The Kerr–Newman metric in the nonsymmetric unified field theory

D. H. BOAL
Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Alta., Canada T6G 2J1

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The method introduced by Newman and Janis for obtaining the metric of a rotating, charged particle in the Einstein–Maxwell theory of gravitation and electromagnetism is examined in the context of the nonsymmetric unified field theory. It is found that a transformation very similar to theirs, when applied to the antisymmetric part of the tensor $g_{\mu\nu}$, will generate the required electromagnetic field associated with the Kerr–Newman metric.

La méthode introduite par Newman et Janis pour obtenir la métrique d'une particule chargée en rotation, dans la théorie de Einstein–Maxwell de la gravitation et de l'électromagnétisme, est examinée dans le contexte de la théorie du champ unifié non symétrique. On trouve qu'une transformation très semblable à la leur, quand on l'applique à la partie antisymétrique du tensen $g_{\mu\nu}$, génère le champ électromagnétique requis associé à la métrique de Kerr–Newman.

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1. Introduction

Some years ago, Newman and Janis (1965) proposed a method for finding the metric for a rotating particle from the spherically symmetric metric of the particle without angular momentum. They first showed that the procedure gave the correct results for a particle without charge (Kerr 1963) and then used the method to predict the metric for a rotating charged mass (Newman *et al.* 1965). Unfortunately, no similar transformation was found for the electromagnetic field.

Recently, we proposed a theory for the unification of gravity and electromagnetism (Moffat and Boal 1975) in which the electromagnetic field is directly related to the antisymmetric part of the metric tensor, $g_{\mu\nu}$. If this theory is correct, then there must exist a similar transformation to Newman's (which applies to the symmetric part of $g_{\mu\nu}$) which will yield the correct electromagnetic field. We wish to present such a transformation here.

2. Summary of the Theory

In the unified theory of gravitation and electromagnetism proposed by Einstein (Einstein 1945; Einstein and Strauss 1946), the tensor $g_{\mu\nu}$ is allowed to be nonsymmetric. However, to obtain physical solutions, we found (Moffat and Boal 1975) that Einstein's Hamiltonian density, $\mathscr{H}$, must be modified to read[1]

[1] $$\mathscr{H}^* = \mathscr{H} + (4\pi/k^2)\sqrt{-g}\, g^{\mu\nu} g_{[\nu\mu]}$$

where

[2] $$\mathscr{H} = \sqrt{-g}\, g^{\mu\nu} \mathsf{R}_{\mu\nu}$$

$\mathsf{R}_{\mu\nu}$ denotes the usual contracted curvature tensor which may also be nonsymmetric. We leave aside for the moment the question of whether $g_{\mu\nu}$ is real nonsymmetric or Hermitian. The constant k is a universal constant having the dimensions of length. We solved the field equations derived by varying the action given by the Hamiltonian in [1] for the spherically symmetric case and found (for real nonsymmetric $g_{\mu\nu}$)

[3] $$\mathrm{d}s^2 = \left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right)\left(1 + \frac{k^2 Q^2}{r^4}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

in polar coordinates, where m and Q denote the mass and charge of the particle, respectively, in units where $G = c = 1$. It was found that the antisymmetric part of the tensor $g_{\mu\nu}$ was related to the electromagnetic tensor $\mathsf{F}_{\mu\nu}$ through the relation

[4] $$g_{[\mu\nu]} = k\mathsf{F}_{\mu\nu}$$

The consequences of the deviation of this result from the Reissner–Nordström solution (Reissner 1916; Nordström 1918) into which it collapses when $k \to 0$, was explored for a variety of phenomena and found to yield only small

[1]We will use the notation $g_{(\mu\nu)} = (1/2)(g_{\mu\nu} + g_{\nu\mu})$ and $g_{[\mu\nu]} = (1/2)(g_{\mu\nu} - g_{\nu\mu})$. The remaining definitions are given in Moffat and Boal (1975), the only change being an insertion of 8π into the Lagrangian to facilitate comparison with the Einstein–Maxwell theory.

effects (Boal and Moffat 1975). It has also been shown (Johnson 1975) that the solution given by [3] generates the correct equations of motion for a charged particle to the lowest nontrivial order of approximation.

