ON THE SOCIETAL BENEFITS OF ILLIQUID BONDS IN THE
LAGOS-WRIGHT MODEL

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ABSTRACT. In the equilibria of monetary economies, individuals may have different intertemporal marginal rates of substitution, so that they would like to engage in additional intertemporal trades of money. Kocherlakota (2003) argues that the role of nominal bonds is to allow individuals to make such trades. This logic, however, appears to fail in standard Lagos and Wright (2005) model. I explain why this is the case and describe how the Lagos-Wright model needs to be modified to induce an essential role for nominal bonds.

1. Introduction

In an intriguing paper, Kocherlakota (2003) provides an explanation for why nominally risk-free bonds are essential in monetary economies. His explanation is based on the idea that in the equilibria of monetary economies, individuals may have different intertemporal marginal rates of substitution. When this is so, individuals will want to engage in additional intertemporal trades of money; and the role of bonds is to allow individuals to make these trades. Moreover, in order for bonds to serve the interest of society, they must be rendered illiquid (in the sense of being costly to exchange for goods). In other words, he demonstrates why it is efficient for bonds to be illiquid rather than liquid.

Kocherlakota (2003) demonstrates his result within the context of a highly simplified model of intertemporal trade. For reasons of analytical tractability, he restricts his environment in a manner that renders an illiquid bond essential for one period only. Moreover, his analysis limits attention to a welfare-improving policy; that is, he does not characterize the optimal intervention. My initial goal was to extend his results along both of these dimensions within the context of the highly tractable quasi-linear environment introduced by Lagos and Wright (2005).

In the version of the Lagos-Wright model I study, I follow Kocherlakota (2003) in assuming that individuals are endowed with output and are subject to idiosyncratic preference shocks that generate a demand for insurance. As in Kocherlakota (2003), privately-issued financial instruments cannot exist; as record-keeping is absent, and as individuals lack commitment. As explained in Kocherlakota (1998), these latter two frictions make a fiat money instrument essential. Following Kocherlakota (2003), I assume competitive spot markets (search frictions are absent); these spot market trades potentially allow individuals to self-insure against idiosyncratic

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risk. I also assume, in line with Kocherlakota (2003), that society cannot impose individual penalties (which, among other things, rules out lump-sum taxation).

Absent intervention (a constant money supply), the stationary monetary equilibrium in the Lagos-Wright model is inefficient relative to the first-best. In other words, individuals have different intertemporal marginal rates of substitution. On the basis of this fact, along with the intuition supplied by Kocherlakota (2003), one might reasonably expect that an intervention of some form is both feasible and desirable. For example, if preference shocks are observable, then the type-contingent transfer policy described by Kocherlakota (2003) should do the trick. If preference shocks are private information, then the illiquid bond described by Kocherlakota (2003) should allow agents to engage in money-bond trades that essentially mimic what might otherwise have been accomplished directly by a type-contingent transfer policy.

It was surprising to discover that zero intervention constitutes a constrained-efficient policy in the Lagos-Wright model. Evidently, differences in intertemporal marginal rates of substitution is only a necessary—but not sufficient—condition to explain why illiquid bonds (or type-contingent transfers) are essential in Kocherlakota’s environment.

As it turns out, the reason for this discrepancy appears to lie in the different spatial structure assumed in each of these environments. In the Lagos-Wright model, I assume that individuals are always together in a central location. In contrast, Kocherlakota (2003) considers households that consist of worker-shopper pairs that travel in a particular manner to different locations. On the surface, this difference appears to be innocuous; in particular, it does not affect the properties of the first-best allocation—which is common across both specifications. However, it turns out to make a big difference for what is implementable in a monetary economy. The main reason for this is because the two environments induce different cash-constraints.

