Pion–pion scattering in a model satisfying crossing symmetry, analyticity, and approximate unitarity[1]

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Phase shifts for π–π scattering are obtained from a model satisfying Mandelstam analyticity, exact crossing symmetry, and approximate unitarity in the range $2m_\pi \leq s^{1/2} \leq 1.3$ GeV. The low energy region is dominated by the ρ, ε, and S^* poles; the $\delta_0{}^0$ phase shift is in good agreement with the data obtained by Protopopescu *et al.* A detailed comparison is made with the available world's data on π–π scattering.

Les déphasages pour la diffusion π–π sont obtenus à partir d'un modèle satisfaisant l'analyticité de Mandelstam, la symétrie de croisement exacte et l'unitarité approximative dans l'intervalle $2m_\pi \leq s^{1/2} \leq 1.3$ GeV. La région de basse énergie est dominée par les pôles ρ, ε et S^*; le déphasage $\delta_0{}^0$ est en bon accord avec les résultats obtenus par Protopopescu *et al.* On fait une comparaison détaillée avec les données disponibles mondialement pour la diffusion π–π.
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1. Introduction

In previous work (Moffat 1971; Curry *et al.* 1971; Moffat and Weisman 1972), we have studied the solutions obtained from a model of meson–meson scattering satisfying the requirements:

(1) Mandelstam analyticity,
(2) crossing symmetry,
(3) resonances in all nonexotic channels (mass spectrum),
(4) Regge behaviour in all channels,
(5) approximate unitarity, and
(6) Adler zero.

The results obtained for π–π scattering below $s^{1/2} \sim 1.3$ GeV result from a mass spectrum determined by just one exchange-degenerate Regge trajectory $\alpha_\rho(t)$ corresponding to the ρ, f^0, g resonances, with a subsidiary spectrum of underlying daughters. Satellite terms are included to guarantee that all ghosts are cancelled. As the Regge trajectories turn over and become constant at very high energies only a finite (but large) number of resonances are contained in the model, and therefore the number of satellites required is also finite.

Recently (Binnie *et al.* 1973) the S^* resonance at $m_{S^*} \sim 997$ MeV has been seen to decay into two pions as well as a $K\bar{K}$ pair. This means that any study of pion–pion scattering should contain a pole in the second sheet corresponding to this resonance. In the following, we shall include a new parent trajectory $\alpha_{S^*}(t)$ in the model, such that the effects of the S^* resonance are taken into account in the scheme. We then find that the resulting phase shifts obtained, when condition (5) is approximately satisfied, agree well with the experimental results of Protopopescu *et al.* (1973). Because of the predicted rapid decrease in the $I = 0$ S-wave absorption parameter $\eta_0{}^0$ just above 1 GeV, we also obtain good agreement with the moment data of Protopopescu *et al.* (1973). These results also confirm the solution, corresponding to the Protopopescu *et al.* (1973) data, found by Basdevant *et al.* (1972, 1973) and also by Pennington and Protopopescu (1973). The scattering length predictions are also in good agreement with the results of Basdevant *et al.* (1973).

It is now clear, as has been suspected all along, that the requirements (1), (2), (4), and (5) are not sufficient to determine the π–π amplitude uniquely. One must, in addition, require a mass spectrum, namely, condition (3). In our approach the spectrum is determined by choosing specific Regge parent and daughter trajectories in the model, *i.e.* we assume beforehand that we know the nature of the angular momentum singularities. Condition (5) is then used to determine all low energy parameters and a unique solution emerges.

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The same methods can be applied to our model to extend the solutions for the phase shifts above $s^{1/2} \sim 1.3$ GeV, but it is suspected that additional threshold cuts may be required to make such an extension successful.

The model considered here has the considerable advantage of possessing global analyticity and crossing properties, a feature not necessarily shared by other model calculations of π–π scattering. Thus all the conditions (1)–(6) are contained within a closed expression for the total amplitude, which is constructed from relatively simple analytic functions. There are no iterations performed at any stage of calculation, so *all* crossing conditions are always satisfied globally during any phase of the calculations.

In Sect. 2, we summarize some of the basic kinematics and amplitudes for π–π scattering. In Sect. 3, the parameters are determined by unitarization and the predictions are compared with the data.

2. Method for Determining the Amplitude

The kinematical conventions we shall use are the same as those in Moffat and Weisman (1972). A summary of the basic properties of the model are given in the Appendix.

The solutions are found by projecting out the partial waves

$$[1] \quad f_l^I(s) = \frac{1}{2}\int_{-1}^{1} \mathrm{d}\cos\theta f^I(s,\theta)P_l(\cos\theta)$$

Here,

$$[2] \quad f^I(s,\theta) = \frac{1}{16\pi s^{1/2}} A^I(s,t,u)$$

are the π–π scattering amplitudes.

From the relation

$$[3] \quad f_l^I(s) = \frac{1}{2iq}[\eta_l^I \exp(2i\delta_l^I) - 1]$$

we can determine the phase shifts. Then η_l^I are the inelasticity parameters, $0 \leq \eta_l^I \leq 1$ and $\eta_l^I = 1$ in the elastic region $4m_\pi^2 \leq s \leq 16m_\pi^2$. Also, δ_l^I is the real phase shift for isospin I in the lth partial wave. From [3] we get

$$[4] \quad \eta_l^I = |1 + 2iqf_l^I(s)|$$

The physical restrictions $\eta_l^I = 1$ in the elastic region and $\eta_l^I \leq 1$ in the inelastic region are found to determine the parameters quite sensitively and to yield solutions in the range $4m_\pi^2 \leq s \lesssim 1.3$ GeV. The parameters are uniquely determined given a definite mass spectrum. Thus the unitarization procedure predicts uniquely a solution of the phase shifts and cross sections once the mass spectrum is provided. An analysis (Zeber 1973) of the stability of the solutions for δ_l^I against small perturbations ($\sim$10–15%) in the η_l^I showed that the phase shifts were insensitive to such variations. Therefore, it is not anticipated that small errors in the unitarity constraints have too large an effect on the stability of the physical solutions.

