Bank Runs Under Optimal Banking Structures

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1 Introduction

2 Physical Environment

The economy consists of a finite number (N) of individuals. The commodity space consists of consumption lotteries over 'current' and 'future' consumption (c, c'). Individuals are identical *ex ante*, but *ex post* become one of two types, $\omega \in \{p, i\}$. An person's type is private information. *Ex post* preferences depend on one's type realization; $U(c, c'; \omega)$. *Ex ante* preferences are represented by an expected utility function $E[U(c, c'; \omega)]$, where E denotes an appropriate expectations operator. There is a distribution over type profiles that is common knowledge; let $\Pi(j)$ denote the *ex ante* probability of $j = 0, 1, 2, ..., N$ patient individuals.

There are three stages of activity. All agents are together in the *ex ante* stage at some centralized location. The *ex post* stage is divided in 'current' and 'future' periods. At the end of the *ex ante* stage, individuals disperse to remote (spatially separated) locations, with every person subsequently returning one at a time to the central location in the current period. The sequence in which people return to the central location is governed by an exogenous and independent random process, with all orderings of the N individuals being equally likely. In the future period, all individuals remain in the central location.

Each person has a perfectly divisible endowment $y > 0$ that can be invested in the *ex ante* period. Assume that capital can be liquidated costlessly in the current period and that unliquidiated capital earns a gross return $R > 1$ in the future period.

3 Incentive Compatible Bank Contract

Following Green and Lin (2003), we model a bank as the planner in a mechanism design problem. In this set up, individuals 'deposit' their endowment with the bank in exchange for a promised allocation $\mathbf{c}$ (i.e., a state-contingent contract). As a part of the contract, people arriving back to the central location in the current period are required to make reports of their type, denoted $\widetilde{\omega} \in \{p, i\}$. Of course, individuals arrive in random order, with the order of arrivals observed by the bank.

Let $\widetilde{\omega}_n \in \{p, i\}$ denote the type report of a person whose realized place in line is n and let $\widetilde{\omega}^n = (\widetilde{\omega}_1, \widetilde{\omega}_2, ..., \widetilde{\omega}_n)$ denote the history of reports. An allocation is given by:

$$\mathbf{c} \equiv \left\{ c_n(\widetilde{\omega}^n), c_n'(\widetilde{\omega}^N) \right\} \text{ for } n = 1, 2, ..., N.$$

Let $\widetilde{\omega}_n = \omega_n$ denote a truthful report. Note that, by construction, the allocation is required to satisfy a *sequential service constraint*. In particular, the current-period consumption allocated to a person with a particular line position cannot be made contingent on future reports.

Definition: Let $E\left[\bullet \mid \omega^{n-1}, \widetilde{\omega}_n\right]$ denote the expectation over the joint distribution of truthful subsequent reports $(\omega_{n+1}, \omega_{n+2}, ..., \omega_N)$ conditional on $(\omega^{n-1}, \widetilde{\omega}_n)$. The stochastic process governing type realizations is said to be *independent* if $E\left[\bullet \mid \omega^{n-1}, \omega_n\right] \equiv E\left[\bullet \mid \omega_n\right]$.

Definition: An allocation $\mathbf{c}$ is *incentive-feasible* if

$$\sum_n c_n'(\widetilde{\omega}^N) \leq R\left(Ny - \sum_n c_n(\widetilde{\omega}^n)\right) \tag{1}$$

and if for all $(\widetilde{\omega}_n, \omega_n) \in \{p, i\} \times \{p, i\}$,

$$\begin{aligned} & E\left[U\left(c_n(\omega^{n-1}, \omega_n), c_n'(\omega^N); \omega_n\right) \mid \omega^{n-1}, \omega_n\right] \\ \geq \quad & E\left[U\left(c_n(\omega^{n-1}, \widetilde{\omega}_n), c_n'(\omega^N_{\widetilde{n}})\, ; \omega_n\right) \mid \omega^{n-1}, \widetilde{\omega}_n\right], \end{aligned} \tag{2}$$

where
$(\omega_1, ..., \widetilde{\omega}_n, ..., \omega_N)$.

Proposition (Wallace): If type realizations are independent, then any incentive-feasible allocation can be implemented as a unique subgame perfect equilibrium of a direct revelation game in which each agent reports his type. (Are there any restrictions placed on U in this proposition?).