In the Lagos-Wright model, it can never be the case that all individuals are cash-constrained. The implication of this is that type-contingent transfers are neutral (so that an illiquid bond is not essential). In contrast, for a sufficiently high rate of inflation, the cash-constraints for all types of individuals can be made to bind in Kocherlakota’s environment. Once these constraints bind, type-contingent transfers are no longer neutral (so that an illiquid bond is essential if type-contingent transfers are unavailable). The details of how this all works is described below. I also demonstrate that in a suitably modified Lagos-Wright model, Kocherlakota’s main result continues to hold in a stationary state. I also demonstrate that there is a policy that implements the first-best allocation.

2. The Environment

The economy is populated by a continuum of *ex ante* identical agents, distributed uniformly on the unit interval. Each period $t = 0, 1, 2, \ldots, \infty$ is divided into two subperiods, labeled *day* and *night*. Agents meet at a central location in both subperiods; in particular, I abstract from the commonly employed assumption of random pairwise meetings in one of the subperiods.
All agents have common preferences and abilities during the day. Let $x_t(i) \in \mathbb{R}$ denote the consumption (production, if negative) of output in the day by agent $i$ at date $t$. The key simplifying assumption is that preferences are linear in this term; the possibility of exchange then implies transferable utility. Output produced in the day is nonstorable, so an aggregate resource constraint implies:

\[ \int x_t(i) di \leq 0; \]

for all $t \geq 0$.

At night, agents realize a shock at the beginning of the night that determines their type $\omega \in \{\omega_l, \omega_h\}$ for the night. Preferences at night are given by $u(c)$; where $u'' < 0 < u'$, $\lim_{c \to 0} u'(c) = \infty$, $\lim_{c \to \infty} u'(c) = 0$ and $0 < \omega_l < \omega_h < \infty$ with

\[ \eta \equiv \omega_h/\omega_l \geq (2\beta^{-1} - 1) > 1. \]

. Refer to an agent $i$ with $\omega_t(i) = \omega_l$ as patient and an agent $i$ with $\omega_t(i) = \omega_h$ as impatient. Types are determined randomly by an exogenous stochastic process. This process is i.i.d. across agents and time; there is no aggregate uncertainty. There is an equal probability of being either type of agent.

All agents have an endowment $0 < y < \infty$ of output at night. As the night good is also nonstorable, there is another aggregate resource constraint given by:

\[ \int c_t(i) di \leq y; \]

for all $t \geq 0$.

As agents are ex ante identical, their preferences can be represented as:

\[ E_0 \sum_{t=0}^{\infty} \beta^t [x_t(i) + \omega_t(i) u(c_t(i))]. \]

where $0 < \beta < 1$. Note that there is no discounting across subperiods.

Weighting all agents equally, a planner maximizes (2.4) subject to the resource constraints (2.1) and (2.3). As utility is linear in $x_t(i)$, agents are indifferent across any lottery over $\{x_t(i) : t \geq 0\}$ that delivers a given expected value. Without loss of generality, a planner may set $x_t(i) = 0$ for all $i$ and all $t \geq 0$.

Assume that all agents of a given type are treated symmetrically and restrict attention to stationary allocations of the form $(c_l, c_h)$. In this case, ex ante welfare is represented by

\[ U(c_l, c_h) = \left( \frac{0.5}{1 - \beta} \right) [u(c_l) + \eta u(c_h)]; \]

Maximizing this welfare function subject to the resource constraint $2y \geq c_l + c_h$ implies that the first-best allocation is characterized by:

\[ u'(c_l^*) = \eta u'(c_h^*) \text{ and } c_l^* + c_h^* = 2y. \]

Clearly, we have $c_h^* > y > c_l^* > 0$.

The physical environment described above corresponds closely to Lagos and Wright (2005); call this the LW Model. The physical environment considered by Kocherlakota (2003), which I call the K Model, is similar to the LW Model along
several dimensions, but differs in the following manner. First, each agent is replaced by a household consisting of two agents, labeled shopper and worker. Households are assigned permanently to one of two groups (of equal measure), labeled group 1 and group 2. Each group consists of an equal measure of patient and impatient types.

As in the LW Model, all agents are centrally located in the day. But while all agents are also centrally located at night in the LW Model, this is not the case in the K Model. In particular, there are two spatially separated locations at night in the K Model, labeled island 1 and island 2. As households enter the night, they realize their types. Workers in group 1(2) remain on island 1(2); while shoppers in group 2(1) travel to island 1(2). The critical difference here, relative to LW Model, is that households cannot consume their own output.