The mass spectrum we shall assume in the present work is fixed by cancelling all odd-daughter poles, such as the ε, ρ', etc., which were found (Moffat 1971) to have negative residues. However, the poles lie in the second sheet and therefore the satellite coefficients are energy dependent, so that some background remains from the cancellation of the residues. This aspect of the problem plays an important role in the calculation of the $I = 0$ S-wave phase shift in the neighbourhood of the ε pole.

3. Comparison of Results with Experiment

The predicted inelasticity parameters for π–π scattering are shown in Fig. 1. The inelasticities obtained show no violation of the unitarity bound above $16m_\pi^2$ except for a small violation ($\sim$10%) in η_0^0 about 700 MeV. Included in the figure are the experimental results of Baton *et al.* (1970), Protopopescu *et al.* (1973), and Hyams *et al.* (1973). The η_0^0 displays the expected dip structure obtained by both Baton *et al.* (1970) and Protopopescu *et al.* (1973) about 1 GeV. Below this energy, the inelasticities of Protopopescu *et al.* (1973) are defined as 1.0, and so are not included on the diagram. The η_1^1 also show a shallow dip, expected by Baton *et al.* (1970) and, to a lesser extent, by Hyams *et al.* (1973). The η_0^2 are slightly less than 1 for energies less than 2 GeV, but return to 1 as the energy approaches 3 GeV, where the cross sections are slowly rising to an asymptotic limit. The principal partial wave phase shifts are shown in Figs. 2–4. The phase shift for the $I = 0$ S wave follows reasonably closely the experimental results of Protopopescu *et al.* (1973). Like Protopopescu *et al.* (1973), we have included a pole in the $I = 0$ S wave which could be identified with the ε resonance. This pole is the remnants of the first ρ daughter trajectory. However, as with the

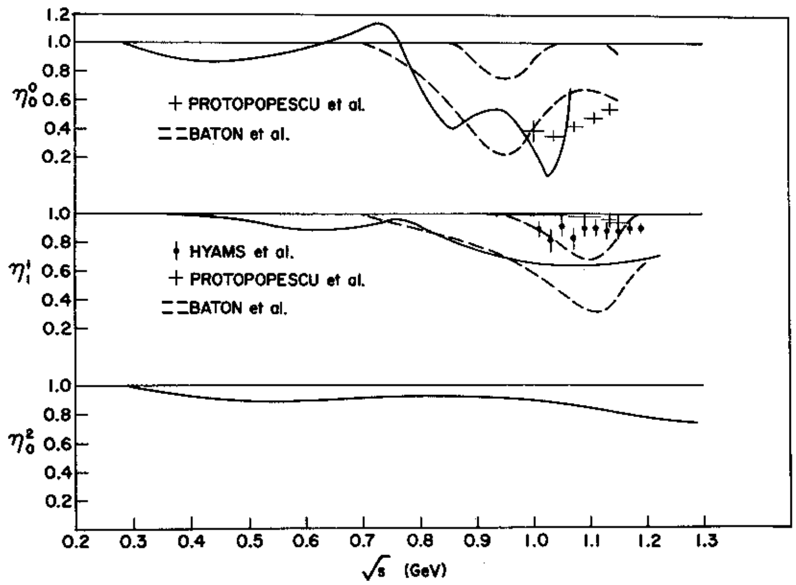

FIG. 1. Absorption parameters (η_l^I) predicted for π–π scattering. Data are from Baton *et al.* (1970), Hyams *et al.* (1973), and Protopopescu *et al.* (1973).

ε pole found by Protopopescu *et al.* (1973) (660 ± 100 MeV), it does not possess the usual characteristic phase shift of $\pi/2$ at the pole position. Rather, δ_0^0 does not pass through 90° until ~850 MeV. The rapid increase to 180° is effected by the S^* trajectory, whose mass and width are found to be

[5] $m_{S^*} = 1020$ MeV; $\Gamma_{S^*} = 50$ MeV

This compares with the S^* pole of Protopopescu *et al.* (1973):

[6] $m_{S^*} = 997 \pm 6$ MeV; $\Gamma_{S^*} = 54 \pm 16$ MeV

Other authors have quoted values of up to 1040 MeV for the S^* mass. In the low energy region, we note that our values of δ_0^0 are lower than the data of Protopopescu *et al.* (1973). Some such disagreement is not unexpected, since Protopopescu *et al.* (1973) force the inelasticity to equal 1.0 up to the $\omega\pi$ threshold, whereas we have allowed the inelasticity to be less than 1. Further, since this data does not include possible systematic errors due to the pole extrapolation from the $\pi^+p \to \pi^+\pi^-\Delta^{++}$ reaction, we include for comparison the phase shifts (down solution) obtained by Baton *et al.* (1970) from $\pi^-p \to \pi^+\pi^-n$, as well as those of Scharenguivel *et al.* (1969), which were obtained from the forward–backward asymmetry parameters (see later).

The δ_1^1 phase shift is in excellent agreement with experiment. Here, we obtain a ρ resonance with

[7] $m_\rho = 765$ MeV; $\Gamma_\rho = 150$ MeV

(the width is defined as the energy interval enclosed by the phase shift passing through 45° and 135°). The width obtained here is somewhat less than that of Protopopescu *et al.* (1973) (~160 MeV) but in agreement with the literature value (150 ± 10 MeV; Particle Data Group (1974)).

The predicted $I = 2$ S wave agrees well with the data of Baton *et al.* (1970), Walker *et al.* (1967), and Colton *et al.* (1971). The $I = 0$ D wave is in agreement with the data of Baton *et al.* (1970), but rises less steeply than the phase shift obtained by Protopopescu *et al.* (1973).

