

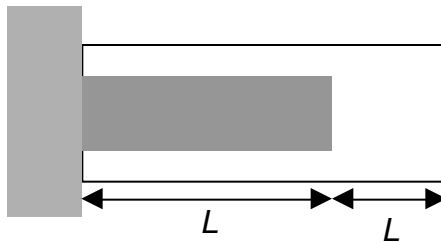
Appendix D extra - Physical properties of matter

This material is a short review of the physical properties of matter as observed on macroscopic length scales, presented at an introductory level.

Thermal expansion

From statistical mechanics, one can establish that the mean kinetic energy of a point molecule in a three-dimensional ideal gas is equal to $3/2 k_B T$. As can be verified experimentally, this predicts that the speed of molecules increases with temperature, with the result that the pressure of an ideal gas rises with temperature. This phenomena applies in general, and for solids, implies that materials usually expand when they are heated (although there are known counterexamples).

The general observation is that the relative expansion is proportional to the change in temperature



In symbols:

$$\Delta L / L = \alpha \Delta T,$$

where L is the original length, ΔL is the change in length and ΔT is the change in temperature. The proportionality constant α is called the coefficient of linear expansion, having dimensions of $[\text{temperature}]^{-1}$. Some typical values

| Material | α ($^{\circ}\text{C}^{-1}$) |
|----------|--------------------------------------|
| steel | 12×10^{-6} |
| brick | 9×10^{-6} |
| rubber | 80×10^{-6} |

Volume expansion

If a material expands in all directions, then its volume expands as well. Having observed that the relative length of a material increases linearly with temperature, we might expect:

$$\Delta V / V = \beta \Delta T$$

where β is the coefficient of volume expansion.

An approximate expression for β can be obtained through the following argument:

- say we have a cube of side L , such that the volume changes from $V = L^3$ to $V = (L + \Delta L)^3$
- then
$$\frac{V}{V} = \frac{[(L + \Delta L)^3 - L^3]}{L^3} = \frac{[L^3 + 3L^2 \Delta L + 3L(\Delta L)^2 + (\Delta L)^3 - L^3]}{L^3} = \frac{[3L^2 \Delta L + 3L(\Delta L)^2 + (\Delta L)^3]}{L^3}$$
- now, if ΔL is small compared to L , then only the first term in this expansion is important, and
$$\frac{V}{V} = 3L^2 \Delta L / L^3 = 3 \left(\frac{\Delta L}{L} \right)$$
- but we know that
$$\frac{\Delta L}{L} = \alpha \Delta T,$$
 so
$$\frac{V}{V} = 3\alpha \Delta T.$$

Comparing with our expression for the volume expansion coefficient, we predict

$$\beta = 3\alpha.$$

Typical values: fluids tend to expand more than solids

| Material | β ($^{\circ}\text{C}^{-1}$) |
|---------------|-------------------------------------|
| ethyl alcohol | 1.1×10^{-3} |
| glycerine | 5.1×10^{-4} |
| water | 2×10^{-4} |

Counter-example: H_2O is most dense at +4 C, which is the temperature at lake bottoms in the winter (or, why lakes don't freeze solid).

Stress and strain

In a first pass through mechanics, one deals with rigid bodies whose shape is unchanged by the application of a force. Many biomaterials are relatively soft, meaning that they deform in response to a force. To describe this deformation, we need to:

- develop a formalism which accounts for how the applied force is distributed over the surface of the body; this leads to the concept of stress
- develop a description of the local or global deformation of the body; this leads us to strain.

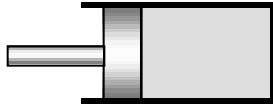
A force is applied over a region of the surface of an extended object, rather than a specific point on its surface. Thus, we talk about quantities like pressure when

discussing fluids, *etc.* But the concept of pressure alone is incomplete: we must also recognize the direction of the applied force with respect to the orientation of the surface; that is, we must allow for shear.

First, we define a general quantity called **stress**, which is the force per unit area:

$$[\text{stress}] = [\text{force}] / [\text{area}]$$

In a situation like



where a fluid (say, a gas) is confined by a cylinder, we find

$$[\text{stress}] = [\text{pressure}].$$

Isn't stress always pressure? Certainly pressure is an example of stress, but there are others, where the applied force is not perpendicular to the surface, to be introduced shortly.

The **strain** characterizes the deformation of an object in response to a stress. Strain is dimensionless, and equals the *relative* change in shape of an object. The relative change in volume is **one example** of the **strain**:

$$\Delta V / V = [\text{strain}],$$

The general relation between stress and strain was proposed by Robert Hooke

$$[\text{stress}] \propto [\text{strain}] \quad (\text{Hooke's law for materials})$$

which applies at small strains and can be used to obtain $F = kx$ for springs (see below). The proportionality constants are called the elastic moduli of the material.