At some level, the difference between these two environments appears innocuous. In particular, note that the first-best allocation for the K Model continues to be characterized by (2.6). I argue below, however, that these two environments have very different implications when both environments are restricted further to make money essential.

I impose the following additional restrictions on these environments. First, agents lack commitment and it is impossible to monitor individual trading histories, so that a fiat money instrument is essential. Second, assume that society cannot impose any penalties on individuals. Among other things, this latter restriction rules out lump-sum taxation (but obviously does not preclude lump-sum transfers of money). Third, I assume that trade among agents occurs on a sequence of competitive spot markets (money for goods).

Kocherlakota (2003) makes one other restriction explicit: agent types at night (whether patient or impatient) are private information. The reason he does so is to rule out the possibility of type-contingent money transfers at night (given the lack of record-keeping, all types would strictly prefer to report a type that results in the highest transfer). Absent this restriction, type-contingent transfers can implement the first-best allocation in the K Model. If type-contingent transfers are ruled out, then an illiquid bond can improve the allocation. None of this will be true in the LW Model.

To put things another way, type-contingent transfers in the LW Model are neutral. Hence, whether types are private information or not is irrelevant. The implication of this is that an illiquid bond can serve no purpose in the LW Model. In contrast, type-contingent transfers are not neutral in the K Model. Hence, whether types are private information or not is critical. If types are observable, then an illiquid bond is not necessary. However, if types are unobservable, then an illiquid bond market can in effect substitute for the missing policy instrument. The critical issue then appears to be whether a type-contingent transfer policy is neutral or not. We can determine whether this is the case or not by assuming that types are observable.
3. Policy and Markets

Society creates and manages a supply of money tokens. These money tokens are durable, divisible, and non-counterfeitable. All households are initially endowed with an equal amount of money. Let $M$ denote the money supply at the beginning of some arbitrary day. The supply of money can be made to grow at some constant rate $\mu \geq 1$, so that $M^+ = \mu M$ (variables with the superscript "+" denote "next period" values).

The new money $(\mu - 1)M$ created over a period may be injected as lump-sum transfers to agents in the day or the night. An educated guess suggests that lump-sum transfers during the day will serve no purpose in this quasi-linear environment (indeed, such a policy is necessarily welfare-reducing). I therefore restrict attention to lump-sum transfers at night. Let where $\tau_j \geq 0$ denote the nominal transfer accruing to an agent of type $j \in \{l, h\}$ at night, so that the government budget constraint is given by

$$0.5\tau_l + 0.5\tau_h = (\mu - 1)M.$$  

Agents are able to trade money for output on a sequence of competitive spot markets. Let $(v_d, v_n)$ denote the price-system prevailing at any given date; where $v_d$ and $v_n$ denote the value of money (measured in units of output) in the day and night, respectively.

4. The LW Model

4.1. Decision-Making. Let $z \geq 0$ denote the cash balances held by a household at the beginning of the day and let $m \geq 0$ denote the money carried forward into the night. The household’s budget constraint during the day is given by,

$$x = v_d(z - m).$$  

Let $V(m)$ denote the ex ante value of entering the night with money $m$ and let $W(z)$ denote the value of entering the day with money $z$. Making use of the budget constraint (4.1), the choice problem may be expressed recursively as

$$W(z) = \max_{m \geq 0} v_d(z - m) + V(m).$$

The demand for money during the day $\hat{m}$ is characterized by,

$$v_d = V'(\hat{m});$$

and the envelope theorem implies,

$$W'(z) = v_d.$$  

Let $V_j(\hat{m})$ denote the value of entering the night with money $\hat{m}$ having realized type $j \in \{l, h\}$. The choice problem at night can be expressed recursively as