The elastic cross sections for $\pi^+\pi^-$ and $\pi^-\pi^0$ are calculated from the partial-wave decomposition summing over the principal partial waves, and are shown in Figs. 5 and 6. The $\pi^+\pi^-$ calculations are in good agreement with the data of Baton *et al.* (1970) and Protopopescu *et al.* (1973). The two sets of data are quite close to each other over the main energy range, except for the range from 300 to 600 MeV, where the Baton *et al.* (1970) data is significantly higher than the Protopopescu *et al.* (1973) data. In this

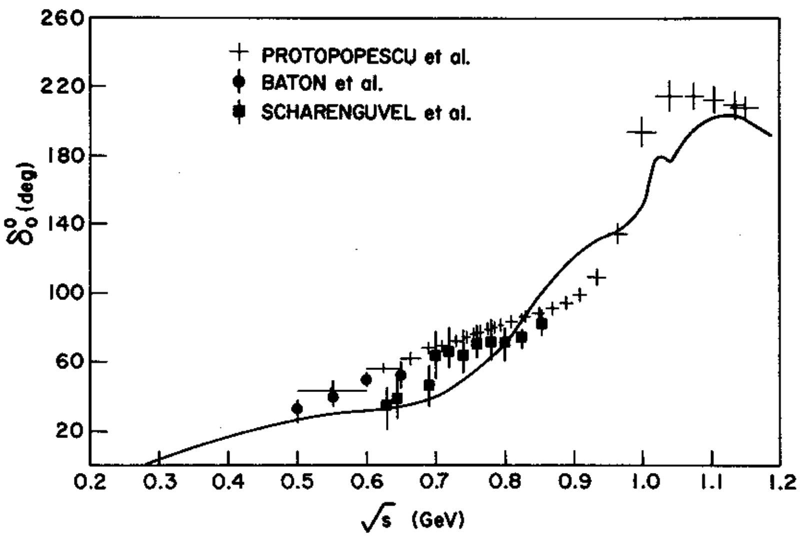

FIG. 2. Phase shift δ_0^0 (notation δ_l^I). Data are from Protopopescu *et al.* (1973), Baton *et al.* (1970) (without absorption), and Scharenguivel *et al.* (1969) (down solution only).

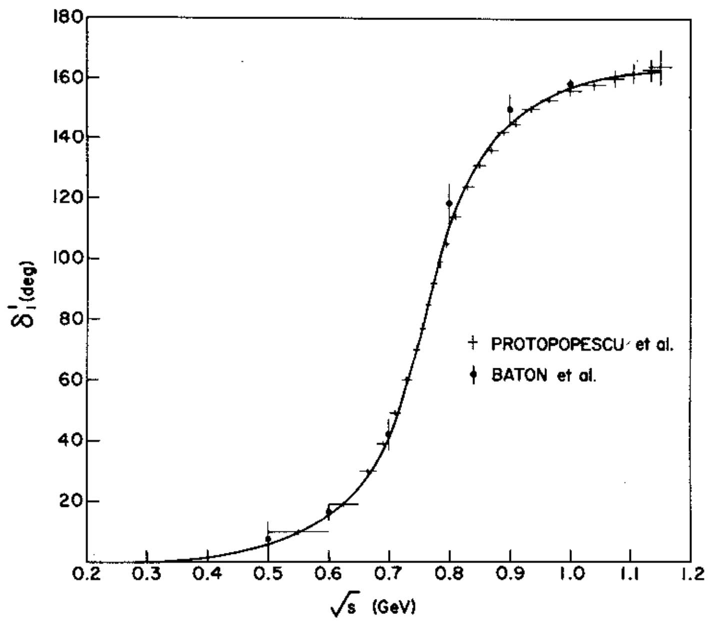

FIG. 3. Phase shift δ_1^1. Data are from Protopopescu *et al.* (1973) and Baton *et al.* (1970) (shown without absorption).

region, we agree with the latter. In the $\pi^-\pi^0$ case, only the data due to Baton *et al.* (1970) is shown and once again there is good agreement.

The charge-exchange cross section $\sigma(\pi^+\pi^- \to \pi^0\pi^0)$ is shown in Fig. 7, as compared with the data of Hagopian *et al.* (1969), Deinet *et al.* (1969), and Sonderegger and Bonamy (1969). Below 700 MeV, there appear to be two distinct sets of data, one having large cross sections ($\sim$15 mb) and one having small ($\sim$5 mb). Our predictions agree with the latter. At about 750 MeV, the data points and our calculations

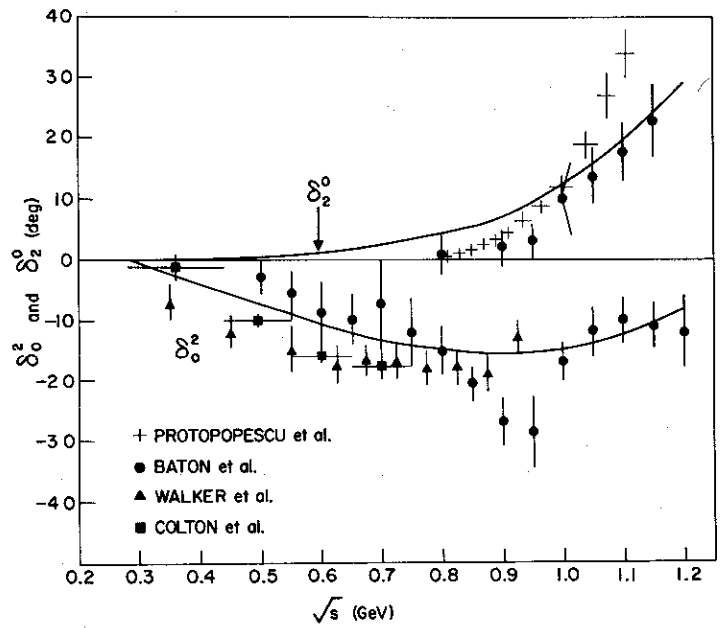

FIG. 4. Phase shifts δ_0^2 and δ_2^0. Data are from Protopopescu *et al.* (1973), Baton *et al.* (1970), Walker *et al.* (1967), and Colton *et al.* (1971).

