^{(k)} = \langle k\, q' | \hat{R} | k\, q \rangle$.

$$\Rightarrow \hat{R}\, \hat{T}_q^{(k)}\, \hat{R}^{-1} = \sum_{q'} T_{q'}^{(k)} \langle k, q' | \hat{R} | k\, q \rangle$$

If we allow $\hat{R}$ to be infinitesimal $\hat{R} = \mathbb{1} - i\epsilon \vec{u} \cdot \hat{\vec{J}} / \hbar$, then

$$\begin{aligned} \hat{R}\, \hat{T}\, \hat{R}^{-1} &= (1 - i\epsilon \vec{u} \cdot \hat{\vec{J}}/\hbar \ldots)\, \hat{T}_q^{(k)} (1 + i\epsilon \vec{u} \cdot \hat{\vec{J}} \ldots) \\ &= \hat{T} - i\epsilon\, [\vec{u} \cdot \hat{\vec{J}}, \hat{T}_q^{(k)}] / \hbar \ldots \end{aligned}$$

and

$$\begin{aligned} &\sum_{q'} \hat{T}_{q'}^{(k)} \langle k\, q' | 1 - i\epsilon \vec{u} \cdot \hat{\vec{J}}/\hbar \ldots | k\, q' \rangle \\ &= \sum_{q'} \hat{T}_{q'}^{(k)} \left[\delta_{qq'} - \frac{i\epsilon}{\hbar} \langle k q' | \vec{u} \cdot \hat{\vec{J}} | k q' \rangle \right] \end{aligned}$$

equating

$$[\vec{u} \cdot \hat{\vec{J}}, \hat{T}_q^{(k)}] = \sum_{q'} T_{q'}^{(k)} \langle k\, q' | \vec{u} \cdot \hat{\vec{J}} | k\, q \rangle$$

We can use this relationship to form the commutators by allowing $\vec{u}$ to point along the x, y, z axis, and taking $u_x + i u_y$ combinations as necessary. For example:

$$\begin{aligned} \vec{u} = (0, 0, 1) \Rightarrow [\hat{J}_z, \hat{T}_q^{(k)}] &= \sum_{q'} \hat{T}_{q'}^{(k)} \langle k\, q' | \hat{J}_z | k\, q \rangle \\ &= \hbar q \sum_{q'} \hat{T}_{q'}^{(k)} \delta_{qq'} \\ &= \hbar q\, \hat{T}_q^{(k)}. \end{aligned}$$

Similarly, we find:

$$[\hat{J}_\pm, \hat{T}_q^{(k)}] = \hbar \{k(k+1) - q(q \pm 1)\}^{1/2} \hbar T_{q\pm 1}^{(k)}$$

Wigner-Eckart Theorem

Now that we have introduced tensor operators, let us evaluate their matrix elements. We let $|\tau J M\rangle$ be some general state of total angular momentum J, with τ being any other quantum numbers appropriate to the description of the system. Using $\hat{R}^{-1} = \hat{R}^{+}$, we have

$$\langle \tau' J' M' | \hat{T}_q^{(k)} | \tau J M\rangle = \langle \tau' J' M' | \hat{R}^{+} \hat{R} T_q^{(k)} R^{+} R | \tau J M\rangle$$

$$= \sum_{\mu'} \sum_{\sigma} \sum_{\mu} \left[D_{\mu' M'}^{(J')*} \langle \tau' J' \mu' | \right] \times$$

$$\times \left[D_{\sigma q}^{(k)} T_\sigma^{(k)} \right] \left[D_{\mu M}^{(J)} | \tau J \mu \rangle \right]$$

$$= \sum_{\mu'} \sum_{\sigma} \sum_{\mu} D_{\mu' M'}^{(J')*} D_{\mu M}^{(J)} D_{\sigma q}^{(k)} \langle \tau' J' M' | T_\sigma^{(k)} | \tau J M \rangle \quad (1)$$

The l.h.s. which we started with does not depend on the rotations R which we have introduced on the right hand side. So we can integrate over the rotational angles etc as follows: apply $\hat{R}$ to $|kq\rangle|jm\rangle = \sum_J \sum_M |k j J M\rangle \times$ C.G.