$$V_j(\hat{m}) = \max_{c_j, z_j^+} \omega_j u(c_j) + \beta W(z_j^+)$$

subject to

$$z_j^+ = \hat{m} + \tau_j + v_n^{-1}(y - c_j) \geq 0.$$
If the cash-constraint (4.6) remains slack, desired consumption is characterized by
\[ v_n \omega_j u'(\hat{c}_j) = \beta v_d^+; \]
where here, I have made use of (4.4). If the cash-constraint (4.6) binds, desired consumption is given by
\[ \hat{c}_j = y + v_n (\hat{m} + \tau_j). \]
By the envelope theorem,
\[ V_0'(\hat{m}) = v_n \omega_j u'(\hat{c}_j). \]
Note that \( V(\hat{m}) = 0.5V_t(\hat{m}) + 0.5V_h(\hat{m}), \) so that by making use of (4.9),
\[ V'(\hat{m}) = 0.5v_n [\omega_l u'(\hat{c}_l) + \omega_h u'(\hat{c}_h)]; \]
which, by (4.4) implies
\[ v_d = 0.5v_n [\omega_l u'(\hat{c}_l) + \omega_h u'(\hat{c}_h)]. \]

4.2. Equilibrium. I restrict attention to stationary equilibria. Among other things, stationarity implies that \( v_n M_f = v_n M, \) so that the equilibrium inflation rate is determined solely by \( \mu; \) i.e.,
\[ \begin{pmatrix} v_n^+ \\ v_n \end{pmatrix} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}. \]
Given (4.12) and a stationary allocation \((\hat{c}_l, \hat{c}_h),\) condition (4.11) implies that
\[ \begin{pmatrix} v_d^+ \\ v_d \end{pmatrix} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}. \]
Now, given some policy \( \mu \geq 1, \) I need to determine which agents (if any) are cash-constrained at night. One case can be ruled out immediately; i.e.,

**Lemma 1.** For any policy \( \mu \geq 1, \) the cash-constraints (4.6) cannot remain slack for both types of households.

**Proof.** If (4.6) remain slack for both patient and impatient households, then their desired consumption at night each satisfy (4.7); which implies \( v_n^+ M_f = v_n M, \) so that the equilibrium inflation rate is determined solely by \( \mu; \) i.e.,
\[ \begin{pmatrix} v_n^+ \\ v_n \end{pmatrix} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}. \]
By condition (4.13), this can only be true if \( \mu = \beta < 1; \) which is a contradiction, as \( \mu \geq 1. \)

By Lemma 1, it must be true that the cash constraint for at least one type must bind. If only one constraint binds, it is straightforward to demonstrate that it will be for the impatient household. When this is the case, then the desired consumption pattern will satisfy
\[ v_n \omega_l u'(\hat{c}_l) = \beta v_d^+; \]
\[ \hat{c}_h = y + v_n (\hat{m} + \tau_h); \]
so that \( \hat{z}_l^+ = \hat{m} + v_n^{-1}(y - \hat{c}_l) > 0 \) and \( \hat{z}_h^+ = 0. \)

Holding the price-system fixed, (4.14) implies that desired consumption for patient households does not depend on their money transfer \( \tau_l \) (they simply save it).
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On the other hand, (4.15) implies that desired consumption for impatient households is increasing in their money transfer \( \tau_h \). This latter observation suggests that it may be possible to alleviate the cash-constraint for impatient households by targeting them directly with a money transfer. Whether such a policy will have this desired effect remains to be seen, as any such policy will have general equilibrium consequences which need to be sorted out.

Let me now characterize the equilibrium allocation at night, assuming that only one cash-constraint binds. First, observe that condition (4.11) implies

\[
\beta v^+ = 0.5\beta v^+ [\omega_1 u'(\hat{c}_l) + \omega_2 u'(\hat{c}_h)];
\]

where here, I have imposed the stationarity condition \((\hat{c}_l, \hat{c}_h) = (c^+_l, c^+_h)\). Combining this latter expression with (4.14) and (4.12), and rearranging terms,

\[
0.5 \beta v^+ = 0.5 \left( \frac{\beta}{\mu} \right) [u'(\hat{c}_l) + \eta u'(\hat{c}_h)].
\]

Market-clearing at night requires

\[
0.5 \hat{c}_l + 0.5 \hat{c}_h = y.
\]

Together, conditions (4.16) and (4.17) characterize the equilibrium allocation \((\hat{c}_l, \hat{c}_h)\) conditional on policy \( \mu \).

