Realism in Economic Model Building

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In the early 1960s one of my teachers wrote on the blackboard an equation which represented a solution to a complicated economic model. I began asking him what each variable and coefficient meant. We reached a stage where the teacher was unable to explain to me what a particular coefficient represented. The teacher then told me to stop asking this question. I asked why. The teacher said that some aspects of mathematics are inherently 'mysterious' and thus beyond explanation. I said I would not accept this prohibition and certainly would not accept that mathematical analysis is mysterious. My reluctance is easy to explain. I learned my mathematics while I was an engineering student. Mathematics was always a means to an end; it was never an end in itself. Admittedly, in our physics classes, I was often embarrassed by my fellow engineering students who would always object to the teacher using letters instead of actual numbers when discussing some physics problem. I was willing to use letters rather than actual numbers. However, as an economics student, I still retained the predisposition that the use of mathematics must be transparent.

In this paper I will be concerned with how I think we should go about the business of economic model building. When building an economic model, it is important that any temporary mathematical device disappear from the results. The easiest way to accomplish this is to never use mathematical objects that do not represent real world objects or processes. However, strict adherence to this easy proscription would make economic model building very difficult. For example, if we accept this easy proscription, could we even assume the existence of a utility function? Does 'utility' represent a real world object?

I will explain why the use of the concept of utility does not necessarily present a problem and how we can use mathematics in a transparent manner. But first, for future reference, I want to identify two other methodological pronouncements in order to distinguish them from my transparency requirement. One is the 'operationalism' of physics and the other is the 'intuitionism' of mathematics.

Operationalism is the simple-minded methodological program that requires all theoretical terms to be defined only by measurement operations. This program was the motivation behind Paul Samuelson's development of revealed preference analysis. Significantly, that analysis began without reference to 'preferences' or 'utility' on the grounds that we should endeavor to develop economic models without recourse to psychology or circular reasoning [see Samuelson 1938]. What Samuelson advocated was his form of operationalism in the sense that he wished to derive explanations of consumer behavior that involved only directly observable and hence measurable entities. Specifically, he wished to invoke postulates expressed only in terms of observable prices and observable quantities of goods purchased. Rather than the more common assumption of utility maximization, his primary postulate was that for any two sets of observed prices and quantities, the consumer must be consistent. To be consistent, if a consumer, who is facing a set of prices, makes a decision to buy one bundle of goods, A, rather than another affordable bundle, B, then that consumer would buy bundle B only when bundle A is unaffordable. Whether the consumer is consistent in this manner is easily determined by calculating the values of the two purchased bundles. I say easily since in both cases the prices and quantities purchased are directly observable. Armed with this postulate, Samuelson claims that one can derive a Slutsky equation for that consumer and thereby be in a position to explain the consumer's demand behavior.

Intuitionism in mathematics was the program for how one should do mathematics. It is more extreme than operationalism. Intuitionism is based on the following three proscriptions:

1. Do not rely on *tertium non datur* (the axiom of the excluded middle: statements are true or false, there is no third status) in the construction of mathematical proofs;
2. Do not rely on actual infinity in any proof; and
3. Do not rely on any foundation for a proof other than that which can be derived from the 'original intuition' of counting.

Proofs following these rules are called 'constructive proofs'. While intuitionism may appeal to some purists, there are few constructive proofs in economics.

Obviously, these proscriptions of mathematical intuitionism are routinely violated in economic model building. Moreover, many of us think the requirements of operationalism are too strong. As I will show, my transparency requirement merely combines a weak version of operationalism with a mild version of intuitionism.

