The basic OLG model

David Andolfatto
Wicksell’s triangle and the OLG model

- Recall our Wicksellian model, extended to $J$ periods

  Preferences
  \[
  U_J = \beta c_J + c_1 \\
  U_1 = \beta c_1 + c_2 \\
  U_2 = \beta c_2 + c_3 \\
  \vdots \\
  U_{J-1} = \beta c_{J-1} + c_J
  \]

  Endowments
  \[
  (0, 0, 0, \ldots, y_J) \\
  (0, y_1, 0, \ldots, 0) \\
  (0, 0, y_2, \ldots, 0) \\
  \vdots \\
  (0, 0, 0, \ldots, y_{J-1})
  \]

- We get a “triangle” when $J = 3$; and a “circle” for $J > 3$

- Standard OLG model when $J = \infty$ (now interpreted as sequence of “young” and “old” agents)
• The type \( J = \infty \) agent is referred to as the “initial old”
  
  – care only for date 1 consumption
  
  – own nothing that is valued by other agents

• All other agents are referred to as “future generations”
  
  – generation “born” at date \( t \) cares for consumption at dates \( t \) and \( t + 1 \)
  
  – i.e., they generally want to consume when “young” and “old”
  
  – the old value what the young have, but not vice-versa (LDC)
The CFH (chapter 1) OLG model

• Time \( t = 1, 2, \ldots, \infty \)

• Let \( \{c^y_t, c^o_t\} \) denote consumption at date \( t \) by the young (\( y \)) and old (\( o \)), respectively

• Initial old care only for \( c^o_1 \); a young agent born at date \( t \) has preferences \( u(c^y_t, c^o_{t+1}) \)

• Each young agent has nonstorable endowment \( y \)

• Let \( N_t \) denote population of young at date \( t \) and assume \( N_t = nN_{t-1} \)
The Golden Rule allocation

• Resource constraint

\[ N_t c^y_t + N_{t-1} c^o_t = N_t y \quad \text{for all } t \geq 1 \]

• A stationary allocation \( \{c^y_t, c^o_t\} = \{c^y, c^o\} \) for all \( t = 1, 2, \ldots, \infty \)

• So resource constraint can be rewritten as

\[ c^y + \frac{c^o}{n} = y \]

• The GR allocation is the solution to \( \max u(c^y, c^o) \) subject to above constraint
• Note: GR allocation does not maximize the welfare of the initial old (although, it improves on what they receive in autarky)
  
  – it is the best feasible stationary allocation for future generations

• If \( u \) has all the usual “nice” properties, then the GR allocation is characterized by

\[
MRS(c^y, c^o) = n
\]

\[
c^y + \frac{c^o}{n} = y
\]
• For the linear preferences used in our Wicksell model, \( u = \beta c^y + c^o \), and \( \beta < n \), GR allocation is given by

\[
\begin{align*}
c^y &= 0 \\
c^o &= ny
\end{align*}
\]

• We use the GR allocation as a benchmark
  
  – it is Pareto optimal (although, not uniquely so)

• Question: what might the outcome of this economy be if agents behave noncooperatively?
  
  – assume lack of commitment
A gift-giving economy

- An interesting aspect of the OLG model is the lack of double coincidence
  - I have argued earlier that this is not necessary to explain monetary exchange, but it is nevertheless realistic
  - e.g., I do not pay for my morning coffee by issuing IOUs redeemable economic lectures
  - it is rather interesting to discover how trade may be sustained even in the absence of any bilateral gains to trade
  - how can it be individually-rational to bestow a gift on someone who will never be in a position to reciprocate?
• A cooperative outcome can be sustained as a noncooperative (Nash) equilibrium

• Consider any allocation \((c^y, c^o)\) such that \(u(c^y, c^o) \geq u(y, 0)\) and \(c^y + n^{-1}c^o \leq y\)

  – such an allocation requires that each young person make a gift \(y - c^y \geq 0\) to the old

• Imagine that whether a gift is made or not is part of the public record (costless record-keeping)

• Moreover, imagine that people agree to disentitle any agent who fails to make a gift (tit-for-tat strategy)
Money as a record-keeping device

- The idea of money as a record-keeping device goes back at least to Ostroy (AER 1973)
  - Townsend (AER 1987) refers to a communication device; Kocherlakota (JET 1998) refers to money as memory

- That is, suppose that society cannot observe individuals making gifts; all that can be observed is money holdings
  - then money can substitute for missing memory

- Note: this “memory” function of money cannot be important in bilateral relationships (why?)
A competitive monetary equilibrium