Note that condition (4.16) implies that the monetary equilibrium corresponds to the first-best only in the event that \( \mu = \beta \); something that is ruled out here by the restriction that \( \mu \geq 1 \).

Conditions (4.16) and (4.17) also imply that \( \hat{c}_l(\mu) \not< y \) and \( \hat{c}_h(\mu) \not< y \) as \( \mu \not< \mu_A \); that is, the risk-sharing properties of the monetary equilibrium allocation deteriorate monotonically with inflation. I state, without proof, a well-known property of the LW Model; i.e.,

**Lemma 2.** Ex ante welfare \( U(\hat{c}_l, \hat{c}_h) \) is monotonically decreasing in \( \mu \in [1, \mu_A] \).

The results so far are predicated on the assumption that the cash-constraint for the patient household remains slack for all \( \mu \in [1, \mu_A] \). To see this, I begin by noting that market-clearing in the day requires

\[
\hat{m} = M.
\]

In this case, the market-clearing condition at night (4.17) implies

\[
0.5 \hat{c}_l + 0.5 [y + v_n(M + \tau_h)] = y;
\]

so that the equilibrium price-level at night must satisfy \( \hat{v}_n = (M + \tau_h)^{-1}(y - \hat{c}_l) \).

To make the point as cleanly as possible, let me for the moment restrict attention to policies of the form \( \tau_l = 0 \) and \( \tau_h = 2(\mu - 1)M > 0 \). In this case, the equilibrium price-level at night is given by

\[
\hat{v}_n = \left[ \frac{y - \hat{c}_l}{(2\mu - 1)M} \right].
\]
Finally, consider the cash-constraint (4.6). Making use of (4.18) and (4.19),

\[(4.20) \quad \hat{z}_l^+ (\mu) = 2\mu M; \]

which is obviously positive for all \( \mu \in [1, \mu_A] \). Hence, it cannot be the case that both cash-constraints bind in the LW Model when money is valued. Because this result is important for what is to follow, I state it explicitly as Lemma 3 with an alternate proof that allows for more general policies.

**Lemma 3.** In any monetary equilibrium, the cash-constraints (4.6) cannot both be binding.

**Proof.** If (4.6) bind for both patient and impatient households, then their desired consumption at night satisfies \( \hat{c}_j = y + v_n (\hat{m} + \tau_j) \) for \( j \in \{l, h\} \). This fact, combined with the market-clearing conditions (4.17) and (4.18), together with the government budget constraint (3.1) implies \( v_n 2\mu M = 0 \); which is impossible when \( v_n > 0 \). \( \Box \)

4.3. **A Neutrality Proposition.**

**Proposition 1.** The competitive monetary equilibrium allocation at night \((\hat{c}_l, \hat{c}_h)\) is invariant to \((\tau_l, \tau_h)\) for any \((\tau_l, \tau_h)\) satisfying the government budget constraint (3.1).

**Proof.** Conditions (4.16) and (4.17) imply that \((\hat{c}_l, \hat{c}_h)\) depends only on the policy parameter \( \mu \). From the government budget constraint (3.1), \( \mu \) depends solely on the aggregate transfer value \( 0.5\tau_l + 0.5\tau_h \) and not on the composition of its components \((\tau_l, \tau_h)\). \( \Box \)

Proposition 1 has the flavor of a neutrality result. That is, in any monetary equilibrium with \( \mu > 1 \) and \((\tau_l, \tau_h)\) both strictly positive, increasing \( \tau_h \) at the expense of lowering \( \tau_l \) will have no real consequences. This is true despite the fact that impatient households are cash-constrained. Indeed, it follows as a corollary to Lemma 2 that the constrained-efficient policy here entails setting \( \mu = 1 \); so that \((\tau_l, \tau_h) = (0, 0)\). That is, any attempt to transfer money to cash-constrained households in the LW Model can only serve to reduce welfare (through the effect such a policy will have on inflation); see also Berentsen, Camera and Waller (2007, pg. 182).