The most obvious extension of this solution would be to include the effects of spin. However, the method employed by Newman *et al.* (1965) for including spin in the less complex Einstein–Maxwell case ($k \to 0$ here) was not the exact solution of the field equations but rather the use of a complex coordinate transformation. The metric of the Reissner–Nordström solution was first expressed in null coordinates

[5] $$ds^2 = \Delta(r)\,du^2 + 2\,du \cdot dr - r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

where

[6] $$\Delta(r) = 1 - (2m/r) + (4\pi Q^2/r^2)$$

The contravariant form of the metric was expressed in terms of a null tetrad by

[7] $$g^{\mu\nu} = (l^\mu n^\nu + l^\nu n^\mu) - (a^\mu b^\nu + a^\nu b^\mu)$$

where

[8] $$l^\mu = \delta_1{}^\mu$$

[9] $$n^\mu = \delta_4{}^\mu - (1/2)\Delta(r)\delta_1{}^\mu$$

[10] $$a^\mu = (1/\sqrt{2}\,r)(\delta_2{}^\mu + (i/\sin\theta)\delta_3{}^\mu)$$

and where b^μ is the complex conjugate of a^μ. The coordinate r is now allowed to be complex. The tetrad vectors $\boldsymbol{l}'^\mu$, $\boldsymbol{n}'^\mu$, and $\boldsymbol{a}'^\mu$ for the metric corresponding to a rotating charged mass, are obtained from [8]–[10] by applying the complex coordinate transformation

[11] $$r' = r + ia\cos\theta$$

[12] $$u' = u - ia\cos\theta$$

[13] $$\theta' = \theta; \quad \phi' = \phi$$

The vector $\boldsymbol{b}'^\mu$ is found by applying the transformation given by [13] and the complex conjugates of [11] and [12]. Here, 'a' is the angular momentum per unit mass. Demanding that r' be real, the new tetrad $\boldsymbol{l}'^\mu$, $\boldsymbol{n}'^\mu$, $\boldsymbol{a}'^\mu$, and $\boldsymbol{b}'^\mu$ gives the rotating analogue of the Reissner–Nordström metric in radiation coordinates.

3. Rotating Electromagnetic Field

The method devised by Newman for obtaining the rotating metric can be extended to the case of the nonsymmetric theory only with difficulty, since the nonsymmetric tensor, $g_{\mu\nu}$, cannot be diagonalized into the symmetric form required by the usual tetrad formalism. It is possible to generalize the tetrad formalism to represent a Hermitian tensor by means of a complex tetrad $t^\mu{}_\alpha$ and its complex conjugate $t^{*\mu}{}_\alpha$, where $g^{\mu\nu}$ becomes (see, for example, Smith (1974))

[14] $$g^{\mu\nu} = t^\mu{}_\alpha t^{*\nu}{}_\beta \eta^{\alpha\beta}$$

and where $\eta^{\alpha\beta}$ is the flat space–time metric. The imaginary part of the metric tensor $g_{\mu\nu}$ (which corresponds to the electromagnetic field in the $k \to 0$ limit) is then just the appropriate antisymmetric combination of the tetrad and its complex conjugate.

Fortunately, Newman was able to first 'test' his transformation on the Schwarzschild metric (to produce the Kerr metric) before applying it to the Reissner–Nordström metric. Because there is, as yet, no analogous solution for the nonsymmetric theory, there is no guarantee that a transformation for $k \neq 0$ would produce the correct result, even though it reduced to the proper electromagnetic tensor in the limit $k \to 0$. On account of this uniqueness problem, we will confine our attention to the $k \to 0$ limit. In this limit, for both the real nonsymmetric and Hermitian cases, $g_{[\mu\nu]}$ decouples from $g_{(\mu\nu)}$ in the field equations. Equation 14 suggests that we represent $g^{[\mu\nu]}$ by the antisymmetric combinations of the same tetrad as represents the symmetric part. Working in the real nonsymmetric case (the same arguments apply for the Hermitian tensor), we find

