ECON 282, Intro Game Theory, (Fall 2008) Christoph Luelfesmann, SFU

Solution Problem Set 2

Due at the beginning of class on Tuesday, Oct. 7. Please let me know if you have problems to understand one of the questions.

Exercise 1. (easy to medium)

Solve Exercise 42.1 in Osborne. Graph and explain your results.

Answer:

We first find the best response functions for both players. Player i maximizes his utility with respect to his action a_i for given a_j.

For player 1, the first order condition for an interior solution $a_1^*(a_2)$ yields (take the derivative and set it equal to zero)

$$\frac{dU_1(a_1, a_2)}{da_1} = a_2 - 2a_1 = 0.$$

(to find this, use the rules of derivation provided in the assignment). Solving for a_1, this yields the BRF

$$a_1^*(a_2) = a_2/2.$$

For player 1, the first order condition is

$$\frac{dU_2(a_1, a_2)}{da_1} = 1 - a_1 - 2a_2 = 0.$$

Solving for a_2, one obtains the BRF

$$a_2^*(a_1) = \frac{1}{2} - a_1/2.$$

In a Nash equilibrium (a_1^*, a_2^*), we must have $a_1^* = a_1^*(a_2^*)$, and $a_2^* = a_2^*(a_1^*)$, in words, both players' actions are best responses to each other. Hence, (a_1^*, a_2^*)

must satisfy the two conditions $a_1^* = a_2^*/2$, and $a_2^* = \frac{1}{2} - a_1^*/2$. Substituting yields $a_2^* = 2/5$ and $a_1^* = 1/5$.

Exercise 2 (medium).

Solve Exercise 42.2 a) in Osborne. Graph and explain your results.

Answer:

Note that both players split 'output' $f(x_1, x_2)$ evenly, and 'working' causes costs. Hence, each player i's utility function is $U_i = (3/2)x_i x_j - x_i^2$. To find player i's BRF, we derive the first order condition for an (interior) maximum,

$$\frac{dU_i(x_i x_j)}{dx_i} = \frac{3}{2}x_j - 2x_i = 0.$$

Solving for x_i, this yields the BRF

$$x_i^*(x_j) = \frac{3}{4}x_j.$$

It should be immediately clear that the conditions $x_1^* = (3/4)x_2^*$ and $x_2^* = (3/4)x_1^*$ hold only for $x_1^* = x_2^* = 0$, which is the unique NE. In equilibrium, no player puts in any effort and receives a utility level of zero.

To show that the equilibrium outcome is worse than the outcome with other action profiles (x_1, x_2), consider $x_1 = x_2 = 1$ where both players 'work hard'. Utilities are then $U_i(1,1) = 3/2 - 1 = 1/2$ which is clearly larger than the equilibrium payoffs. (Note that this game 'resembles' the Prisoner's dilemma, even though players do not really have dominant strategies. Still, for any 'cooperative' effort put in by the other player, each player would like to provide less effort than his partner.)

Exercise 3 (easy to harder).

Consider the linear Cournot model (linear demand $P = \alpha - Q$, constant unit costs c) with 2 identical firms.

(a) (easy) Find the Nash equilibrium (i.e., the Cournot market outcome) when $\alpha < c$, and explain.

Answer: If $\alpha < c$, the largest price that any consumer is willing to pay (which is α) is still lower than the unit costs of production. Hence, no firm will enter the market because no matter how much it produces, price will always be below unit costs.

(b) (medium) Show whether or not the Cournot outcome (as presented in class) yields maximal total profits for the firms in the market (total profits are the sum of firm profits, $\Pi = \Pi_1 + \Pi_2$).

Answer: It does not (as formally shown in part c), see below). Intuition: in the Cournot model, firms compete with each other, which leads to larger output (and lower prices). Reducing output (and hence, raising prices) would boost profits. More formally this is true because both firms cause a 'negative externality' on each other: if firm i raises q_i, it hurts firm j (because price is lowered) but does not care about it is concerned only about its own profits. Because of this externality, equilibrium prices are not profit maximizing.