1. Notions such as utility are unproblematic mathematical tools

The notion of utility is a convenient means of expressing the idea of maximization even though one cannot observe utility. Moreover, its use does not preclude reaching conclusions that do not require reference to the notion of utility. For an elementary example, let us consider a person buying two goods, X and Y, with a fixed budget, B. As usual, we postulate that the consumer has the following utility function, $f(X, Y)$ which is simple mapping from any point in goods-space to a unique scalar U, that is

$$U = f(X, Y) \qquad [1]$$

Note that this equation is not just an arbitrary mathematical function but it intentionally represents the idea that by consuming particular quantities of X and Y the consumer obtains a particular quantitative level of satisfaction indicated by the scalar U. As usual, we next say that the consumer's budget is entirely spent for the given prices of the goods, P_X and P_Y, that is, the budget is allocated as follows

$$B = P_X{\cdot}X + P_Y{\cdot}Y \qquad [2]$$

Finally, we claim that the consumer chooses a pair of quantities, X^* and Y^*, such that U is maximum. From this we can conclude that such a consumer will purchase a pair of quantities where the ratio of the given prices will equal the 'marginal rate of substitution' (MRS). Note Samuelson rejected the use of MRS if it is interpreted as the slope of an indifference curve. But, the term MRS can easily be considered shorthand for the answer to the following question that hypothetically can be put to the consumer. 'If you are given one less unit of X (ΔX = 1), how many units of Y (ΔY = ?) would I have to give back to you in order that your level of satisfaction is the same?' Using this terminology, we can express our conclusion as follows. The consumer purchases X and Y such that

$$P_X \,/\, P_Y = \Delta Y \,/\, \Delta X \qquad [3]$$

Note that our conclusion makes no reference to either utility or the marginal rate of substitution. Moreover, everything in equation [3] is observable *in principle*. That is, we do not see the two ratios; rather, we see the numerators and denominators which are in principle all observable quantities, although the ΔX and ΔY are the results of a thought experiment and not actually a direct observation.

We justify our conclusion by noting the primary necessary condition for maximization: if U is maximized, then $dU/dX = 0$. And since U depends on Y as well as X, and X and Y are directly related by equation [2], we can calculate dU/dX using the partial derivatives $\partial U/\partial X$ and $\partial U/\partial Y$, neither of which is directly observable. Thus the necessary condition for maximizing U subject to equation [2] is

$$dU/dX = \partial U/\partial X + \partial U/\partial Y{\cdot}(-P_X/P_Y) = 0 \qquad [4]$$

which, using the unobservable partial derivatives, implies

$$\partial U/\partial Y{\cdot}P_X = \partial U/\partial X{\cdot}P_Y \qquad [5]$$

We also note, using the same unobservable partial derivatives, that the definition of MRS implies

$$\partial U/\partial Y{\cdot}\Delta Y \equiv \partial U/\partial X{\cdot}\Delta X \qquad [6]$$

Using equations [5] and [6], we can eliminate the unobservable partial derivatives and thereby derive our conclusion, equation [3]. And, by the way, given the elimination of the unobservable partial derivatives, we are left with a statement that is in principle refutable and testable.

This example also illustrates that I am not strictly following the proscriptions of intuitionism. Partial derivatives are usually defined using some notion of infinity or infinitesimals. Although infinitesimals are not necessary for some alternative proofs of equation [3], the use of the mathematical notions of infinity or infinitesimals are used usually for *convenience*. Furthermore, many algebraic manipulations used to rearrange equations are themselves proven using indirect proofs based on an invocation of *tertium non datur*. If we want the convenience of using calculus and algebra, it might be very difficult to strictly follow the proscriptions of intuitionism.

2. Convenience vs realism: the Lagrange multiplier

So, if utility is not problematic, what mathematical notions are? One problematic notion is the Lagrange multiplier. The Lagrange multiplier is a clever device used to transform the constrained maximization problem represented by equations [1] and [2] into an unconstrained, one-equation maximization problem. To understand the Lagrange multiplier tool, note that the dimension in which both sides of equation [1] are measured is units of utility, so-called utils. Similarly, both sides of equation [2] are measured in units of money. So, how can we combine two equations that are measured in different dimensional units and get one equation for which each side is measured in the same dimensional units. The trick is to define a coefficient that transforms dollars into utils. That is, let us define a new function, $V = g(X, Y)$, which is the sum of equations [1] and [2] while recognizing also that [2] represents the exhaustion of the budget. That is,

$$V = g(X, Y) = f(X, Y) + \lambda(B - P_X X + P_Y Y) \qquad [7]$$

where λ is the coefficient of transformation from money-space to util-space. With equation [7] the maximization problem becomes merely one of maximizing V by treating λ as an additional variable. Thus, the necessary conditions for maximization of V are

$$dV/dX = \partial U/\partial X - \lambda P_X = 0 \qquad [8a]$$
$$dV/dY = \partial U/\partial Y - \lambda P_Y = 0 \qquad [8b]$$
$$dV/d\lambda = B - P_X X + P_Y Y = 0 \qquad [8c]$$