- A competitive equilibrium is an allocation $x$ and price system $\phi$ that satisfy two conditions:
  
  1. Given $\phi$, the choice of $x$ maximizes utility s.t. budget constraint
  2. Given $x$, the price system $\phi$ is consistent with market-clearing

- Let $v_t$ denote the price of money (measured in units of goods) at date $t$
  
  - alternatively, let $p_t = 1/v_t$ denote the price-level (the price of goods measured in units of money)

- A price-system here is given by an infinite sequence $\phi = \{v_1, v_2, v_3, \ldots\}$
Budget constraints and choice problem

• Let \( m_t \geq 0 \) denote the money acquired by a young person; then we have

\[
p_t c_t^y + m_t = pty
\]

\[
p_{t+1} c_{t+1}^o = m_t
\]

• Or, combining...

\[
c_t^y + \Pi_{t+1} c_{t+1}^o = y
\]

where \( \Pi_{t+1} = p_{t+1}/p_t \) is the expected inflation rate (gross)

• Note: \( \Pi_{t+1} = v_{t+1}/v_t \) is the expected real rate of return on money (gross)
• So, conditional on an inflation forecast \( \Pi_{t+1} \), a young person of generation \( t \) chooses \((c^y_t, c^o_{t+1})\) to maximize \( u(c^y_t, c^o_{t+1}) \) subject to budget constraint above.

• For the usual preferences, we have the usual characterization of the solution to this choice problem:

\[
\begin{align*}
MRS(c^y_t, c^o_{t+1}) &= \Pi_{t+1}^{-1} \\
\frac{c^y_t}{c^o_t + \Pi_{t+1} c^o_{t+1}} &= y
\end{align*}
\]

• Note: \( \Pi_{t+1}^{-1} \) is like a relative price here (explain)

• Assume that the initial old are endowed with \( M \) units of money, so \( c^o_1 = v_1 M/N_0 \)
Market-clearing and equilibrium

• Note that the solution above implies a demand for real money balances

\[ q_t = y - c_t^y \geq 0 \]

• At each date, the old are in possession of the money supply and the young wish to acquire it

  – the young will sell some of their output to the old for their money

  – the exchange rate between money and goods is \( v_t \)

  – market-clearing requires that for all \( t = 1, 2, ..., \infty \) we have

\[ v_t M = N_t q_t \]
• If $v_t > 0$ for all $t \geq 1$, then market-clearing condition above implies

$$\Pi_{t+1} = \left[ \frac{N_t q_t}{N_{t+1} q_{t+1}} \right] = \left[ \frac{1}{n} \right] \left[ \frac{q_t}{q_{t+1}} \right]$$

• We restrict attention to stationary equilibrium; i.e., $q_t = q_{t+1} = q$; in which case,

$$\Pi = 1/n \text{ with } v_1 = Nq/M$$

where $q = y - c^y$ satisfies

$$MRS(c^y, c^o) = n$$

$$c^y + \frac{c^o}{n} = y$$

• Note: competitive monetary equilibrium corresponds to GR allocation!
Money neutrality

- Money is said to be *neutral* if allocations are independent of $M$
  - thought experiment is a one-time permanent increase in $M$
  - does anybody (in the model) care?
  - need to distinguish between short-run and long-run neutrality

- Analogy: think of a firm that wants to double its outstanding shares (permanently)
  - key question: what is new share issue used for?
• The standard assumption is that new money is injected by way of a “heli-copter drop”
  
  – a lump-sum transfer of money
  
  – a proportional transfer is always neutral (absent money illusion)

• In our model, a lump-sum transfer of new money $\Delta M$ is neutral in the long-run, but not neutral in the short-run

• Because initial young get some money, they don’t have to work as hard (give up as much output) to acquire money
  
  – redistribution of purchasing power away from initial old to initial young
Example 1: Neutrality

- Consider economy above, with population $N = 1$ and initial $M$ held by initial old.