(c) (harder) Suppose now both firms can secretly collude and agree on a total output level, say Q^*, that maximizes their total surplus (each firm then produces half of that output level.) Which level Q^* will they agree upon, and what are the associated profits for each firm? Are consumers better or worse off compared to the outcome in b)? Explain why the collusive outcome is not a Nash equilibrium in the Cournot game (in which firms cannot make such a binding agreement).

Answer: In order to find Q^*, we must derive the Q that maximizes total profits, $\Pi(Q)$. Those are

$$\Pi(Q) = P(Q)Q - cQ = (\alpha - Q)Q - cQ.$$

The first order condition for the total-profit maximizing Q reads

$$\frac{d\Pi(Q)}{dQ} = \alpha - 2Q - c = 0,$$

or $Q^* = (\alpha - c)/2$. (Note that this is exactly the quantity that each of the duopoly firms produces if its opponent produces nothing. (WHY?) Notice that $Q^* < q_1^c + q_2^c = Q^c$, the total output in the Cournot solution. Since a total output of Q^* maximizes total profits and this profit is shared evenly between both firms (as in the Cournot model), they must both be better off. Moreover, the collusive outcome yields a higher market price, which is bad news for consumers. Fortunately, the collusive outcome is not a Nash equilibrium (i.e., it cannot easily be sustained): given that the other player j produces $q_j = Q^*/2$, firm i's best response is always a quantity larger than $q_i = Q^*/2$.

Exercise 4 (harder).

Please solve exercise 59.2. Show that (for some values of f) there exists more than one Nash equilibrium. Find all these Nash equilibria, and explain your findings. (Hint: note that a firm does not enter the market if - given what the other firm does - it makes losses. The presence of fixed costs thus affects its best response function relative to a situation where no such costs exist).

Answer:

Since fixed costs f do not affect the derivatives of each player's profit function, best response quantities are exactly the same as in a model without fixed costs, with one qualification: firm i may now decide not to enter the market at all. Whether it wants to enter the market depends on what it expects its opponent j to do, because a larger q_j reduces market prices and therefore, firm i's revenues for any given output level. Specifically, a firm's profits decrease along its best response function (that is, the larger q_j, the smaller not only $q_i^*(q_j)$ but also Π_i).

Let $\hat{q}_j$ the output level of firm j for which firm i's profit when playing its best response quantity $q_i^*(\hat{q}_j)$ is zero. (Formally, $\Pi_i(q_i^*(\hat{q}_j), \hat{q}_j) = 0$.) Notice that larger fixed costs f lead $\hat{q}_j$ to be smaller. If j's output exceeds $\hat{q}_j$, firm i certainly prefers not to produce at all in order to avoid fixed costs, so its best response becomes $q_i = 0$. Accordingly, firm i's best response function in presence of fixed costs f is as in the standard Cournot model for $q_j < \hat{q}_j$, and it is $q_i^*(q_j) = 0$ for $q_j \geq \hat{q}_j$.

This new issue may give rise to new equilibria: suppose f is large enough that $\hat{q}_j < (\alpha - c)/2$. If firm j plays $q_j = (\alpha - c)/2$, firm i's best response is then not to enter the market, i.e., to choose $q_i = 0$. Conversely, for $q_i = 0$, firm j's best response is $q_j^*(0) = (\alpha - c)/2$. Taken together, $(q_i = 0, q_j = (\alpha - c)/2)$ is a Nash equilibrium. (If both firms have fixed costs f, two such equilibria in which only one firm produces coexist). In addition, and if f is such that $\hat{q}_j > q_j^C = (\alpha - c)/3$, the 'standard' Cournot equilibrium $q_1^C, q_2^C)$ also remains an equilibrium: If firm i believes that firm j plays q_j^C, its best response is certainly to play q_i^C, and vice versa. (Note: explaining these things using a graph is actually a lot easier! Please consult your class notes.)