Wallace's proposition suggests that relaxing the assumption of independence may open the door for multiplicity. Before going down this path, however, we report a related result from Green and Lin (2003). These authors place some additional restrictions on preferences that were also employed by Diamond and Dybvig (1983); e.g.,

$$U(c, c'; \omega) = \begin{cases} u(c + c') & \text{if} \quad \omega = p; \\ Au(c) & \text{if} \quad \omega = i; \end{cases} \tag{3}$$

where $u(x) = (1 - \alpha)^{-1}[x^{1-\alpha} - 1]$, with $\alpha > 1$ and $A \geq 1$. Thus, 'impatient' individuals only value current consumption; while 'patient' individuals view current and future consumption as perfect substitutes.

Remark 1: Note that the restriction $\alpha > 1$ here may not be necessary if $A > 1$.

Proposition (Green and Lin): If type realizations are independent and if $A = 1$, then the solution to maximizing (3) subject to (1) constitutes the unique subgame perfect equilibrium of a direct-revelation game in which each agent reports his type.

This result too asserts that multiplicity is ruled out under the optimal contract. Since the optimal contract satisfies incentive-feasibility, the Green-Lin proposition follows as a corollary to the Wallace proposition. The only difference here is that Wallace places fewer restrictions on preferences and does not restrict attention to the optimal contract (i.e., any incentive-feasible allocation is considered).

One interesting aspect of the Green-Lin solution is that the incentive constraints do not bind. We suspect that the reason for this lies in their assumption that $A = 1$. The analysis in Peck and Shell (2003) suggests that these incentive constraints are likely to bind for large enough values of A. Of course, even for incentive constraints that bind, Wallace's proposition asserts the uniqueness of equilibrium. Nevertheless, we have reason to believe that multiplicity is likely when the incentive constraints bind in an environment that relaxes the assumption of independence.

4 Non-Independence

In what follows, we restrict attention to the case $N = 3$ and assume the Diamond-Dybvig preferences in (3). Unlike Wallace (2004), who considers any incentive-feasible allocation, we focus on an optimal allocation. In the present context, optimality implies the following set of results (to be proved).

Lemma 1: $c_n = 0$ for any n reporting $\widetilde{\omega}_n = p$; and $c'_n = 0$ for any n reporting $\widetilde{\omega}_n = i$ (it will never be optimal to give patient people current consumption; and it will never be optimal to give impatient people future consumption).

Lemma 2: $c'_n = c'$ for all n (equal treatment for patient individuals).

Lemma 3: $\widetilde{\omega}_n = \omega_n$ for all $\omega_n = i$ (an impatient person will always tell the the truth).

Lemma 4: c' is strictly increasing in $\#(p)$ reported in $\widetilde{\omega}^N$ (future consumption is an increasing function of the number of patient people).

Lemma 5: c_n is increasing (c' is decreasing) in the parameter A.

4.1 Objective Function

We already have $\Pi(j)$, which denotes the *ex ante* probability of $j = 0, 1, 2, 3$ patient individuals. Given that all orderings of arrivals are equally likely, given the results from Lemmas 1 and 2, and substituting in the feasibility constraints, the planner's objective function can be written as follows:[1]

$$
\begin{aligned}
&E[U(c, c'; \omega)] = \Pi(3)\{3u(Ry)\} \qquad (4)\\
&+(1/3)\Pi(2)\left\{\begin{array}{c} Au(c_3(p,p,i)) + 2u(R(1/2)[3y - c_3(p,p,i)]) \\ +Au(c_2(p,i)) + 2u(R(1/2)[3y - c_2(p,i)]) \\ +Au(c_1(i)) + 2u((1/2)R[3y - c_1(i)]) \end{array}\right\} \\
&+(1/3)\Pi(1)\left\{\begin{array}{c} Au(c_2(p,i)) + Au(c_3(p,i,i)) + u(R[3y - c_2(p,i) - c_3(p,i,i)]) \\ +Au(c_1(i)) + Au(c_3(i,p,i)) + u(R[3y - c_1(i) - c_3(i,p,i)]) \\ +Au(c_1(i)) + Au(c_2(i,i)) + u(R[3y - c_1(i) - c_2(i,i)])) \end{array}\right\} \\
&+\Pi(0)\left\{Au(c_1(i)) + Au(c_2(i,i)) + Au(3y - c_1(i) - c_2(i,i))\right\}.
\end{aligned}
$$

Thus, the optimization problem boils down to choosing an appropriate current-period 'withdrawal schedule' consisting of six unknowns: $c_1(i), c_2(p,i), c_2(i,i), c_3(p,p,i), c_3(p,i,i), c_3(i,p,i)$, with $c_3(i,i,i) = 3y - c_1(i) - c_2(i,i)$.