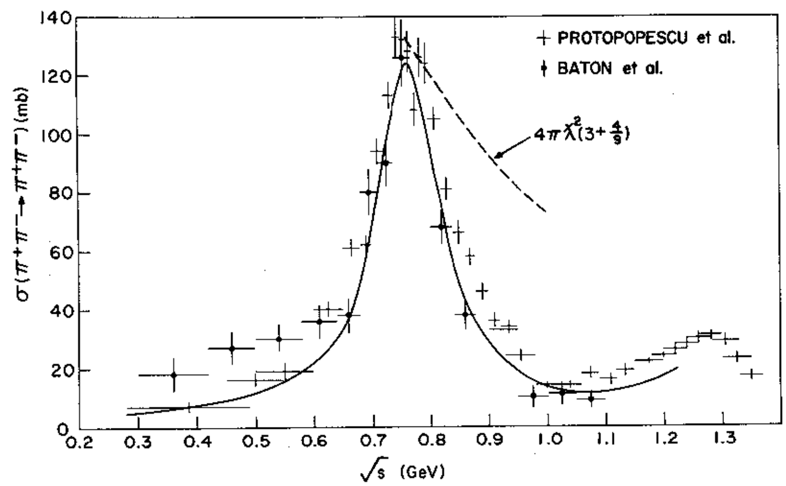

FIG. 5. Elastic cross section for $\pi^+\pi^-$ as a function of centre-of-mass energy. Data are from Protopopescu *et al.* (1973) and Baton *et al.* (1970). Unitarity limit for $S + P$ waves is also shown.

converge to about 8 mb. Above this energy, our calculated cross section falls off like that of Sonderegger and Bonamy (1969), but not so sharply as that of Hagopian *et al.* (1969).

We can obtain total cross sections quite handily by means of the optical theorem. This gives a relation between the total cross section for a given initial configuration of particles and the imaginary part of the scattering amplitude in the forward direction, *i.e.*

$$[8] \qquad \sigma_{TOT}(s) = \frac{4\pi}{q}\,\mathrm{Im}\, f(s, 0)$$

The total and elastic cross section for $\pi^+\pi^-$

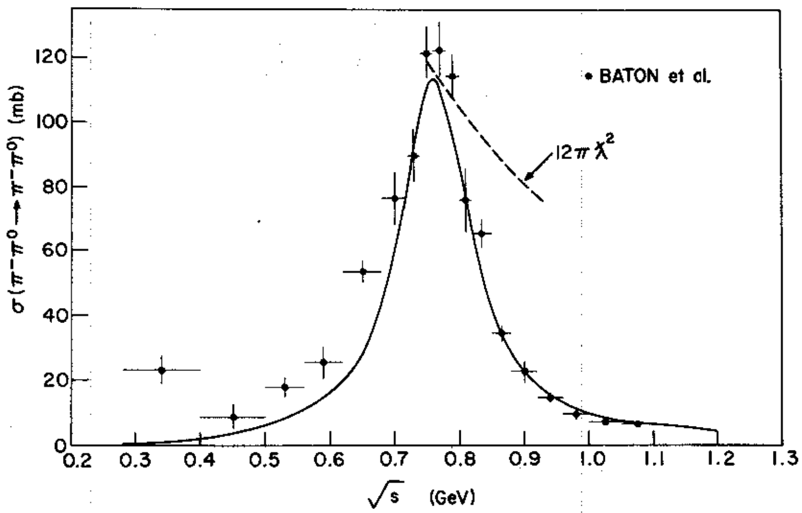

FIG. 6. Elastic cross section for $\pi^-\pi^0$. Data are from Baton *et al.* (1970). Unitarity limit for *P* wave is also shown.

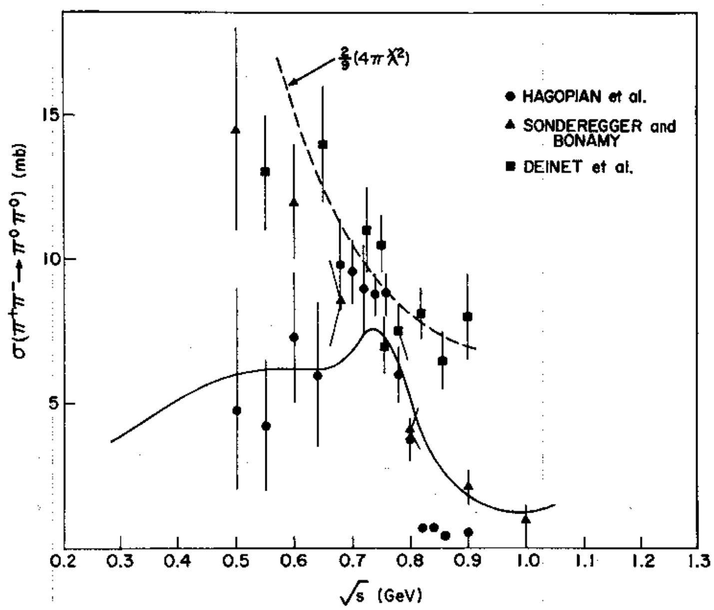

FIG. 7. Charge exchange cross section $\sigma(\pi^+\pi^- \to \pi^0\pi^0)$. Data are from Hagopian *et al.* (1969), Deinet *et al.* (1969), and Sonderegger and Bonamy (1969).

scattering above 2 GeV is shown in Fig. 8. By comparison, we show the elastic cross section of Caso *et al.* (1971), and their estimated total cross section, obtained by assuming the amplitude to be purely imaginary above 1.2 GeV.

