Elasticity in two dimensions

Chapters 3 and 4 of *Mechanics of the Cell*, as well as its Appendix D, contain selected results for the elastic behavior of materials in two and three dimensions. The description is brief and not entirely complete, with some of the results relegated to problem sets. The purpose of these supplementary notes is to present the basic formalism for two-dimensional systems with high (6-fold or isotropic), medium (4-fold) and low (2-fold) symmetry. The connection between the elastic moduli presented here, and models for microscopic interactions such as springs and tethers, is made in the text.

General properties of elastic constants

We start by reviewing a suite of elastic variables taken from Appendix D. The strain tensor u_{ij} quantifies the deformation of an object through derivatives of the displacement vector **u** as a function of position **x**. The definition of u_{ij} ,

$$u_{ij} = 1/2 \, [\partial u_i/\partial x_j + \partial u_j/\partial x_i + \Sigma_k \, (\partial u_k/\partial x_i)(\partial u_k/\partial x_j)], \qquad (1)$$

highlights its symmetry under exchange $i \leftrightarrow j$. For small deformations, the quadratic terms in Eq. (1) can be dropped, leaving

$$u_{ij} \quad 1/2 \, [\partial u_i/\partial x_j + \partial u_j/\partial x_i]. \qquad (2)$$

The stress tensor σ_{ij} is the force per unit area applied to an object, with indices indicating the direction of the force and the normal to the surface element. The force **F** is calculated through

$$F_i = \Sigma_j \, \sigma_{ij} \, a_j. \qquad (3)$$

where the vector **a** of a surface element has a magnitude equal to its area and a direction locally perpendicular to the surface.

At small deformations, stress and strain are linearly related through Hooke's law, which has the general form

$$\sigma_{ij} = \Sigma_{k,l}\, C_{ijkl}\, u_{kl}, \qquad (4)$$

where C_{ijkl} are a set of elastic constants. The change in free energy *density* $\mathcal{F}$ resulting from a deformation is quadratic in *u* for modest changes

$$\Delta\mathcal{F} = 1/2\, \Sigma_{i,j,k,l}\, C_{ijkl}\, u_{ij}\, u_{kl}. \qquad (5)$$

In principle, the tensor C_{ijkl} has a prodigious number of components, but many of these are related by symmetry. Considering only the symmetry of u_{ij} and its product in Eq. (5), the $2^4 = 16$ terms in two dimensions can be reduced to 6 independent components:

$$\begin{array}{l} C_{xxxx} \qquad C_{yyyy} \qquad C_{xxyy} = C_{yyxx} \\ \quad C_{xyxy} = C_{xyyx} = C_{yxyx} = C_{yxxy} \\ \quad C_{xxxy} = C_{xxyx} = C_{xyxx} = C_{yxxx} \\ \quad C_{yyyx} = C_{yyxy} = C_{yxyy} = C_{xyyy}. \end{array} \qquad (6)$$

For materials with 6-fold, 4-fold or 2-fold symmetry, the operation $x \leftrightarrow -x$ or $y \leftrightarrow -y$ forces *C*'s with an odd number of *x* or *y* indices to vanish, which are the bottom two lines in Eq. (6). Consequently, the systems that we consider here have a maximum of 4 elastic constants at low symmetry, and 2 at high symmetry, with $\Delta\mathcal{F}$ possessing the general form

$$\Delta\mathcal{F} = (1/2)\, [\, C_{xxxx}u_{xx}^2 + C_{yyyy}u_{yy}^2 + 2C_{xxyy}u_{xx}u_{yy} + 4C_{xyxy}u_{xy}^2\,], \qquad (7)$$

where the numerical prefactors in each term are obvious from Eq. (6). Symmetries of the material provide further constraints on the elastic constants.

<u>6-fold symmetry</u>

The easiest way to analyze systems of 6-fold (or full rotational) symmetry is to introduce complex linear combinations of the Cartesian coordinates:

$$\xi \equiv x + iy \qquad \eta \equiv x - iy, \qquad (8)$$

which are clearly complex conjugates. A rotation about the *xy* coordinate origin by an angle θ changes changes the coordinates (ξ, η) to

$$\xi \rightarrow \xi \exp(i\theta) \qquad \eta \rightarrow \eta \exp(-i\theta), \tag{9}$$

as can be verified by the usual rotation of *x* and *y* coordinates. Here, six-fold symmetry demands the moduli be invariant under rotations through $\theta = \pi/3$, or $\xi \rightarrow \xi\exp(i\pi/3)$ and $\eta \rightarrow \eta\exp(-i\pi/3)$. The only non-zero components of C_{ijkl} that remain unchanged by this transformation must contain ξ and η the same number of times, since $\exp(i\theta)\exp(-i\theta) = 1$. This leaves only the pair $C_{\xi\xi\eta\eta}$ and $C_{\xi\eta\xi\eta}$, and terms related by symmetry of the indices, as independent components. Thus, the change in the free energy density $\Delta\mathcal{F}$ from Eq. (5) is reduced to

$$\Delta\mathcal{F} = 2C_{\xi\eta\xi\eta} u_{\xi\eta} u_{\xi\eta} + C_{\xi\xi\eta\eta} u_{\xi\xi} u_{\eta\eta}. \tag{10}$$