[15] $$-(1/k)g^{[\mu\nu]} = (Q/2r^2)[(l^\mu n^\nu - l^\nu n^\mu) - (a^\mu b^\nu - a^\nu b^\mu)] + \text{complex conjugate}$$

where we have included the complex conjugate to insure that $g^{[\mu\nu]}$ remains real (see also Newman and Janis (1965)). The only nonzero elements of $\mathsf{F}^{\mu\nu} = (1/k)g^{[\mu\nu]}$ are

[16] $$F^{14} = -F^{41} = -Q/r^2$$

The coordinate r is now allowed to be complex. To produce the tetrad for the electromagnetic tensor of the rotating charge, we apply the same transformation as before to l^μ, n^μ, and a^μ. To find $\boldsymbol{b}'^\mu$, we apply [11] and [13] plus the complex conjugate of [12]. Thus, we have

$$l'^{\mu} = \delta_1{}^{\mu}$$
$$n'^{\mu} = \delta_4{}^{\mu} - (1/2)\Delta'(r)\delta_1{}^{\mu}$$
$$[17] \quad a'^{\mu} = (1/\sqrt{2}r)(ia \sin\theta(\delta_4{}^{\mu} - \delta_1{}^{\mu}) + \delta_2{}^{\mu} + (i/\sin\theta)\delta_3{}^{\mu})$$
$$b'^{\mu} = (1/\sqrt{2}\bar{r})(-ia \sin\theta(\delta_4{}^{\mu} + \delta_1{}^{\mu}) + \delta_2{}^{\mu} - (i/\sin\theta)\delta_3{}^{\mu})$$

where

$$[18] \quad \Delta'(r) = 1 + m\left(\frac{1}{r} + \frac{1}{\bar{r}}\right) + \frac{4\pi Q^2}{r\bar{r}}$$

and where $\bar{r}$ is the complex conjugate of r. As might be expected from [14], the vector b'^{μ} is no longer the complex conjugate of a'^{μ} since the complex tetrad formalism allows for four vectors plus their complex conjugates. We now replace r and θ in [17] by the new coordinates of [11] and [13], and then demand that r' be real. Substituting the resultant tetrad into [15], we find the only nonzero elements of $\mathsf{F}^{\mu\nu}$ above the diagonal are

$$F^{13} = -\frac{Qa}{\rho^6}\left(r^2 - a^2\cos^2\theta\right)$$
$$F^{14} = -\frac{Q}{\rho^4}\left(r^2 - a^2\cos^2\theta\right) \times \left(1 + \frac{a^2\sin^2\theta}{\rho^2}\right) \quad [19]$$
$$F^{23} = \frac{2Qar\cos\theta}{\rho^6\sin\theta}$$
$$F^{24} = \frac{2Qa^2r\sin\theta\cos\theta}{\rho^6}$$

where

$$[20] \quad \rho^2 = r^2 + a^2\cos^2\theta$$

and where we have dropped the primes from the new coordinates.

4. Conclusion

The elements of the tensor $\mathsf{F}^{\mu\nu}$ given by [19] are just those of the electromagnetic field of a rotating charged mass (Misner *et al.* 1973). A similar transformation could also be applied to the Hermitian case to obtain the same tensor. Whether this transformation can be applied to the Hermitian case for nonzero k has yet to be established, since there may be several different formulations which reduce to [19] when k vanishes.

Despite the striking similarity between the formal procedures for finding $g^{(\mu\nu)}$ and $g^{[\mu\nu]}$ in the limit of vanishing k, neither Newman's (Newman *et al.* 1965; Newman and Janis 1965) prescription for the symmetric part nor the prescription described here for the antisymmetric part can be rigorously derived as yet. We must await exact solutions to other field configurations before accepting their universality.

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