Notes on Exchange Rate Volatility in an Equilibrium Asset Pricing Model

Manuelli and Peck (IER, 1990)

1 Model

Time is discrete and the horizon is infinite; $t = 1, 2, ..., \infty$. There are two economies, each of which are populated by a sequence of two-period-lived overlapping generations (with an initial old generation). Each economy features a constant population of $2N$ agents. A representative young agent has preferences defined only over old-age consumption, c_{t+1}. Preferences are represented by:

$$E_t u(c_{t+1}),$$

where $u'' \leq 0 < u'$ with $\lim_{c \to 0} u'(c) = \infty$ and E_t is an expectations operator conditional on information available at date t. Young agents have a nonstorable endowment $y_t > 0$. This endowment follows an exogenous stochastic process.

Assume that the two countries both receive the same endowment shock. Let $(2M, 2M^*)$ denote the fixed stocks of domestic and foreign money, respectively (with each money endowed evenly among the initial old in both economies). Let (v_t, v_t^*) denote the value of domestic and foreign currency, respectively, measured in units of output. Likewise, let (q_t, q_t^*) denote the quantity of real money balances of the domestic and foregin money held by a domestic agent. The choice problem facing a representative young agent at date t can be stated as follows:

$$\max E_t u(c_{t+1})$$

subject to:

$$\begin{aligned} y_t &= q_t + q_t^*; \\ R_{t+1} q_t + R_{t+1}^* q_t^* &\geq c_{t+1}; \end{aligned}$$

where $R_{t+1} = (v_{t+1}/v_t)$ and $R_{t+1}^* = (v_{t+1}^*/v_t^*)$. The initial old trivially consume $c_1 = [v_1 M + v_1^* M^*]/N$.

Note that since agents trivially save their entire endowment, their choice problem essentially boils down to a portfolio decision; i.e.,

$$\max E_t u\left(R_{t+1}^* y_t + \left[R_{t+1} - R_{t+1}^*\right] q_t\right),$$

so that q_t^D satisfies:

$$E_t \left[R_{t+1} - R_{t+1}^*\right] u'\left(R_{t+1}^* y_t + \left[R_{t+1} - R_{t+1}^*\right] q_t^D\right) = 0.$$

Note further that since preferences and endowments are identical across countries, consumption will also be the same. If the exchange rate is constant (or if agents are risk-neutral), there is some indeterminacy in the optimal portfolio but all such portfolios buy the same (expected) consumption. If the exchange rate fluctuates, then there is a unique optimal portfolio (if agents are risk-averse) so, again, consumption is the same across countries. Since all equilibria involve no net trade of goods between countries, we can without loss focus on symmetric equilibria where agents in each country hold the same portfolio (so that we need not introduce notation to distinguish between domestic and foreign agents). In this case, we have:

$$\begin{aligned} v_t M &= N q_t^D; \\ v_t^* M^* &= N q_t^{*D}. \end{aligned}$$

For simplicity, let us normalize such that $M = M^* = N = 1$, so that $q_t^D = v_t$ and $q_t^{*D} = v_t^*$. Now, from the bugdet constraint:

$$\begin{aligned} c_{t+1} &= R_{t+1} q_t^D + R_{t+1}^* q_t^{*D}; \\ &= R_{t+1} v_t + R_{t+1}^* v_t^*; \\ &= v_{t+1} + v_{t+1}^*. \end{aligned}$$

The relevant money market clearing condition is given by:

$$v_t M + v_t^* M^* = N y_t \ \forall \ t;$$

Since this condition must hold at all dates, we have:

$$\frac{v_{t+1} + v_{t+1}^*}{v_t + v_t^*} = \frac{y_{t+1}}{y_t}.$$

The equation above represents the (gross) real rate of return on a diversified (symmetric) portfolio (i.e., in equilibrium, each young agent purchases one unit of domestic money and one unit of foreign money). Combining the money market clearing condition with the budget constraint reveals that $c_{t+1} = y_{t+1}$; so we have the following restriction on equilibrium rates of return:

$$E_t \left[R_{t+1} - R_{t+1}^* \right] u' \left(y_{t+1} \right) = 0. \tag{1}$$

This equation may alternatively be written as:

$$E_t \left[R_{t+1} - R_{t+1}^* \right] E_t u'(y_{t+1}) + cov_t(\left[R_{t+1} - R_{t+1}^* \right], u'(y_{t+1})) = 0. \tag{2}$$

2 Constant Exchange Rate Equilibria

Assume that $y_t = y$, so that there is no 'fundamental' uncertainty. This restriction does not rule out the possibility of randomness in 'non-fundamentals.' In this section, however, assume that R_t and R_t^* are deterministic. In this case, condition (1) reduces to:

$$R_{t+1} = R_{t+1}^*.$$

Let e_t denote the value of foreign money measured in units of domestic money, so that $v_t^* = e_t v_t$. Then this latter equation can be written as:

$$R_{t+1} = \left(\frac{e_{t+1}}{e_t} \right) R_{t+1};$$

which implies:

$$e_{t+1} = e_t = e.$$

In other words, in the absence of any uncertainty, the equilibrium exchange rate must be constant. From the money market clearing condition,

$$\begin{aligned} v_t + v_t^* &= y; \\ (1 + e_t) v_t &= y; \end{aligned}$$

so that the (common) equilibrium rate of return on money is given by:

$$R_{t+1} = R_{t+1}^* = \frac{(1 + e_{t+1}) y}{(1 + e_t) y} = 1.$$

Note that since both currencies earn the same rate of return $R = R^* = 1$, the optimal portfolio decision at the individual level is indeterminate. That is, both currencies are literally perfect substitutes; since each is as good as the other, individuals are willing to hold any combination of (q, q^*) such that $q + q^* = y$ (note: here, y corresponds to the demand for *total* real balances). Since $v_t = q_t$, we can rewrite the money market clearing condition as:

$$(1 + e) q = y.$$

But since q is indeterminate, so is e. In other words, any e satisfying $0 < e < \infty$ is consistent with an equilibrium in which both currencies are willingly held. This is just the Kareken and Wallace (1981) indeterminacy result.