Since the role of an illiquid bond in Kocherlakota (2003) is to mimic the welfare-enhancing role of type-contingent transfers, it follows that there are no societal benefits associated with an illiquid bond in the LW Model. This is somewhat surprising in that the two environments are similar along several dimensions. What then, might possibly account for these two very different conclusions?

5. **The K Model**

5.1. **Decision-Making.** In the K Model, decision-making in the day corresponds precisely to the LW Model; in particular, money demand \( \hat{m} \) continues to be characterized by (4.3) and \( W'(z) = v_d \). The choice problem at night, however, differs in a subtle, but important way.
A K Model household carries money \( \hat{m} \) to the night market. At this point, the household type is realized; the worker stays at home to produce output \( y \), while the shopper travels to the foreign location to purchase output \( c_j \). This structure induces the following cash-constraint,

\[(5.1) \quad z_j^+ = \hat{m} + \tau_j - v^{-1}_n c_j \geq 0 \]

for \( j \in \{ l, h \} \). Note that the cash-constraint (5.1) in the K Model differs from that of the LW Model; see (4.6). The key difference is that households cannot consume their own output in the K Model. An implication of this is that both types of households may potentially be cash-constrained.

If a K Model household is not cash-constrained at night, then desired consumption is determined by

\[(5.2) \quad v_n \omega_j u'(\hat{c}_j) = \beta v_d^+; \]

which corresponds to (4.7) in the LW Model. If a K Model household is cash-constrained at night, then desired consumption is determined by

\[(5.3) \quad \hat{c}_j = v_n (\hat{m} + \tau_j); \]

which differs from the corresponding condition in the LW Model; see (4.8).

It is easy to verify that the same envelope conditions apply at night, so that condition (4.11) continues to hold here; i.e.,

\[ v_d = 0.5 v_n \left[ \omega_l u'(\hat{c}_l) + \omega_h u'(\hat{c}_h) \right]. \]

5.2. Equilibrium. Without loss of generality, I restrict attention here to transfer policies of the form \( \tau_l = 0, \tau_h = 2(\mu - 1)M \geq 0 \).

It is easy to verify that Lemma 1 applies to the K Model as well; hence, at least one of the cash-constraints will bind. If this is the case, then as in the LW Model, it will bind for the impatient household. Using conditions (5.2) and (5.3), this implies

\[(5.4) \quad v_n \omega_l u'(\hat{c}_l) = \beta v_d^+; \]

\[(5.5) \quad \hat{c}_h = v_n (\hat{m} + 2(\mu - 1)M). \]

The equilibrium allocation at night in this case corresponds exactly to the LW Model; i.e., see (4.16) and (4.17). Hence, any difference between the these two models will depend critically on whether both cash-constraints may conceivably bind in the K Model.

We already know that the policy \( \mu = 1 \) is constrained-efficient in the LW Model and that at the constrained-efficient allocation, impatient households are cash-constrained while patient households are not. The first question to ask is whether this is also the case for the K Model when \( \mu = 1 \).

If patient households are not cash-constrained, their desired consumption is determined by (5.4). If impatient households are cash-constrained, then their desired consumption is determined by (5.5). In equilibrium, \( \hat{m} = M \). Market-clearing at night then implies

\[ 0.5 \hat{c}_l + 0.5 \hat{c}_h M = y; \]
so that the equilibrium price-level at night must satisfy
\[ \hat{v}_n = \left[ \frac{2y - \hat{c}_t}{M} \right]. \]
Utilizing (5.1), this then implies
\[ \hat{z}_t^+ = M - \hat{v}_n^{-1} \hat{c}_t = \left[ 1 - \left( \frac{\hat{c}_t}{2y - \hat{c}_t} \right) \right] M > 0. \]
That is, when \( \mu = 1 \), patient households are not cash-constrained so that the resulting allocation in this case corresponds to the LW Model.