4.2 Incentive Constraints

From Lemma 3, we know that we do not have to check the incentive constraints for impatient individuals (as long as they get some strictly positive amount of consumption). Thus, we need only consider the incentive constraints for patient individuals. Here, we have three sets of IC constraints; each associated with a particular line-placing.

For the third (patient) person in line, the following must hold:

$$
\begin{aligned}
c'(i,p,p) &= (1/2)R[3y - c_1(i)] \geq c_3(i,p,i); \qquad (5)\\
c'(p,i,p) &= R(1/2)[3y - c_2(p,i)] \geq c_3(p,i,i); \\
c'(i,i,p) &= R[3y - c_1(i) - c_2(i,i)] \geq c_3(i,i,i); \\
c'(p,p,p) &= Ry \geq c_3(p,p,i).
\end{aligned}
$$

[1] See Appendix 1 for details.

Note that since $R > 1$, the third constraint in the system above is automatically satisfied (hence, we can ignore it).

Consider now the second (patient) person in line. Let $\pi(\omega_3 \mid \omega_2, \omega_1)$ denote the probability that the last person in line is ω_3, conditional on a history (ω_2, ω_1). Conditional on the history $\omega_1 \in \{p, i\}$, the second patient person in line $(\omega_2 = p)$ will tell the truth if:

$$\begin{aligned} \omega_1 &= p : \pi(p \mid p, p)u(c'(p, p, p)) + \pi(i \mid p, p)u(c'(p, p, i)) \geq u(c_2(p, i)); \quad (6) \\ \omega_1 &= i : \pi(p \mid p, i)u(c'(i, p, p)) + \pi(i \mid p, i)u(c'(i, p, i)) \geq u(c_2(i, i)). \end{aligned}$$

Finally, consider the first person in line. Let $\pi(\omega_2, \omega_3 \mid \omega_1)$ denote the probability of future types (ω_2, ω_3) conditional on ω_1. Then the following must hold:

$$\begin{aligned} \pi(p, p &\mid p)u(c'(p, p, p)) + \pi(p, i \mid p)u(c'(p, p, i)) \quad (7) \\ +\pi(i, p &\mid p)u(c'(p, i, p)) + \pi(i, i \mid p)u(c'(p, i, i)) \geq u(c_1(i)). \end{aligned}$$

Thus, all together there are six IC constraints to consider (ignoring the one in (5) that holds automatically).

4.3 No-Bank-Run Conditions

Consider the solution to the problem above. Since the IC constraints above hold, we know that the last person in line will always tell the truth. The question, however, is whether there are circumstances in which the first and second persons in line have an incentive to jointly lie (if they do, then we have a bank-run equilibrium).

Let us consider the second person in line. Suppose he observes the report $\widetilde{\omega}_1 = i$. In a truth-telling equilibrium, this report will be taken at face value. But let us now suppose that (for some unexplained reason), the second (patient) person **believes** that the first (patient) person is a liar (recall that impatient agents will always tell the truth).

Let $\phi(\omega_1 \mid p) = \Pr[\omega_1 \mid \omega_2 = p]$. Thus, $\phi(i \mid p)$ is the probability attached by the second person on the first person's report being truthful (which only happens in the event that the first person is indeed impatient). With complementary probability, the first person is patient and perceived to be a liar. Under this scenario, the second person in line will view truth-telling a dominant strategy if and only if:

$$\begin{aligned} \phi(i &\mid p)\left[\pi(p \mid p, i)u(c'(i, p, p)) + \pi(i \mid p, i)u(c'(i, p, i))\right] \quad (8) \\ +[1 - \phi(i &\mid p)]\left[\pi(p \mid p, p)u(c'(i, p, p)) + \pi(i \mid p, p)u(c'(i, p, i))\right] \\ &\geq u(c_2(i, i)). \end{aligned}$$

Remark 2: Note that under the assumption of independence, we have $\pi(p \mid p,i) = \pi(p \mid p,p)$. In other words, the second person's conditional forecast over future truthful reports in no way depends on the history of type realizations (prior to the second person's own realization). Hence, under independence, the no-bank-run condition (8) corresponds to the second IC constraint in (6).