Equation [8c] is the equivalent of equation [2]. Using equations [8a] and [8b] and eliminating λ, we get equation [5]. Again, using the definition of MRS (i.e., the identity [6]), we get our original conclusion. As Samuelson noted regarding the Lagrange multiplier,

> this is an analytical 'trick' whose justification lies in its equivalence with expressions which can be rigorously derived by more roundabout methods. (1947/65, p. 132)

An example of the 'more roundabout' method is merely our original argument using equations [1] to [6].

This 'trick' has its limitations if one tries to treat it as a real variable. One very common attempt is to interpret the λ as the 'marginal utility of money'. The difficulty with this interpretation is that it is inconsistent with the notion which equation [1] purports to represent. Specifically, equation [1] represents the idea that a consumer obtains a particular level of satisfaction by consuming the particular quantities of X and Y. There is no mention of money in this utility function. Surely, one does not eat money; one eats X and Y. In other words, the interpretation of λ as the 'marginal utility of money' does not correspond to a real world entity. A similar mistake is made by those who augment equation [7] by adding some sort of term for the allocation of a limited amount of time and its corresponding transformation coefficient μ which transforms units of time into utils. The μ is then interpreted as the 'marginal utility of time'. Again, this is illegitimate since there is no 'time variable' in equation [1].

3. Convenience vs. purpose: the linear-homogeneous production function

One of the most convenient tools for analyzing the production function of a perfectly competitive firm (or of an economy) is the notion of a 'linear-homogeneous production function'. Consider a production function for good X which has as its inputs, labor L and capital K: $X = h(L, K)$. If $h(L, K)$ is a linear-homogeneous production function, then we can say that at every combination of quantities of inputs, if inputs are doubled, output will double. Moreover, at every such input point, Euler's theorem holds. That is,

$$X = MPP_L \cdot L + MPP_K \cdot K \quad \text{at all } L, K \text{ and } X = h(L, K) \qquad [9]$$

where the MPPs are the respective marginal products, $\partial X/\partial L$ and $\partial X/\partial K$. If we assume that equation [9] holds and we also assume the firm is maximizing profit with respect to both L and K, there would never be a role for competition. To see this, consider the conditions for profit maximization where profit is the excess of revenue over cost.

If the input prices are P_L and P_K, then for profit to be maximum for the amount of labor used, the following two equations must be true:

$$MPP_L = P_L/P_X \qquad [10a]$$

$$MPP_K = P_K/P_X \qquad [10b]$$

The role of competition in our textbooks is to enable us to say that free entry into any industry assures that in the long run revenue will not exceed cost. The reasoning is simply that positive profit is an incentive for new firms to enter into the profit-making industry and to exit less profitable industries. Advocates of the competitive market system (as a basis for organizing society) rely heavily on this reason. They say that by relying on the competitive market system we are assured that not only will excess profits not be made, but prices will be the lowest possible (profit being zero but maximized implies the lowest average cost since the price is average revenue). Moreover, which industries are making profits depends on the demands of consumers. But is the competitive market system necessary for this result in models where productions functions are linear-homogeneous? No, it is not. To see this, consider the definition of (total) profit, *TP*, measured in units of *X*:

$$TP \equiv X - (P_L/P_X)\cdot L - (P_K/P_X)\cdot K \qquad [11]$$

Thus if $TP = 0$, then

$$X = (P_L/P_X)\cdot L + (P_K/P_X)\cdot K \qquad [12]$$

If we say equations [9], [10a] and [10b] are all true, then we can use equations [10a] and [10b] to substitute for MPP_L and MPP_K in equation [9] and thereby obtain equation [12] *without assuming the existence of competition*.