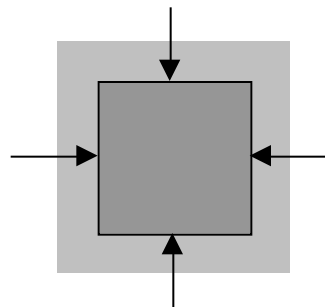
Elastic moduli

For isotropic materials (same in all directions), there are two fundamental elastic moduli, the bulk modulus B and the shear modulus μ .

Bulk modulus

This modulus characterizes the change in the relative volume induced by the change in an isotropic pressure P

$$P = B \left(\Delta V / V \right)$$

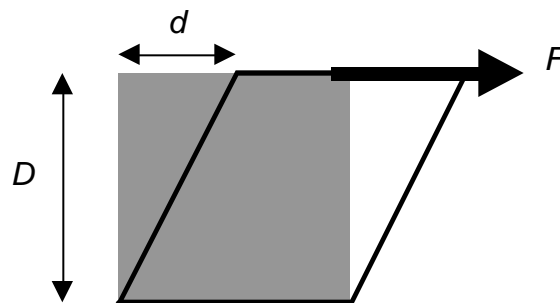


Because the strain is dimensionless, B has the same units as pressure; *i.e.*, it has units of energy density. Examples of bulk moduli

| material | B (J/m ³) |
|-------------|-------------------------|
| tungsten | 20×10^{10} |
| aluminum | 7×10^{10} |
| polystyrene | 0.5×10^{10} |
| water | 0.22×10^{10} |

Shear modulus

For this deformation, the stress may be applied parallel to a surface



Here, the shear strain is d/D , and the stress-strain relation reads

$$F/A = \mu d/D.$$

For solids, the shear modulus is often about half of the bulk modulus. For fluids, the shear modulus vanishes. Some examples:

| material | μ (J/m ³) |
|----------|---------------------------|
| tungsten | 15×10^{10} |
| bone | 8×10^{10} |
| lead | 0.54×10^{10} |
| fluids | 0 |

Young's modulus

The Young's modulus arises for uniaxial stress. For example, apply a force F to the end of a bar with cross sectional area A and initial length L . The stress-strain relation is

$$F/A = Y (\Delta L / L)$$



Typical values of Young's modulus are:

| material | Y (J/m ³) |
|----------------|-------------------------|
| tungsten | 36×10^{10} |
| pyrex glass | 6.2×10^{10} |
| bone (tension) | 1.6×10^{10} |
| nylon | 0.37×10^{10} |
| polystyrene | 0.14×10^{10} |
| proteins | 0.05×10^{10} |

From our stress - strain relationship, we can determine the force constant of a rod in terms of its Young's modulus and geometry. Equating the stretch x with the displacement from equilibrium L , our stress-strain equation reads:

$$F/A = Yx/L$$

which can be rearranged as

$$F = (YA/L) x$$

Now, this equation looks just like Hooke's law $F = kx$, with the force constant k being

$$k_{sp} = YA/L.$$

Example: stretching of a tendon

A typical dimension of the tendon in your ankle is

$$\begin{aligned} [radius] &= 1/3 \text{ cm} \\ \rightarrow A &= R^2 = (1/3 \times 10^{-2})^2 = 1/3 \times 10^{-4} \text{ m}^2 \\ [length] &= L = 1/3 \text{ m} \end{aligned}$$

The Young's modulus of tendon is 10^9 J/m³, with some variation (10^8 also quoted), so

$$k_{sp} = YA/L = 10^9 \times 1/3 \times 10^{-4} / (1/3) = 10^5 \text{ J/m}^2.$$

If a force of 800 N (weight of an adult male) is placed on this tendon, it stretches by

$$\begin{aligned} x &= F/k_{sp} = 800 / 10^5 = 0.8 \times 10^{-2} \text{ m} \\ \text{or} \\ x &= 0.8 \text{ cm.} \end{aligned}$$

Example: stretching of bone

Let's repeat this calculation but with a much stiffer material, bone. Major bones in our bodies have a larger dimension than tendons. Let's use

$$\begin{aligned} \text{--> } [radius] &= 1 \text{ cm} \\ A &= R^2 = (10^{-2})^2 = 1 \times 10^{-4} \text{ m}^2 \\ [length] &= L = 1/3 \text{ m} \end{aligned}$$

$$Y = 1.6 \times 10^{10} \text{ J/m}^3 \quad (\text{typical value for bone under tension})$$

Hence,

$$k_{sp} = YA/L = 1.6 \times 10^{10} \times 1 \times 10^{-4} / (1/3) = 4.8 \times 10^7 \text{ J/m}^2.$$

Under the same force as the tendon, the stretch is

$$x = F/k_{sp} = 800 / 4.8 \times 10^7 = 1.7 \times 10^{-5} \text{ m}$$

or

$$x = 0.017 \text{ mm.} \quad (\text{tiny})$$