Result 1: The NBR condition (8) can only be violated if $\pi(p \mid p,i) > \pi(p \mid p,p)$.

Proof: Let $X = u(c'(i,p,p))$; $Y = u(c'(i,p,i))$; and $Z = u(c_2(i,i))$. As well, let $\alpha = \pi(p \mid p,i)$ and $\beta = \pi(p \mid p,p)$.Then the second IC constraint in (6) can be written as:

$$\alpha X + (1-\alpha)Y \geq Z;$$

while a violation of the NBR condition (8) can be written as:

$$\phi\left[\alpha X + (1-\alpha)Y\right] + (1-\phi)\left[\beta X + (1-\beta)Y\right] < Z.$$

Since $X > Y$ (by Lemma 4) for these two conditions to hold simultaneously, we need $\alpha > \beta$. [Note that these two conditions are guaranteed to hold for any $\alpha > \beta$ if the IC constraint is binding].

Conjecture 1: By Lemma 5, there should exist a parameter value $A > 1$ such that the IC constraints in (6) bind. [Note: computational results reveal that all the IC constraints will bind for large enough A].

So, we should be able to find parameters for A and $\pi(p \mid p,i), \pi(p \mid p,p)$ such that the NBR condition is violated for the second patient person in line under the optimal contract and under the assumption that he believes that the first patient person in line is a liar. The next question to address is whether it would ever make sense for the first patient person in line to misreport his type.

Suppose that the NBR condition (8) is violated and suppose that the second patient person in line **believes** that the first patient person in line is a liar. The question is whether this belief can be a self-fulfilling prophesy. Suppose that the first patient person in line anticipates that the second patient person in line believes he is a liar. Does the first patient person in line then have an incentive to fulfil this expectation? The answer is 'no' as long as the following condition is met:

$$\begin{aligned}\pi(p,p \mid p)u(c'(p,i,p)) + \pi(p,i \mid p)u(c'(p,i,i)) &+ \\ \pi(i,p \mid p)u(c'(p,i,p)) + \pi(i,i \mid p)u(c'(p,i,i)) &\geq u(c_1(i)).\end{aligned} \tag{9}$$

Result 2: Suppose that the IC constraint (7) binds. Then the NBR condition (9) is violated (this makes use of Lemma 4).

Results 1 and 2 suggest the following. Find a parameter A such that the relevant IC constraints bind for the first and second person in line. At the same time, choose a probability structure such that $\pi(p \mid p,i) > \pi(p \mid p,p)$. Then we have a situation in which the NBR conditions for both the first and second patient persons in line are jointly violated. Hence, there exists a second bank-run equilibrium where, under the optimal contract, the first and second patient persons in line misreport their types.

5 An Example

Consider the following parameterization:

$$\begin{aligned} u(c) &= \frac{[c^{1-\theta}-1]}{1-\theta};\ \text{with } \theta = 1.5; \\ \Pi(0) &= \Pi(1) = \Pi(2) = 1/3;\ \Pi(3) = 0; \\ R &= 1.10;\ \text{and } y = 1. \end{aligned}$$

Given this probability structure, the relevant conditional probabilities are given by:

$$\begin{aligned} \pi(p \quad &| \quad p,i) \equiv \alpha = \frac{\Pi(2)}{\Pi(1)+\Pi(2)}; \\ \pi(p \quad &| \quad p,p) \equiv \beta = \frac{\Pi(3)}{(1/3)\Pi(2)+\Pi(3)}; \\ \pi(p,p \quad &| \quad p) \equiv \gamma_1 = \frac{\Pi(3)}{(1/3)\Pi(1)+(2/3)\Pi(2)+\Pi(3)}; \\ \pi(p,i \quad &| \quad p) \equiv \gamma_2 = \frac{(1/3)\Pi(2)}{(1/3)\Pi(1)+(2/3)\Pi(2)+\Pi(3)}; \\ \pi(i,p \quad &| \quad p) \equiv \gamma_3 = \frac{(1/3)\Pi(2)}{(1/3)\Pi(1)+(2/3)\Pi(2)+\Pi(3)}; \\ \pi(i,i \quad &| \quad p) \equiv \gamma_4 = \frac{(1/3)\Pi(1)}{(1/3)\Pi(1)+(2/3)\Pi(2)+\Pi(3)}. \end{aligned}$$

Notice that since $\Pi(3) = 0$, we automatically have $\alpha > \beta$.