From the definitions in Eq. (8), the products of the strain tensors in Eq. (10) have the Cartesian representation:

$$u_{\xi\xi} = u_{xx} - u_{yy} + 2iu_{xy}, \qquad u_{\eta\eta} = u_{xx} - u_{yy} - 2iu_{xy}, \qquad u_{\xi\eta} = u_{xx} + u_{yy}, \tag{11}$$

permitting $\Delta\mathcal{F}$ to be expanded as

$$\Delta\mathcal{F} = 2C_{\xi\eta\xi\eta}\ (u_{xx} + u_{yy})^2 + C_{\xi\xi\eta\eta}\ \{(u_{xx} - u_{yy})^2 + 4u_{xy}{}^2\}. \tag{12}$$

Now, the combinations of strain tensors in Eq. (12) correspond to specific deformations: the first combination is a pure dilation (with no shear) while the second is a shear (with no dilation). The elastic moduli corresponding to these deformation modes are the area compression modulus K_A and the shear modulus μ, written with the normalization

$$\Delta\mathcal{F} = (K_A/2)(u_{xx}+u_{yy})^2 + \mu\ \{(u_{xx}-u_{yy})^2/2 + 2u_{xy}{}^2\} \quad \text{(six-fold symmetry)}. \tag{13}$$

Comparing Eqs. (12) and (13) leads to the identification:

$$K_A = 4C_{\xi\eta\xi\eta} \qquad \mu = 2C_{\xi\xi\eta\eta}. \tag{14}$$

Young's modulus (Y) and Poisson's ratio (σ_p) are often used as an alternative pair to K_A and μ. These are more directly related to the experimental procedure of measuring strain as a function of stress, as we now show. As is established in most introductory books on continuum mechanics, the stress tensor can be obtained from

$$\sigma_{ij} = \partial \mathcal{F} / \partial u_{ij}. \tag{15}$$

Applied to the free energy of Eq. (13), this relation yields, for the diagonal components of the stress tensor:

$$\sigma_{xx} = (K_A/2)\, 2\, (u_{xx}+u_{yy}) + \mu\, 2\, (u_{xx}-u_{yy})/2 = (K_A+\mu)u_{xx} + (K_A-\mu)u_{yy}\ , \tag{16a}$$
$$\sigma_{yy} = (K_A/2)\, 2\, (u_{xx}+u_{yy}) - \mu\, 2\, (u_{xx}-u_{yy})/2 = (K_A-\mu)u_{xx} + (K_A+\mu)u_{yy}\ . \tag{16b}$$

Now, Y and σ_p can be determined by observing the strain arising from a uniaxial stress imposed upon an object. Suppose that the force is applied along the x-axis, such that the stress tensor has the form:

$$\begin{pmatrix} \sigma_{xx} & 0 \\ 0 & 0 \end{pmatrix}$$

Then, the left-hand side of Eq. (16b) vanishes, permitting u_{yy} to be solved in terms of u_{xx}. The (negative) ratio of u_{yy} to u_{xx} is Poisson ratio: the relative compression in the transverse direction compared to the relative stretch in the longitudinal direction, where transverse and longitudinal refer to the direction of the applied force

$$\sigma_p = -u_{yy}/u_{xx}\ . \tag{17}$$

Solving Eq. (16b) for u_{yy}/u_{xx} ,

$$\sigma_p = (K_A-\mu) / (K_A+\mu). \tag{18}$$

Young's modulus is defined from the stress - strain relationship under uniaxial stress

$$\sigma_{xx} = Yu_{xx}. \tag{19}$$

With the substitution $u_{yy} = -\sigma_p u_{xx}$, Eq. (16a) reads

$$\sigma_{xx} = [(K_A+\mu) - \sigma_p(K_A-\mu)]u_{xx}. \tag{20}$$

The Young's modulus is contained in the square brackets on the right hand side of this equation. Substituting Eq. (18) to eliminate σ_p gives

$$Y = 4K_A\mu \,/\, (K_A+\mu). \tag{21}$$

Note that all of these results apply to isotropic materials as well.

4-fold symmetry

The restriction that the free energy be invariant under rotations by $\theta = \pi/3$ (6-fold symmetry) or arbitrary θ (full rotational symmetry) reduces the number of independent elastic moduli to 2, which we have chosen as the pairs (K_A, μ) or (Y, σ_p). We now consider the less-restrictive situation of 4-fold symmetry, as represented by, for example, a network of identical springs joined at right angles. Of the moduli present in Eq. (7), the symmetry of the material now adds only the relation $C_{xxxx} = C_{yyyy}$, reducing the free energy to

$$\Delta\mathcal{F} = (1/2)\,[\, C_{xxxx}(u_{xx}^2+u_{yy}^2) + 2C_{xxyy}u_{xx}u_{yy} + 4C_{xyxy}u_{xy}^2 \,]. \tag{22}$$

This expression can be recast in terms of combinations of the strain tensor describing specific deformations, just as we did with Eq. (13). For the geometry of a square, these are:

$u_{xx}+u_{yy}$	dilation, the only non-zero term if all lengths change proportionally with no change in angles
$u_{xx}-u_{yy}$	pure shear, the only term remaining if expansion and contraction are equal in orthogonal directions with no change in angles
u_{xy}	simple shear, the only term remaining if angles change without a change in length.