The predicted total cross sections for $\pi^0\pi^0$ and $\pi^+\pi^+$ scattering are shown in Fig. 9. The total cross section for $\pi^0\pi^0$ scattering is, of course, not reliably known. It is purely $I = 0$ and $I = 2$ and, therefore, shows the expected enhancements due to the ε and S^* poles. As the Pomeron becomes dominant at high energies, the cross section rises to the asymptotic value of ~ 25 mb obtained with all 2π processes. The $\sigma(\pi^+\pi^+)$ is seen to be

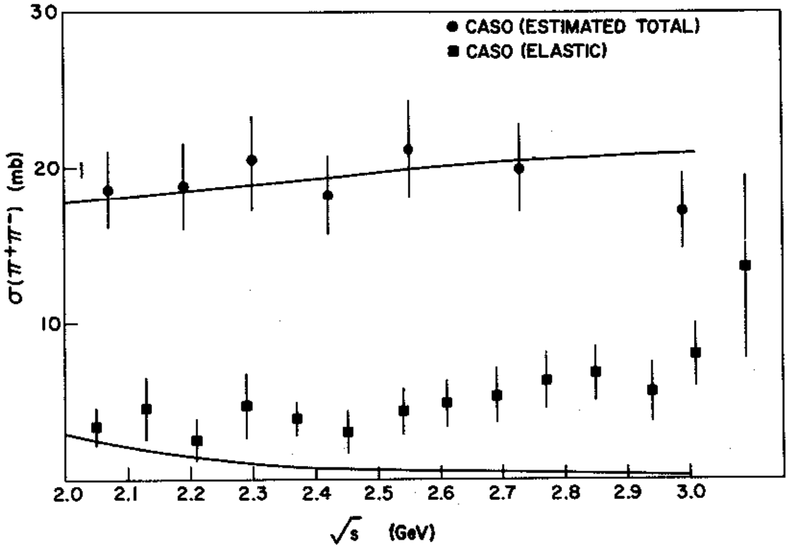

FIG. 8. Elastic and total cross sections for $\pi^+\pi^-$. Data are from Caso *et al.* (1971). Total cross section estimated by Caso *et al.*

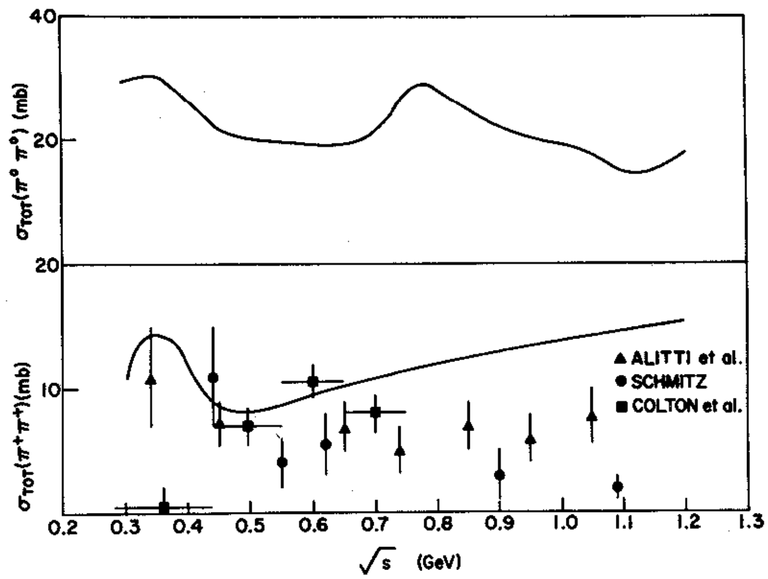

FIG. 9. Total cross sections for $\pi^0\pi^0$ and $\pi^+\pi^+$. Data are from Alitti *et al.* (1965), Schmitz (1964), and Colton *et al.* (1971).

in reasonably good agreement with the data of Alitti *et al.* (1965), Schmitz (1964), and Colton *et al.* (1971), although the data would appear to approach a rather lower asymptotic limit than the one we do.

In Fig. 10, we show our prediction of the forward–backward asymmetry for $\pi^+\pi^-$ on the mass shell. We obtain these values from the equation

$$[9] \qquad \frac{F - B}{F + B} = \frac{3\mathrm{Re}\,[f_1(s)f_0^*(s)]}{|f_0(0)|^2 + 3|f_1(s)|^2}$$

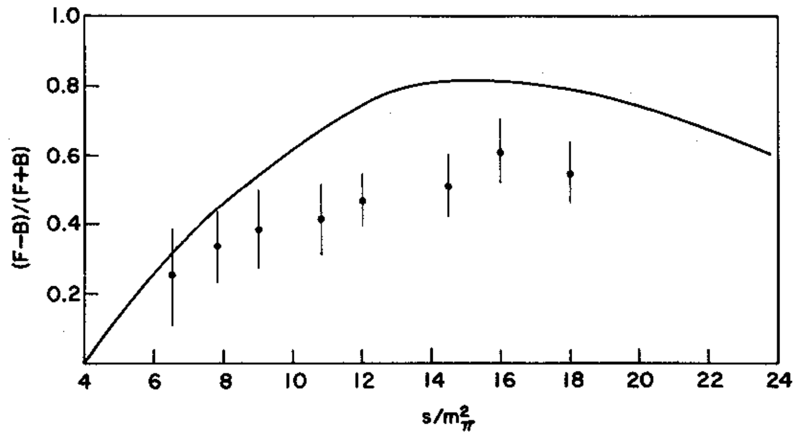

FIG. 10. Forward–backward asymmetry parameters $(F - B)/(F + B)$ for $\pi^+\pi^-$ on the mass shell. Data are from Scharenguivel *et al.* (1970).

where we have neglected D and higher waves. The data of Scharenguivel *et al.* (1970) are shown for comparison.