5.1 Objective Function

We wish to solve for six unknowns: $\mathbf{x} = (x_1, ..., x_6)$, where:

$x_1 = c_1(i);\ x_2 = c_2(i,i);\ x_3 = c_2(p,i);\ x_4 = c_3(p,p,i);\ x_5 = c_3(i,p,i);\ x_6 = c_3(p,i,i)$.

The objective function is given by:

$$
\begin{aligned}
&U(\mathbf{x}) = \Pi(3)\{3u(Ry)\} \\
&+(1/3)\Pi(2)\left\{\begin{array}{c} Au(x_4) + 2u((R/2)[3y - x_4]) \\ +Au(x_3) + 2u((R/2)[3y - x_3]) \\ +Au(x_1) + 2u((R/2)[3y - x_1]) \end{array}\right\} \\
&+(1/3)\Pi(1)\left\{\begin{array}{c} Au(x_3) + Au(x_6) + u(R[3y - x_3 - x_6]) \\ +Au(x_1) + Au(x_5) + u(R[3y - x_1 - x_5]) \\ +Au(x_1) + Au(x_2) + u(R[3y - x_1 - x_2])) \end{array}\right\} \\
&+\Pi(0)\left\{Au(x_1) + Au(x_2) + Au(3y - x_1 - x_2)\right\}.
\end{aligned}
$$

5.2 Incentive Constraints

1. $(R/2)[3y - x_1] \geq x_5$.
2. $(R/2)[3y - x_3] \geq x_6$.
3. $Ry \geq x_4$.
4. $\beta u(Ry) + (1 - \beta)u((R/2)[3y - x_4]) \geq u(x_3)$.
5. $\alpha u((R/2)[3y - x_1]) + (1 - \alpha)u(R[3y - x_1 - x_5]) \geq u(x_2)$.
6. $\gamma_1 u(Ry) + \gamma_2 u((R/2)[3y - x_4]) + \gamma_3 u((R/2)[3y - x_3]) + \gamma_4 u(R[3y - x_3 - x_6]) \geq u(x_1)$.

5.3 Results[2]

We report results for two values of $A \in \{1, 2\}$. For $A = 1$, none of the IC constraints bind (this is consistent with Green and Lin); for $A = 2$, all of the IC constraints bind.

	$A = 1$	$A = 2$
$c_1(i)$	1.00698	1.02780
$c_2(i, i)$	1.00039	1.02833
$c_2(p, i)$	1.01584	1.04500
$c_3(p, p, i)$	1.02127	1.10000
$c_3(i, p, i)$	1.01237	1.08471
$c_3(p, i, i)$	1.00780	1.07525
$c_3(i, i, i)$	0.99262	0.94387

[2] Fernando Martin wrote the Fortran code that performed these computations; see Appendix 3.

5.4 No-Bank-Run Conditions

The parameter ϕ is given by:

$$\phi = \frac{(1/3)\left[\Pi(1) + \Pi(2)\right]}{(1/3)\Pi(1) + (2/3)\Pi(2) + \Pi(3)}.$$

For the allocation $\mathbf{x}(A)$ computed above, evaluate the following two expressions:

$$\begin{aligned}
\Omega_2 &= \phi\left[\alpha u((R/2)[3y - x_1]) + (1-\alpha)u(R[3y - x_1 - x_5])\right] \\
&\quad + (1-\phi)\left[\beta u((R/2)[3y - x_1]) + (1-\beta)u(R[3y - x_1 - x_5])\right]; \\
\Omega_1 &= \gamma_1 u((R/2)[3y - x_3]) + \gamma_2 u(R[3y - x_3 - x_6]) + \gamma_3 u((R/2)[3y - x_3]) + \gamma_4 u(R[3y - x_3 - x_6]).
\end{aligned}$$

Calculate whether the following inequalities hold:

$$\begin{aligned}
\Omega_2 &\geq u(x_2); \\
\Omega_1 &\geq u(x_1).
\end{aligned}$$

If these last two inequalities are violated, then we have a bank-run equilibrium. We find that these two inequalities hold for $A = 1$ (which is consistent with Green and Lin). However, these two inequalities appear to be violated for $A = 2$. If the analysis and computations are correct, we have our bank run equilibrium.