- Let $(v^*, q^*)$ denote steady-state equilibrium

  $$MRS(y - q^*, q^*) = 1 \text{ and } v^* = q^*/M$$

- Easy to see that money is long-run neutral ($q^*$ does not depend on $M$)

- Is the equilibrium path invariant to changes in $M$ for the initial old?
• Initial old consume \( c_1^o = v_1 M \)

• Initial young demand \( q_1 \) satisfying

\[
MRS(y - q_1, \frac{v_2}{v_1} q_1) = \frac{v_2}{v_1}
\]

• Conjecture that \( v_t = v^* \) for all \( t \geq 2 \) (verify this later); in which case

\[
MRS(y - q_1, \frac{v^*}{v_1} q_1) = \frac{v^*}{v_1}
\]

• Market-clearing implies \( v_1 M = q_1 \) and \( v^* M = q^* \)
• Combining...

\[
MRS(y - q_1, \frac{q^*/M}{q_1/M}q_1) = \frac{q^*/M}{q_1/M}
\]

\[
MRS(y - q_1, q^*) = \frac{q^*}{q_1}
\]

• We know that \(q_1 = q^*\) is a solution

  – \(q_1\) is also independent of \(M\) (short-run neutrality holds)

• Note that increasing \(M\) for the initial old is equivalent here a proportional increase in individual money holdings (remember, the initial young start with zero money balances)
Example 2: Non-neutrality

- Suppose we increase the initial $M$ to $\mu M$, where $\mu > 1$

- We established earlier that if the new money $\Delta M = (\mu - 1)M$ was given to the initial old, money is neutral (the price-level would rise by the factor $\mu$)

- But what if we give $(\mu - 1)M$ to the initial young instead?

- We know that money is neutral in the long-run, but what about in the short-run?
• The intuition is pretty clear: there will be a transfer of wealth from initial old to initial young (so money is not neutral in the short-run)

• How can we show this formally?

• Let’s start with the new steady-state—we know that \( \hat{q} = q^* \) and \( \hat{v} = \hat{q}/(\mu M) < v^* \)

• What happens in the transition to the steady-state?

• The initial young have a different budget constraint:

\[
\begin{align*}
c_1^y &= y + v_1(\mu - 1)M - q_1 \\
c_2^o &= (v_2/v_1)q_1
\end{align*}
\]
or, combining...

\[ c_1^y + \left(\frac{v_1}{v_2}\right) c_2^o = y + v_1(\mu - 1)M \]

- It is easy to show, for a given inflation rate, that \( v_1\Delta M > \Delta q_1 > 0 \) (assume normal goods)
  - the supply of money increases more than the demand for money
  - market-clearing condition at date 1
    \[ v_1M + v_1(\mu - 1)M = q_1 \]

- \( v_1\Delta M > \Delta q_1 > 0 \) implies \( v_1 \) must fall (price-level must rise)
• Initial old $c^0_1 = v_1 M$, so anything that causes $v_1 \downarrow$ makes them worse off

• Now, let’s check the dynamics...

\[
\begin{align*}
MRS(y + v_1(\mu - 1)M - q_1, \frac{v_2}{v_1}q_1) &= \frac{v_2}{v_1} \\
MRS(y - q_2, \frac{v_3}{v_2}q_2) &= \frac{v_3}{v_2} \\
MRS(y - q_3, \frac{v_4}{v_3}q_3) &= \frac{v_4}{v_3} \\
&\vdots \\
MRS(y - q^*, q^*) &= 1
\end{align*}
\]
• Conjecture that the new steady-state is reached in period 3, so that

\[
MRS(y + v_1(\mu - 1)M - q_1, \frac{v_2}{v_1}q_1) = \frac{v_2}{v_1}
\]

\[
MRS(y - q_2, \frac{v^*}{v_2}q_2) = \frac{v^*}{v_2}
\]

\[
MRS(y - q^*, q^*) = 1
\]

• Substitute in market-clearing conditions \( v_t = q_t/(\mu M) \)...

\[
MRS(y + \frac{q_1}{\mu M}M - q_1, \frac{q_2/(\mu M)}{q_1/(\mu M)}q_1) = \frac{q_2/(\mu M)}{q_1/(\mu M)}
\]

\[
MRS(y - q_2, \frac{q^*/(\mu M)}{q_2/(\mu M)}q_2) = \frac{q^*/(\mu M)}{q_2/(\mu M)}
\]

\[
MRS(y - q^*, q^*) = 1
\]

• Simplifying...
\[ MRS(y - (1/\mu)q_1, q_2) = \frac{q_2}{q_1} \]
\[ MRS(y - q_2, q^*) = \frac{q^*}{q_2} \]
\[ MRS(y - q^*, q^*) = 1 \]

- Looks like \( q_2 = q^* \) is a solution to the 2nd equation, so simplify further...

\[ MRS(y - (1/\mu)q_1, q^*) = \frac{q^*}{q_1} \]

- So \( q_1 > q^* \)

- Moreover, observe that \( q_1 \) depends positively on \( \mu \), the size of the one-time lump-sum money injection to the initial young \( \Delta M = (\mu - 1)M \)