Next, we show our prediction for the $\pi^+\pi^-$ moments in Figs. 11 and 12. The expression for the moments is given by

[10]
$$Y_1{}^0 = \left[\left(\frac{3}{\pi}\right)^{1/2} \mathrm{Re}\,(S^*P) + 2\left(\frac{3}{\pi}\right)^{1/2} \mathrm{Re}\,(P^*D)\right]\frac{4\pi\lambda\!\!\!^{-2}}{\sigma_{\pi\pi}}$$
$$Y_2{}^0 = \left[\frac{3}{(5\pi)^{1/2}}|P|^2 + \left(\frac{5}{\pi}\right)^{1/2} \mathrm{Re}\,(S^*D) + \frac{5}{7}\left(\frac{5}{\pi}\right)^{1/2}|D|^2\right]\frac{4\pi\lambda\!\!\!^{-2}}{\sigma_{\pi\pi}}$$

where

[11] $$\sigma_{\pi\pi} = 4\pi\lambda\!\!\!^{-2}(|S|^2 + 3|P|^2 + 5|D|^2)$$

(We neglect F and higher waves.)

The $Y_1{}^0$ moment shows all the gross features of the experimental data of Protopopescu *et al.* (1973). The most important of these is the rapid drop in value about the S^* pole, and the subsequent recovery at about 1.2 GeV. The drop reaches a minimum of about −0.13, which is lower than the minimum of −0.05 indicated by the data. At lower energies, we find our prediction to be somewhat higher than the data, again partly due to our freeing of the inelasticity as described in the discussion of $\delta_0{}^0$. Figure 12 shows the $Y_2{}^0$ moment. Agreement over the entire energy range appears to be good. With both moments, we find an asymptotic value of about 0.3 (excluding F waves).

Finally, summing up the energy dependent behaviour of the $I = 0$ S wave, we show an Argand plot of our predictions up to 1.1 GeV in Fig. 13.

We find the S-wave threshold parameters to be

[12] $$a_0 = 0.22\,m_\pi{}^{-1};\quad a_2 = -0.039\,m_\pi{}^{-1}$$

Since our value of a_2 is rather smaller than the current algebra result of $-0.06\,m_\pi{}^{-1}$, we are led to a larger value of the scattering length ratio

[13] $$a_0/a_2 = -5.7$$

compared to Weinberg's (1966) value of −3.5. For comparison with experimental data, the reader is referred to Fig. 14. We show both the range of a_0 and a_2 allowed by experiment (error bars are omitted) and also the solution bands allowed by Basdevant *et al.* (1972), as well as Le Guillou *et al.* (1971).

4. Conclusion

We have succeeded in finding solutions for π–π scattering in a model satisfying Mandelstam analyticity, exact crossing symmetry, and approximate unitarity up to ~1.2–1.3 GeV. The results that we have obtained are in good agreement with the available data, both phase shifts and cross sections. Further, the asymptotic values for the total cross section are quite

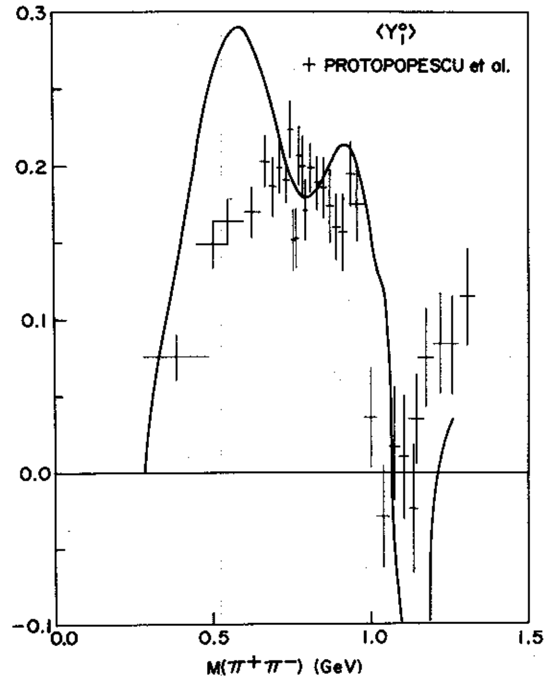

FIG. 11. Moment Y_1^0 for $\pi^+\pi^-$. Data are from Protopopescu *et al.* (1973).

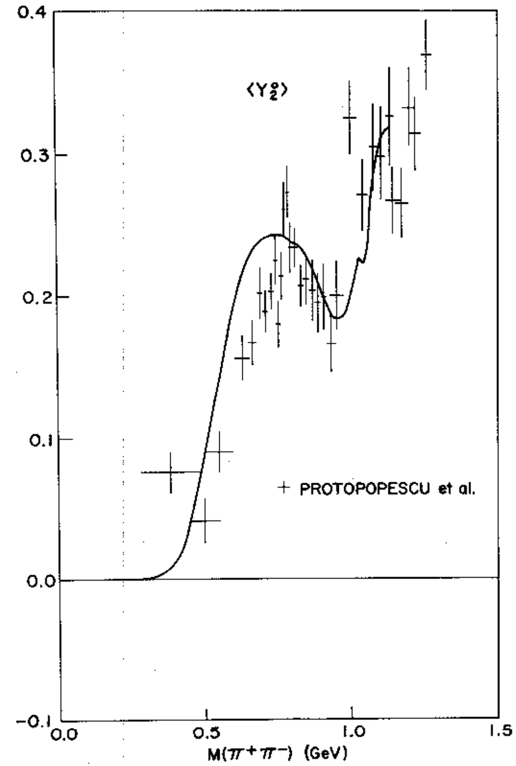

FIG. 12. Moment Y_2^0 for $\pi^+\pi^-$. Data are from Protopopescu *et al.* (1973).

reasonable, and are found to rise with energy to their asymptotic limit, much in the same way as most data for high energy cross sections now appear to be doing. In the low energy region, once we have imposed the mass spectrum and unitarity constraints on the model, we can determine the parameters of the model from the $I = 0$ and $I = 1$ phase shifts, inelasticities, and cross sections. This allows us to predict the $I = 2$ amplitude and the scattering lengths. In particular, the model provides a unique prediction of the off-mass-shell extrapolation required to obtain the $I = 0$ and $I = 2$ scattering lengths.