6 Appendix 1: Objective Function

Given that all orderings of arrivals are equally likely, we have the following probabilities:

- (i, i, i) occurs w,p. $\Pi(0)$;
- $(p, i, i), (i, p, i), (i, i, p)$ each occur w.p. $(1/3)\Pi(1)$;
- $(p, p, i), (p, i, p), (i, p, p)$ each occur w.p. $(1/3)\Pi(2)$;
- (p, p, p) occurs w.p. $\Pi(3)$.

Invoking the results from Lemmas 1 and 2, the planner's objective function can be written as follows:

$$\begin{aligned}
E[U(c, c'; \omega)] &= \Pi(3)\{3u(c'(p, p, p))\} \\
&\quad +(1/3)\Pi(2)\left\{\begin{array}{c} Au(c_3(p, p, i)) + 2u(c'(p, p, i)) \\ +Au(c_2(p, i)) + 2u(c'(p, i, p)) \\ +Au(c_1(i)) + 2u(c'(i, p, p)) \end{array}\right\} \\
&\quad +(1/3)\Pi(1)\left\{\begin{array}{c} Au(c_2(p, i)) + Au(c_3(p, i, i)) + u(c'(p, i, i)) \\ +Au(c_1(i)) + Au(c_3(i, p, i)) + u(c'(i, p, i)) \\ +Au(c_1(i)) + Au(c_2(i, i)) + u(c'(i, i, p)) \end{array}\right\} \\
&\quad +\Pi(0)\left\{Au(c_1(i)) + Au(c_2(i, i)) + Au(c_3(i, i, i))\right\}.
\end{aligned}$$

Thus, the allocation consists of the following objects:

$$\begin{array}{l}
c_1(i), c_2(i, i), c_2(p, i), c_3(i, i, i), c_3(p, i, i), c_3(i, p, i), c_3(p, p, i); \\
c'(p, p, p), c'(p, p, i), c'(p, i, i), c'(i, p, i), c'(i, i, p), c'(p, i, p), c'(i, p, p).
\end{array}$$

The future consumptions can be substituted out by invoking the feasibility constraints:

$$\begin{aligned}
c'(p, p, p) &= Ry; \\
c'(p, p, i) &= R(1/2)[3y - c_3(p, p, i)]; \\
c'(p, i, i) &= R[3y - c_2(p, i) - c_3(p, i, i)]; \\
c'(i, p, i) &= R[3y - c_1(i) - c_3(i, p, i)]; \\
c'(i, i, p) &= R[3y - c_1(i) - c_2(i, i)]; \\
c'(p, i, p) &= R(1/2)[3y - c_2(p, i)]; \\
c'(i, p, p) &= (1/2)R[3y - c_1(i)].
\end{aligned}$$

And, as well, we can substitute out for $c_3(i, i, i) = 3y - c_1(i) - c_2(i, i)$.

7 Appendix 2: Conditional Probabilities

Let's start with $\pi(\omega_1 = i \mid \omega_2 = p)$. How can the probability of this event be expressed in terms of the $\pi(j)$? Here are the various events and their unconditional probabilities:

- (i, i, i) occurs w,p. $\Pi(0)$;
- $(p, i, i), (i, p, i), (i, i, p)$ each occur w.p. $(1/3)\Pi(1)$;
- $(p, p, i), (p, i, p), (i, p, p)$ each occur w.p. $(1/3)\Pi(2)$;
- (p, p, p) occurs w.p. $\Pi(3)$.