Imagine now that y_t follows a stochastic process. The question I ask here is whether $e_t = e$ is still an equilibrium. To see whether this is the case, assume that $e_t = e$. In this case, $R_{t+1} = R_{t+1}^*$; which obviously implies $E_t \left[R_{t+1} - R_{t+1}^* \right] = 0$. In this scenario, the (common) equilibrium rate of return on money is given by:

$$R_{t+1} = R_{t+1}^* = \frac{y_{t+1}}{y_t}.$$

Of course, this rate of return is random, so that individuals attain an expected utility $E_t u(y_{t+1})$. From the money market clearing condition, we have:

$$(1 + e) v_t = y_t.$$

Once again, the two currencies are viewed as perfect substitutes. Thus, for a fixed e, individuals are perfectly willing to hold $q_t = v_t$ and $q_t^* = v_t^*$ units of domestic and foreign money (respectively) in their portfolios, where

$$\begin{aligned} v_t &= \left(\frac{1}{1+e}\right) y_t; \\ v_t^* &= \left(\frac{e}{1+e}\right) y_t. \end{aligned}$$

In this equilibrium, both the domestic and foreign price-levels fluctuate in exactly the same way. The rate of return on money fluctuates, but since the exchange rate is fixed, the returns on both monies fluctuate in exactly the same way. Thus, in this model, changing fundamentals need not induce fluctuations in the equilibrium exchange rate. As a corollary, it follows that bi-lateral fixed exchange rate system–or a currency union–is consistent with efficiency.

3 Volatile Exchange Rate Equilibria with Constant Fundamentals

Let us return now to the case in which there is no 'fundamental' uncertainty, so that $y_t = y$. Now, since $u'(y)$ is a constant, we can write the restriction (1) as:

$$\begin{aligned} E_t \left[R_{t+1} - R_{t+1}^* \right] &= 0; \\ E_t \left[\frac{v_{t+1}}{v_t} - \frac{e_{t+1}}{e_t} \frac{v_{t+1}}{v_t} \right] &= 0; \end{aligned}$$

which reduces to:

$$E_t e_{t+1} v_{t+1} = e_t E_t v_{t+1}.$$

From the money market clearing condition, $v_t + v_t^* = y$; or $(1+e_t)v_t = y$. Recall, as well, that $q_t = v_t$ and $q_t^* = v_t^*$. Now, let's follow Manuelli and Peck (1990), and define the variable α_t by:

$$v_t \equiv \alpha_t y.$$

Here, α_t has the interpretation of being the fraction of one's (real) saving allocated to the domestic currency. We want to restrict attention to situations where both currencies are always valued; i.e.,

$$0 < \alpha_t < 1.$$

Now, use this definition and the money market clearing condition to write $e_t = (1-\alpha_t)/\alpha_t$. Substituting this into our earlier restriction:

$$\begin{aligned} E_t e_{t+1} v_{t+1} &= e_t E_t v_{t+1}; \\ E_t \left(\frac{1-\alpha_{t+1}}{\alpha_{t+1}}\right) \alpha_{t+1} y &= \left(\frac{1-\alpha_t}{\alpha_t}\right) E_t \alpha_{t+1} y; \\ \alpha_t E_t (1-\alpha_{t+1}) &= (1-\alpha_t) E_t \alpha_{t+1}; \end{aligned}$$

which reduces to:

$$\alpha_t = E_t \alpha_{t+1}. \tag{3}$$

Proposition: (Manuelli and Peck, 1990). *Any stochastic process* $\{\alpha_t\}_{t=1}^{\infty}$ *satisfying* $0 < \alpha_t < 1$ *and condition (3) constitutes an equilibrium (with both monies valued); with* $v_t = q_t = \alpha_t y$; $v_t^* = q_t^* = (1 - \alpha_t)y$ *and* $e_t = (1 - \alpha_t)/\alpha_t$.

Remark: Note that (3) implies that each country's respective price-level follows a martingale (with the price-level in each country being perfectly negatively correlated with each other); in particular, it does not imply that the exchange rate itself is a martingale. In fact, the theory predicts that $(1 + e_t)^{-1}$ follows a martingale; i.e., one can rewrite (3) as:

$$(1 + e_t)^{-1} = E_t(1 + e_{t+1})^{-1}.$$

3.1 Example

Let $\{\alpha_t\}_{t=1}^{\infty}$ follow the process:

$$\alpha_{t+1} = \alpha_t + \eta_{t+1},$$

where $\eta_{t+1} \sim U[-\kappa_t, \kappa_t]$ and $\kappa_t \equiv \min\{\alpha_t, 1 - \alpha_t\}$. Note that $E\left[\eta_{t+1} \mid \alpha_t\right] = 0$ so that $E\left[\alpha_{t+1} \mid \alpha_t\right] = \alpha_t$, as required (note: we are free to choose $0 < \alpha_1 < 1$ arbitrarily).