The main aberration from the phase shifts of Protopopescu *et al.* (1973) is in the $I = 0$ S wave at low energy. The discrepancy may be partly experimental, in that the inelasticity parameter is fixed at 1.0, and this may result in perhaps a 10% change in the experimental values of the phase shift. However, this would not account for all of the observed difference, and for this we must look more deeply into both the extrapolation procedure for extracting the data, and into the model itself. We leave discussion of the former aside as beyond the bounds of this paper.

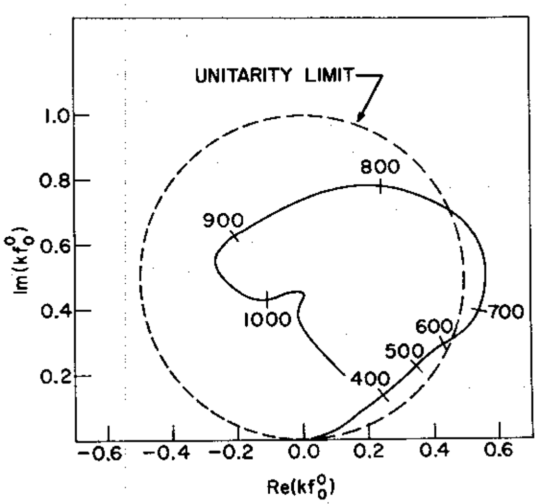

FIG. 13. Argand plot of $I = 0$ S wave calculated from model.

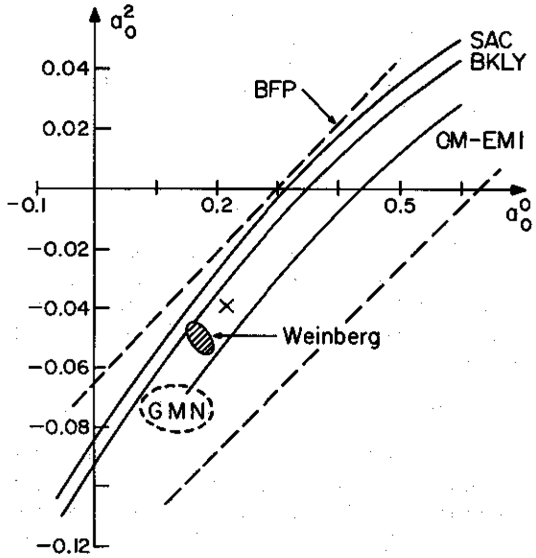

FIG. 14. Plot of a_0^2 vs. a_0^0. Dashed lines indicate boundary of solutions obtained by Basdevant *et al.* (1972) (BFP) and Le Guillou *et al.* (1971) (GMN). The Weinberg (1966) predictions are also included. The full lines indicate universal curves correlating the scattering lengths for the Saclay (SAC), Berkeley (BKLY), and CERN-Munich (CM-EM1) phase shifts, from Basdevant *et al.* (1973). The calculation of the model is marked with a cross. Data are from Baton *et al.* (1970) (SAC), Protopopescu *et al.* (1973) (BKLY), and Grayer *et al.* (1973) (CM-EM1).

As for the latter, a clear defect of the model is that η_0^0 is predicted to be less than unity, even in the elastic region. The form of the double spectral functions is not correct in the elastic region (Moffat 1971) due to the spectral function boundaries being approximated by step functions. This could account for the defects in the elastic region. We do not anticipate any major corrections to our predicted values for the scattering lengths due to errors in the elastic unitarity conditions. In the model, we have included only the 4π cut in the analytic structure, and have omitted the 6π, 8π, 10π, ... and $2K$, $4K$, ... cuts as contributing only in a small way. It now appears (Protopopescu *et al.* 1973) that coupling to the $K\bar{K}$ channel is of some importance (although we have been able to obtain most of the gross features of this coupling in the present model) and inclusion of the $K\bar{K}$ cut might provide enough freedom to allow a small change in the amplitude at low energies.

From the model, we learn from examining the relative values of γ_ρ and γ_{S^*} that the S^* is much more weakly coupled to the $\pi\pi$ system than the ρ meson is. This is compatible with evidence found by Nagels *et al.* (1973) and Binnie *et al.* (1973).

In summary, we have solved a model which is exactly crossing symmetric and possesses satisfactory analyticity properties. However, it must be unitarized phenomenologically, and must, like most models, have the mass spectrum inserted at the outset.

Acknowledgments

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Appendix

We shall summarize some of the basic formulas already presented in previous papers (Moffat 1971; Curry *et al.* 1971; Moffat and Weisman 1972). The total π–π amplitude is defined by

[A.1] $$A^I(s, t, u) = F^I(s, t, u) + P^I(s, t, u)$$

where F^I denotes the isospin amplitude corresponding to purely nondiffractive (resonant) contributions, while P^I denotes the Pomeranchukon amplitude. The amplitudes of definite isospin for π–π scattering are determined in the s channel by

[A.2] $$\begin{aligned} A_S^{I=0} &= \tfrac{3}{2}[F(s,t) + F(s,u)] - \tfrac{1}{2}F(t,u) + A_P(t,s) + A_P(t,u) + A_P(u,s) \\ &\quad + A_P(u,t) + 3[A_P(s,t) + A_P(s,u)] \\ A_S^{I=1} &= F(s,t) - F(s,u) + A_P(t,s) + A_P(t,u) - A_P(u,s) - A_P(u,t) \\ A_S^{I=2} &= F(t,u) + A_P(t,u) + A_P(t,s) + A_P(u,s) + A_P(u,t) \end{aligned}$$

The resonant amplitude $F(s, t) = F(t, s)$ is given by

[A.3] $$\begin{aligned} F(s,t) = {} & -\gamma_\rho(s)\Gamma(1-\alpha_\rho(s))(w_\rho(t)^{\alpha_\rho(s)} + \{d_1 + d_2[1-\alpha_\rho(s)] + d_3[2-\alpha_\rho(s)][1-\alpha_\rho(s)]\} \\ & \times w_{1\rho}{}^{\alpha_\rho(s)-1} + d_4[1-\alpha_\rho(s)]w_{1\rho}(t)^{\alpha_\rho(s)-2}) + \gamma_{S^*}(s)\Gamma(-\alpha_{S^*}(s))w_{S^*}(t)^{\alpha_{S^*}(s)}[s \leftrightarrow t] \end{aligned}$$

where we have omitted S^* satellite terms since the S^* daughters are not of significant size.