From Baye's rule, we have:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

The unconditional probability of a p in the second spot is given by:

$$(1/3)\Pi(1) + (2/3)\Pi(2) + \Pi(3).$$

The joint probability of observing $(\omega_1, \omega_2) = (i, p)$ is given by:

$$\pi(i, p) = (1/3)[\Pi(1) + \Pi(2)].$$

Thus, if follows that:

$$\phi(\omega_1 = i \mid \omega_2 = p) = \frac{(1/3)[\Pi(1) + \Pi(2)]}{(1/3)\Pi(1) + (2/3)\Pi(2) + \Pi(3)}.$$

Now, let's consider the conditional probability $\pi(\omega_3 = p \mid \omega_2 = p, \omega_1 = i)$. The unconditional probability of observing $(\omega_1, \omega_2) = (i, p)$ is given by:

$$(1/3)\Pi(1) + (1/3)\Pi(2).$$

The joint probability of observing $\omega_3 = p$ is given by:

$$(1/3)\Pi(2).$$

Therefore, we have:

$$\pi(\omega_3 = p \mid \omega_2 = p, \omega_1 = i) = \frac{\Pi(2)}{\Pi(1) + \Pi(2)}.$$

Now, let's consider the conditional probability $\pi(\omega_3 = p \mid \omega_2 = p, \omega_1 = p)$. The unconditional probability of observing $(\omega_1, \omega_2) = (p, p)$ is given by:

$$(1/3)\Pi(2) + \Pi(3).$$

The joint probability of observing $\omega_3 = p$ is given by $\Pi(3)$. Therefore, we have:

$$\pi(\omega_3 = p \mid \omega_2 = p, \omega_1 = p) = \frac{\Pi(3)}{(1/3)\Pi(2) + \Pi(3)}.$$

The following conditional probabilities are used for the first person in line. Consider now the conditional probability $\pi(\omega_2 = p, \omega_3 = p \mid \omega_1 = p)$. The unconditional probability of $\omega_1 = p$ is given by:

$$(1/3)\Pi(1) + (2/3)\Pi(2) + \Pi(3).$$

The joint probability of $(\omega_2, \omega_3) = (p, p)$ is given by $\Pi(3)$. Therefore,

$$\pi(\omega_2 = p, \omega_3 = p \mid \omega_1 = p) = \frac{\Pi(3)}{(1/3)\Pi(1) + (2/3)\Pi(2) + \Pi(3)}.$$

Similar calculations reveal:

$$\begin{aligned}
\pi(\omega_2 &= p, \omega_3 = i \mid \omega_1 = p) = \frac{(1/3)\Pi(2)}{(1/3)\Pi(1) + (2/3)\Pi(2) + \Pi(3)}; \\
\pi(\omega_2 &= i, \omega_3 = p \mid \omega_1 = p) = \frac{(1/3)\Pi(2)}{(1/3)\Pi(1) + (2/3)\Pi(2) + \Pi(3)}; \\
\pi(\omega_2 &= i, \omega_3 = i \mid \omega_1 = p) = \frac{(1/3)\pi(1)}{(1/3)\Pi(1) + (2/3)\Pi(2) + \Pi(3)}.
\end{aligned}$$