The simple shear mode is how we usually think of shear - an old barn leaning to the side, for example. In terms of these modes, $\Delta\mathcal{F}$ is

$$\Delta\mathcal{F} = (K_A/2)(u_{xx}+u_{yy})^2 + (\mu_p/2)(u_{xx}-u_{yy})^2 + 2\mu_s u_{xy}^2 \quad \text{(four-fold symmetry).} \quad (23)$$

Comparing Eqs. (22) and (23) leads immediately to the identification:

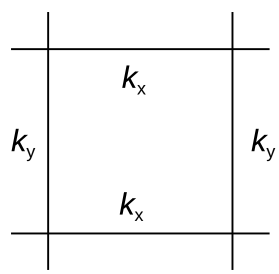
$$K_A = (C_{xxxx} + C_{xxyy}) / 2$$
$$\mu_p = (C_{xxxx} - C_{xxyy}) / 2 \qquad \text{(pure shear)} \qquad (24)$$
$$\mu_s = C_{xyxy} \qquad \text{(simple shear).}$$

The combinations (Y, σ_p) can be used in place of (K_A, μ_p), although a complete description of the elastic behavior still requires μ_s. The mathematics for finding (Y, σ_p) follows the same steps as Eqs. (16) - (21), yielding

$$Y = 4K_A\mu_p / (K_A+\mu_p) \qquad (25a)$$
$$\sigma_p = (K_A-\mu_p) / (K_A+\mu_p). \qquad (25b)$$

2-fold symmetry

Several microscopic representations of systems with just 2-fold symmetry are found in the text, including four-fold networks with inequivalent spring constants, as in:



With $k_x \neq k_y$, the Young's moduli clearly depend upon direction. Consequently, the area compression modulus is ambiguously defined because the application of an isotropic stress generates an anisotropic strain, $u_{xx} \neq u_{yy}$. To emphasize the different definitions:

- in $K_A^{-1} = (\partial A/\partial\tau)/A$, the deformation is not shape-preserving if τ is isotropic
- in $\Delta\mathcal{F} = (K_A/2)(u_{xx}+u_{yy})^2 + \ldots$ the K_A term can only be isolated from other modes by means of an anisotropic stress

Experimentally, systems with low symmetry are often approached by separately measuring their Young's moduli along different directions.

As far as the energy density is concerned, anisotropy suggests that we drop K_A in favor of K_x and K_y, where two K's are required because of the lower symmetry:

$$\Delta\mathcal{F} = K_x u_{xx}^2/2 + K_y u_{yy}^2/2 + \mu_p(u_{xx}-u_{yy})^2/2 + 2\mu_s u_{xy}^2.$$

However, this remains unsatisfactory, as a pure shear mode ($u_{xx} = -u_{yy}$, $u_{xy} = 0$) now receives contributions from μ_p, K_x and K_y. Thus, we return to Eq. (7) and do just a small adjustment ($\mu_s = C_{xyxy}$) to write

$$\Delta\mathcal{F} = K_x u_{xx}^2/2 + K_y u_{yy}^2/2 + K_{xy} u_{xx} u_{yy} + 2\mu_s u_{xy}^2. \quad (26)$$

Experimentally, one can measure a Young's modulus Y and Poisson ratio σ_p in each direction, replacing three elastic constants K_x, K_y and K_{xy} by the combinations (Y_x, $\sigma_{p,x}$) and (Y_y, $\sigma_{p,y}$), which contain four variables. As above, the relationships amongst these quantities can be found from the uniaxial stress problem. We start with $\sigma_{ij} = \partial\mathcal{F} / \partial u_{ij}$ to find

$$\sigma_{xx} = K_x u_{xx} + K_{xy} u_{yy} \quad (27a)$$
$$\sigma_{yy} = K_y u_{yy} + K_{xy} u_{xx} \quad (27b)$$

Then, considering each axis in turn

1. Stress along the x-axis ($\sigma_{xx} \neq 0$, $\sigma_{yy} = 0$). Imposing $\sigma_{yy} = 0$ on Eq. (27b) leads to

$$\sigma_{p,x} = K_{xy}/K_y, \quad (28)$$

where the Poisson ratio for stress in the x-direction is $\sigma_{p,x} = -u_{yy}/u_{xx}$. Combining Eqs. (27a) and (28) gives $\sigma_{xx} = (K_x - K_{xy}^2/K_y)u_{xx}$, from which

$$Y_x = (K_x K_y - K_{xy}^2)/K_y. \quad (29)$$

2. Stress along the y-axis ($\sigma_{xx} = 0$, $\sigma_{yy} \neq 0$). By similar reasoning, we find

$$\sigma_{p,y} = K_{xy}/K_x, \quad (30)$$
$$Y_y = (K_x K_y - K_{xy}^2)/K_x. \quad (31)$$