$$\Rightarrow \underbrace{\hat{R}\, |k q\rangle |j m\rangle}_{\text{rotation operating on two states}} = \sum_J \sum_M \hat{R}\, |k\, j\, J\, M\rangle \underbrace{(k\, j\, q\, m\, | J\, M)}_{\text{C-G coeff.}}$$

$$\sum_{q'} \sum_{m'} |k q'\rangle |j m'\rangle D_{q' q}^{(k)}(R)\, D_{m' m}^{(j)}(R) = \sum_{M'} \sum_J \sum_M |k\, j\, J\, M'\rangle D_{M' M}^{(J)}(R) \times$$

$$\times (k\, j\, q\, m | J M)$$

On the r.h.s. if we substitute $|k\, j\, J\, M'\rangle = \sum_{q' m'} |k q'\rangle |j m'\rangle (k\, j\, q' m' | J M')$

Then we can compare coefficients of $|kq'\rangle|j m'\rangle$ on the left and right-hand sides.

This gives

$$D^{(k)}_{q'q}(R)\, D^{(j)}_{m'm}(R) = \sum_{M'}\sum_{J}\sum_{M} (k\, j\, q'\, m' | J\, M')\, D^{(J)}_{M'M}(R) \times (k\, j\, q\, m | J\, M) \quad ②$$

[This is sometimes referred to as the Clebsch-Gordan series].
Left multiply by $[D^{J}_{M'M}(R)]^*$, and integrate over the rotation angles α, β, γ : $dR = d\alpha \sin\beta\, d\beta\, d\gamma$ [integral may have to range over 4π].

$$\int dR \left[D^{(J)}_{M'M}(R)\right]^* D^{(k)}_{q'q}(R)\, D^{(j)}_{m'm}(R)$$

$$= \sum_{M'}\sum_{J}\sum_{M} (k\, j\, q'\, m' | J\, M')(k\, j\, q\, m | J\, M) \underbrace{\int dR\, D^{(J)}_{M'M}(R) \left[D^{(J)}_{M'M}(R)\right]^*}_{\frac{1}{2J+1}\int dR}$$

by orthonormality. [Not proven in Ballentine].

$$\therefore \int dR \left\{D^{(J)}_{M'M}(R)\right\}^* D^{(k)}_{q'q}(R)\, D^{(j)}_{m'm}(R)$$

$$= \sum_{M'}\sum_{J}\sum_{M} (k\, j\, q'\, m' | J\, M')(k\, j\, q\, m | J\, M) \frac{1}{2J+1} \int dR. \quad ③$$

Finally, substitute ③ into ① by integrating over the angles $\int dR$. We end up with

$$\langle \tau'\, J'\, M' | T^{(k)}_{q} | \tau\, J\, M \rangle = \sum_{\mu'}\sum_{\sigma}\sum_{\mu} \frac{1}{2J'+1} (J\, k\, \mu\, \sigma | J'\, \mu') \langle \tau'\, J'\, \mu' | T^{(k)}_{\sigma} | \tau\, J\, \mu \rangle \times \underbrace{(J\, k\, M\, q | J'\, M')}$$

This is just a constant as far as the sum is concerned. $\therefore$ take out front. Rewrite the sum as $\langle \tau' J' \| T^{k} \| \tau J \rangle$

$$\Rightarrow \langle \tau'\, J'\, M' | T^{(k)}_{q} | \tau\, J\, M \rangle = \langle \tau'\, J' \| T^{(k)} \| \tau\, J \rangle (J\, k\, M\, q | J'\, M')$$

This expression is known as the Wigner-Ekhart theorem. The quantity $\langle \| T \| \rangle$ is referred to as a reduced matrix element. What we have done is replace a general matrix element depending on $M_{J'}', M_J, q, k$, in favour of one depending on J' k J and a Clebsch-Gordan coefficient.

Products of Tensor Operators

The last topic which we wish to cover under angular momentum is the behaviour of $\hat{T}^{(k_1)}_{q_1} \times \hat{T}^{(k_2)}_{q_2}$. No time left.