The amplitude $A_P(t, s) \neq A_P(s, t)$ describes the Pomeranchukon contribution. The trajectories $\alpha_{\rho,S^*}(s)$ are determined by the real analytic function

[A.4] $$\alpha_i(s) = a_i + \frac{b_i s - c_i(4m_\pi^2 - s)^{1/2}}{\{1 + [(4m_\pi^2 - s)/\Delta]^{1/2}\}^2}$$

where $i = \rho, S^*$. The parameters of the ρ and S^* trajectories are obtained from the conditions $\mathrm{Re}\,\alpha_\rho(m_\rho^2) = 1$, $\mathrm{Re}\,\alpha_{S^*}(m_{S^*}^2) = 0$, the widths of the ρ, f^0, S^* resonances, and the Adler condition $\alpha_\rho(m_\pi^2) = 1/2$ to be (where we have taken the slopes of the trajectories to be equal)

[A.5] $$\begin{aligned} a_\rho &= 0.509; & b_\rho &= 0.85 \\ a_{S^*} &= -0.900; & b_{S^*} &= 0.85 \\ c_\rho &= 0.110; & c_{S^*} &= 0.0275 \\ & \Delta_\rho^{1/2} = \Delta_{S^*}^{1/2} = 100 \end{aligned}$$

in GeV units. Also we have

[A.6] $$\gamma_\rho(s) = \gamma_\rho \frac{\alpha_\rho(s) - \frac{1}{2}}{[1 + x_\rho(s)]^{2b\Delta}} \times \exp\{-g_\rho[\alpha_\rho(s) - \alpha_\rho(0)]^2\}$$

[A.7] $$\gamma_{S^*}(s) = \gamma_{S^*} \frac{\alpha_{S^*}(s) - \alpha_{S^*}(m_\pi^2)}{[1 + x_{S^*}(s)]^{2b\Delta}} \times \exp\{-g_{S^*}[\alpha_{S^*}(s) - \alpha_{S^*}(0)]^2\}$$

The constants γ_i and g_i are given by

[A.8] $$\begin{aligned} \gamma_\rho &= 159; & g_\rho &= 1.45 \\ \gamma_{S^*} &= 1.41; & g_{S^*} &= 1.45 \end{aligned}$$

and

[A.9] $$x_i(s) = (4m_\pi^2 - s)^{1/2}/\Delta_i^{1/2}$$

Moreover,

[A.10] $$\begin{aligned} w_\rho(s) &= w_{S^*}(s) \\ &= A + Bs + C(16m_\pi^2 - s)^{1/2} \end{aligned}$$

where $A = 2.72$, $B = -b$, and $C = 0.086$ in GeV units.

The satellite function remains

[A.11] $w_{1\rho}(s) = A_1 + Bs + C(16m_\pi^2 - s)^{1/2}$

with $A_1 = 2.28$.

The satellite coefficients are

[A.12]
$$d_1 = -2.778 - 0.50 \times \exp\{-1.85[f(s) - f(0.98)]^2\}$$
$$d_2 = 2.30 \exp\{-0.41[f(s) - f(1.70)]^2\}$$
$$d_3 = -0.30 \exp\{-0.40[f(s) - f(2.55)]^2\}$$
$$d_4 = -5.15 \exp\{-0.19[f(s) - f(1.65)]^2\}$$

where

[A.13] $$f(s) = \frac{s^2}{\{1 + [(16m_\pi^2 - s)/\Delta]^{1/2}\}^4}$$

The Pomeranchukon amplitude $A_P(t, s)$ is given by

[A.14] $$A_P(t, s) = \gamma_P(t) \ln\left[1 + \left(\frac{4m_\pi^2 - s}{s_0}\right)^{1/2}\right] \times [w_P(s)^{\alpha_P(t)} + d]$$

where

[A.15] $d = -0.345$; $s_0 = 1/|B_P|$

and

[A.16] $w_P(s) = A_P + B_P s + C_P(16m_\pi^2 - s)^{1/2}$

We have

[A.17] $A_P = 0.35$; $B_P = -1$; $C_P = 0.02$

The trajectory $\alpha_P(s)$ is determined by

[A.18] $$\alpha_P(s) = 1 + \frac{bs}{\left[1 + \left(\frac{16m_\pi^2 - s}{\Delta}\right)^{1/2}\right]^2}$$

with $b = 1$. Moreover,

[A.19] $$\gamma_P(s) = \frac{\gamma_P[\alpha_P(s) - \alpha_P(m_\pi^2)] \exp\{-g_P[\alpha_P(s) - \alpha_P(0)]^2\}}{\left[1 + \frac{(16m_\pi^2 - s)^{1/2}}{2\Lambda}\right]^{4b^{1/2}\Lambda}}$$

where

[A.20]
$$\gamma_P = -4000$$
$$g_P = 2\Delta^{1/2} \text{ (in magnitude)}$$
$$\Lambda = \Delta \text{ (in magnitude)}$$

The asymptotic values of the trajectories are $\alpha_\rho(\pm\infty) \simeq \alpha_{S^*}(\pm\infty) \simeq \alpha_P(\pm\infty) \simeq -b\Delta$, and therefore $\gamma_\rho(s) \to 0$, $\gamma_{S^*}(s) \to 0$, and $\gamma_P(s) \to 0$ as $s \to \pm\infty$. The asymptotic properties of the Pomeranchukon and the meaning of the constants A_i, B_i, and C_i and [A.16] were discussed by Moffat (1971).