The next question to ask is whether the cash-constraint for patient households in the K Model necessarily remains slack at higher rates of inflation. Consider an arbitrary inflation rate \( \mu > 1 \) and continue to assume that impatient households are cash-constrained while patient households are not. In this case, market-clearing at night implies
\[ 0.5\hat{c}_t + 0.5v_n(M + 2(\mu - 1)M) = y; \]
so that the equilibrium price-level at night must satisfy
\[ \hat{v}_n = \left[ \frac{2y - \hat{c}_t}{(2\mu - 1)M} \right]. \]
Utilizing (5.1), this then implies
\[ \hat{z}_t^+(\mu) = [1 - (2\mu - 1)\theta(\mu)] M; \]
where
\[ \theta(\mu) \equiv \left( \frac{\hat{c}_t(\mu)}{2y - \hat{c}_t(\mu)} \right) \in (0, 1]. \]

Observe that \( \theta(\mu) \not> 1 \) as \( \mu \not> \mu_A \). Unlike the LW Model, \( \hat{z}_t^+(\mu) \) here is monotonically decreasing in \( \mu \). The question that concerns us here is whether it is possible for \( \hat{z}_t^+(\mu) \) to reach zero before \( \mu \) reaches \( \mu_A \). Evidently, the answer is yes.

**Lemma 4.** There exists a unique \( \mu_0 \in (1, \mu_A) \) satisfying \( \hat{z}_t^+(\mu_0) = 0. \)

**Proof.** As \( \hat{c}_t(\mu) \not> y \) as \( \mu \not> \mu_A \), it follows that \( \theta(\mu) \not> 1 \) as \( \mu \not> \mu_A \). Define the function \( \phi(\mu) \equiv (2\mu - 1)^{-1}. \) Clearly, \( \phi(1) = 1 > \theta(1). \) Moreover, \( \phi(\mu) \downarrow 0 \) as \( \mu \not> \mu_A \). Hence, there is a unique \( \mu_0 \) satisfying \( \theta(\mu_0) = \phi(\mu_0) \), with \( \mu_0 \in (1, \mu_A) \). \( \square \)

The lemma above is critical. What it implies is that the cash-constraints for both types of households will bind for a sufficiently high rate of inflation. When both cash-constraints bind, the desired consumption allocation is no longer described by (5.4) and (5.5); instead, it is described by
\[
\begin{align*}
\hat{c}_t &= v_n\hat{m}; \\
\hat{c}_h &= v_n(\hat{m} + 2(\mu - 1)M).
\end{align*}
\]
In equilibrium, \( \hat{m} = M \) and market-clearing at night implies
\[ 0.5v_nM + 0.5v_n(M + 2(\mu - 1)M) = y; \]
so that the equilibrium price-level at night satisfies
\[ \hat{v}_n = \left[ \frac{y}{\mu M} \right]. \]
Combining these restrictions, the equilibrium allocation at night in the K Model for an inflation rate in the range \( \mu \in [\mu_0, \mu_A] \) is characterized by

\[
\hat{c}_l(\mu) = \left( \frac{1}{\mu} \right) y; \tag{5.9}
\]

\[
\hat{c}_h(\mu) = \left( 2 - \frac{1}{\mu} \right) y. \tag{5.10}
\]

Conditions (5.9) and (5.10) make it plain that, for a sufficiently high rate of inflation, a type-contingent transfer policy in the K Model is not neutral in the sense I described above for the LW Model. In the K Model, transfers directed at impatient households redirect purchasing power away from patient households; i.e., in the socially desirable direction. In contrast, a similar policy in the LW Model has the exact opposite effect that works through an entirely different mechanism (i.e., the effect of anticipated inflation on desired saving behavior).

5.3. The Societal Benefits of Type-Contingent Transfers. The analysis above demonstrates that type-contingent transfers of money to impatient households may potentially improve ex ante welfare. Whether this potential can be realized remains to be seen. In what follows, I adopt the parametric restriction on preferences used by Kocherlakota (2003) and demonstrate that an appropriate type-contingent transfer policy can implement the first-best allocation.