The lesson to be learned from this is that if you truly believe that the competitive market system is necessary to achieve important social objectives such as zero profit and lowest possible prices, then you should avoid building models which presume equation [9] necessarily holds. Unfortunately, linear-homogeneous production functions are commonplace in international trade models. And the primary, and possibly only, reason for using such an assumption is that it makes the mathematical analysis much easier. For those economists advocating a competitive market system, such an assumption may even be counter-productive.

4. When the middle must be excluded: Bayesian Econometrics

While the demands of the intuitionist mathematicians are rather extreme, there are some places where their shoes might be seen to pinch economists' feet. One concerns the prohibition of the reliance on the so-called Axiom of the Excluded Middle (*tertium non datur*). What is excluded by excluding *tertium non datur*? This axiom says that to be admissible into a logical argument, a statement must be true or false. That is, there is no third status such as a probability value. In addition to *tertium non datur*, Aristotle's axioms of logic also include the Axiom of Non-contradiction. This axiom says that to be admissible, a statement cannot be *both* true and false. Used together, these two axioms say admissible statements are *either* true *or* false (and not both). It is this combination that allows for indirect proofs (or proof by contradiction). An indirect proof proceeds by stating a set of given assumptions, A_1 to A_N, all of which are taken to be absolutely true. The question of the proof asks whether a specific conclusion C follows from those assumptions. The technique begins by assuming C is false and then proceeds to show that this leads to a contradiction. Since contradictions are not allowed by the Axiom of Non-contradiction, C cannot be false. By the *tertium non datur* axiom, the only other possibility is that C is true. Hence the indirect proof of C.

Now, most econometricians directly contradict the *tertium non datur* axiom. Any argument or proof that involves an assumption that is considered neither absolutely true nor absolutely false cannot be used to provide an indirect proof. The so-called subjectivist approach to econometrics, for example, Bayesian econometrics, clearly falls into this category. This is not to say that Bayesian econometrics needs to be rejected. But what I am saying is double edged. On the one hand, we cannot use Bayesian econometrics in a model that purports to provide an indirect proof of some important economic proposition or lemma. On the other hand, and more importantly, any economic model which relies on theorems whose assumed truth status can only be indirectly proven, cannot be used with assumptions that violate the *tertium non datur* axiom! This is particularly problematic for those economic model builders who without examination accept theorems provided by sophisticated mathematicians who may not have cared how their theorems are used.

5. When the realism matters: the Inductive Principle

The demand of the intuitionist mathematicians which excludes reliance on infinity as an actual number seems to be completely ignored today. It is a common viewpoint of sophisticated mathematicians that it does not matter what you assume so long as your proof is logically sound. Nevertheless, the idea of infinity remains problematic when building models about real world phenomena.

Clearly, we cannot avoid discussing infinity in economics. The notion of an infinite set is regularly used to prove theorems, lemmas or propositions that are alleged to be true at finite places or times. Nevertheless, there are two problems which we cannot continue to ignore. One problem concerns the use of infinity as an actual number and the method used to avoid that problem. The other concerns how economists model knowledge and learning.

The main problem with the idea of infinity occurs when we treat infinity as a number because we cannot at the same time presume the applicability of the laws of elementary arithmetic. This is usually not a concern for economists as they are more interested in infinite sets rather than infinity as a number. Mathematicians have cleverly avoided the embarrassment of being unable to use arithmetic by using the ‘principle of induction’ (sometimes called ‘mathematical induction’). This principle says, if a given proposition is true for the first case and is true for the Nth case, then it must be true for the N+1 case. It is important to recognize that this is an assumption. It is used for such fundamental concerns as the formal definition of real numbers. Unfortunately, some mathematicians think one can prove that the principle of induction is true. But the proof is, itself, circular as it depends on the formal definition of real numbers!

Where infinity is a problem in economics is in any model that attempts to address the knowledge requirements of the model’s decision makers. It is common for model builders to presume that decision makers learn inductively. They presume this even though the theory of knowledge that would support this presumption is 350 years old and was refuted 200 years ago. This is the well-known problem of induction which has been haunting philosophers for two centuries. In the most general form that economists will understand, the problem with infinity concerns *proofs* that involve integration (or a series summation) over infinite space or time horizons. Learning by induction is just one example of this since it presumes that if one has an infinite amount of time one could prove the truth of one’s knowledge using observations alone (i.e., no assumptions). Of course, all infinite horizon models are open to question.

The lesson here is that we should always remember that an infinite amount of time is an impossible amount of time. Relying on infinity to establish a proof amounts to an admission that the proof is impossible! With this heavy message under our belts, I close with two mundane matters of pedagogy.

6. Model parameters vs. realism: the marginal propensity to consume

Many Principles textbooks discuss the ‘marginal propensity to consume’ as if it is referring to a characteristic of human nature. Consider the elementary textbook’s model of aggregate consumption:

$$C = a + bY \qquad [13]$$

where C is the level of aggregate consumption and Y is aggregate income. Textbooks will tell us that b is a parameter that represents the psychological propensity to spend an extra dollar of income. As a parameter, textbooks use b in their calculations of such things as the investment multiplier or the foreign-trade multiplier. This use leads students to think that somehow we are using measured characteristics of the economy much like physicists use physical constants such as the gravitational constant. The error here is that treating b as a parametric constant fails to indicate that it is a mathematical artifact of our assumption of a linear relationship. What would be the marginal propensity to consume if the relationship were assumed to be quadratic? For example, if

$$C = a + bY + eY^2 \qquad [14]$$

Of course, we could say that the marginal propensity to consume in this case is $b + 2eY$ and thus it is obviously not a constant. But would students be so impressed with our scientific prowess? I think teachers should go to much greater lengths to make clear to students that by discussing parameters of our models we are dealing with our inventions and not with measured natural characteristics of the economy.

7. Notions that depend on the truth of our theory: indifference curves

My final example of misuse of mathematics is the textbook specification of the indifference curve. Now an indifference curve is nothing more than a locus of all points in the consumer's goods-space which yield the same level of utility. That is, the indifference curve is an iso-utility locus. But textbooks say much more than this about the definition of an indifference curve.

Textbooks specify that indifference curves do not cross. They also say indifference curves are convex to the origin. Yet, nothing in the definition of the indifference curve implies such limitations. So, on what grounds do textbooks impose these limitations? Typically, crossing or non-convex indifference curves are rejected because they are inconsistent with our assumption that consumers are 'rational', that is, that consumers are utility maximizers. So, students are thus led to believe that *true* indifference curves are convex and non-crossing.

This is a simple example which again shows how students are misled by teachers and some textbook writers to confuse the properties of models designed to represent our ideas with the nature of the real world. While using the restricted notion of an indifference curve may seem harmless, it causes the explanation of consumer behavior to be circular and thus untestable. To be testable (at least, in principle), the notions used in our expansions must have meanings that are independent of our explanation. The lesson to be learned: Be careful not to impose the *assumed* truth of our assumptions onto the nature of the mathematical tools we choose to employ in our model of our explanation.

8. Is mathematics merely a language?

Many, maybe most, economic theorists who depend heavily on mathematics are very defensive when anyone criticizes how mathematics is used in economics. They usually resort to the view that mathematics is merely a language. And in this sense, the use of mathematical tools is defended by saying that their primary virtue is that they force practitioners to be explicit and at least minimally rigorous.

I think this defense undersells mathematics. Since the time of Marshall and Jevons, mathematics has shaped the nature of economics ideas. And for this reason it is unfortunate that economic theorists today are only interested either in refining and generalizing their models or in finding new and more ingenious ways to model something which we already understand. When economic theorists become more concerned in ensuring that their models and modelling tools make realistic sense, mathematics will once again be in a position to help us to move on to new ideas that might advance our understanding of the economic world.

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