For log preferences, the first-best allocation is given by

\[
c^*_l = \left( \frac{2}{1 + \eta} \right) y; \]

\[
c^*_h = \left( \frac{2\eta}{1 + \eta} \right) y.
\]

Hence, the monetary allocation (5.9) and (5.10) correspond to the first best for an inflation rate equal to \( \mu^* \) satisfying

\[
\mu^* = 0.5(1 + \eta) > 1.
\]

It remains to verify that \( \mu^* \geq \mu_0 \). Recall that \( \mu_0 \) satisfies \( \hat{z}_l^+(\mu_0) = 0 \); or \( \theta(\mu_0) = \phi(\mu_0) \). An expression for \( \theta(\mu) \) can be derived by using the parametric form for preferences together with (4.16) and (4.17); i.e.,

\[
\theta(\mu) = \left( \frac{\mu - 0.5\beta}{0.5\beta \eta} \right).
\]

Recall that \( \phi(\mu) \equiv (2\mu - 1)^{-1} \). Setting \( \theta(\mu) = \phi(\mu) \) results in the following quadratic equation

\[
\mu_0 = 0.25 \left[ (1 + \beta) \pm ((1 + \beta)^2 + 4(\eta - 1))^{1/2} \right].
\]

For the purpose of illustration, assume that \( \eta = (2\beta^{-1} - 1) \), so that the restriction (2.2) is satisfied weakly.\(^1\) A simple calculation then reveals that the positive solution is \( \mu_0 = 1 < \mu^* \).

The analysis here corroborates what is reported in Kocherlakota (2003); namely, that if type-contingent transfers are admissible, then the first-best allocation is

\(^1\)In this case, the constrained-efficient allocation in the LW Model is the autarkic allocation.
implementable. As explained by Kocherlakota (2003), the feasibility of such a policy depends critically on the fact that types are observable; if they are not, then patient households have an incentive to misrepresent themselves in order to receive the money transfer (the lack of memory prevents recipients of the transfer to be punished in any manner in the future).

When types are private information, Kocherlakota (2003) demonstrates that there is a policy that can improve welfare. The policy entails issuing an additional type of token that represents a risk-free claim against future money. It is absolutely critical that this second token (a bond) be made illiquid in the sense that it cannot be used as payment for at least some goods. In the present context, this can be done by issuing one-period bonds in the day and allowing agents to swap money for bonds at the beginning of the night—just after their type realizations—but before goods market trading. In equilibrium, patient agents are willing to swap some of their money for the bonds held by impatient agents. The relative illiquidity of bonds implies that they must trade at a discount. The resulting transfer of purchasing power at night (from patient to impatient agents) exactly mimics what might otherwise have been accomplished with type-contingent transfers of money.

6. Conclusion

Both Lagos and Wright (2005) and Kocherlakota (2003) present models of intertemporal trade that are similar along several dimensions. Agents in both models desire insurance against idiosyncratic shocks. The lack of memory and commitment implies that money is essential; that is, agents are able to self-insure by purchasing and selling output on a sequence of competitive money-goods markets. Absent intervention, the resulting equilibrium in both environments is inefficient relative to the first-best. In the Lagos and Wright (2005) environment, there is no intervention that can improve welfare. In contrast, there is an intervention in the environment considered by Kocherlakota (2003) that can improve welfare. It is not immediately clear what accounts for these two very different conclusions.

What I have demonstrated above is that the key difference appears to be the presence of an additional spatial friction in the K Model that induces a more restrictive cash-constraint on households. For sufficiently high inflation rates, it is possible that both types of households are cash-constrained in the K Model; this can never be the case in the LW Model studied here. When both types of households are cash-constrained, type-contingent transfers can redirect resources in the socially desirable direction. When types are private information, an illiquid bond can achieve the same thing. Hence, the K Model is an environment that rationalizes the societal benefits of illiquid bonds. No such rationalization exists for the LW Model.
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7. References


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