8 Appendix 3: Fortran Code

```
!*************************************************************************************
  ! icc
  ! by Fernando Martin (2005)
  !
  ! Constrained Optimization Problem
  !
  !*************************************************************************************
  Module environment
  Implicit none
  ! Precision parameters
  integer, parameter :: prec = 8                ! double precision
  ! Parameters
  real(prec), parameter :: A = 2.000_prec
  real(prec), parameter :: R        = 1.100_prec
  real(prec), parameter :: theta    = 1.500_prec
  real(prec), parameter :: y        = 1.000_prec
  real(prec), parameter :: pi(4)    = (/ 1/3.0_prec, 1/3.0_prec, 1/3.0_prec, 0.0_prec/)
  ! Other variables
  integer :: i, j
  real(prec) :: alpha, beta, gamma1, gamma2, gamma3, gamma4, phi
  ! Non-linear solver and interpolation settings
  real(prec) :: guess(6), sol(6), fvalue, xguess, xsol
  real(prec) :: xscale(6) = 1.0_prec
  ! Upper and lower bound
  real(prec) :: XLB(6) = 0.00001_prec
  real(prec) :: XUB(6) = 10000.0_prec
  Contains
  !*******************************************************************************
  !    FUNCTIONS
  !*******************************************************************************
  ! Utility function
```

```
real(prec) function u(c)
    Implicit none
    real(prec) :: c
    if(theta==1) then
        u = dlog(c)
    else
        u = (c**(1-theta) - 1) / (1-theta)
    end if
end function u
! ic1
real(prec) function ic1(x1,x2,x3,x4,x5,x6)
    Implicit none
    real(prec) :: x1, x2, x3, x4, x5, x6
    ic1 = R * (3*y-x1) / 2 - x5
end function ic1
! ic2
real(prec) function ic2(x1,x2,x3,x4,x5,x6)
    Implicit none
    real(prec) :: x1, x2, x3, x4, x5, x6
    ic2 = R * (3*y-x3) / 2 - x6
end function ic2
! ic3
real(prec) function ic3(x1,x2,x3,x4,x5,x6)
    Implicit none
    real(prec) :: x1, x2, x3, x4, x5, x6
    ic3 = R * y - x4
end function ic3
! ic4
real(prec) function ic4(x1,x2,x3,x4,x5,x6)
    Implicit none
    real(prec) :: x1, x2, x3, x4, x5, x6
    ic4 = beta * u(R*y) + (1-beta) * u(R*(3*y-x4)/2) - u(x3)
```

```
end function ic4
! ic5
real(prec) function ic5(x1,x2,x3,x4,x5,x6)
    Implicit none
    real(prec) :: x1, x2, x3, x4, x5, x6
    ic5 = alpha * u(R*(3*y-x1)/2) + (1-alpha) * u(R*(3*y-x1-x5)) - u(x2)
end function ic5
! ic6
real(prec) function ic6(x1,x2,x3,x4,x5,x6)
    Implicit none
    real(prec) :: x1, x2, x3, x4, x5, x6
    ic6 = gamma1 * u(R*y) + gamma2 * u(R*(3*y-x4)/2) + gamma3 * u(R*(3*y-x3)/2) + gamma4 * u(R*(3*y-x3-x6)) - u(x1)
end function ic6
! objective function
real(prec) function Utility(x1,x2,x3,x4,x5,x6)
    Implicit none
    real(prec) :: x1, x2, x3, x4, x5, x6
    Utility = pi(4) * u (R*y) &
            & + pi(3) / 3 * ( A*u(x4) + 2*u(R*(3*y-x4)/2) + A*u(x3) + 2*u(R*(3*y-x3)/2) + A*u(x1) + 2*u(R*(3*y-x1)/2) ) &
            & + pi(2) / 3 * ( A*u(x3) + A*u(x6) + u(R*(3*y-x3-x6)) + A*u(x1) + A*u(x5) + u(R*(3*y-x1-x5)) + A*u(x1) + A*u(x2) + u(R*(3*y-x1-x2)) ) &
            & + pi(1) * ( A*u(x1) + A*u(x2) + A*u(3*y-x1-x2) )

end function Utility
! Bank run 1
real(prec) function Omega1(x1,x2,x3,x4,x5,x6)
    Implicit none
    real(prec) :: x1, x2, x3, x4, x5, x6
    Omega1 = phi * ( alpha*u(R*(3*y-x1)/2) + (1-alpha)*u(R*(3*y-x1-x5)) )&
        & + (1-phi) * ( beta *u(R*(3*y-x1)/2) + (1-beta) *u(R*(3*y-x1-x5)) )
end function Omega1
```

```
! Bank run 2
real(prec) function Omega2(x1,x2,x3,x4,x5,x6)
      Implicit none
      real(prec) :: x1, x2, x3, x4, x5, x6
      Omega2 = gamma1 * u(R*(3*y-x3)/2) + gamma2 * u(R*(3*y-x3-x6)) +
gamma3 * u(R*(3*y-x3)/2) + gamma4 * u(R*(3*y-x3-x6))
end function Omega2
End Module environment
!***************************************************************************************
!
! SUBROUTINES AND EXTERNAL FUNCTIONS
!
!***************************************************************************************
!***************************************************************************************
! FCN subroutine
!***************************************************************************************
subroutine FCN(nc,ne,nv,x,active,F,E)
      use environment
      Implicit none
      integer :: nv, nc, ne
      logical :: active(nc)
      real(prec) :: x(nv), F, E(nc)

      F = -Utility(x(1),x(2),x(3),x(4),x(5),x(6))
      if (active(1)) E(1) = ic1(x(1),x(2),x(3),x(4),x(5),x(6))
      if (active(2)) E(2) = ic2(x(1),x(2),x(3),x(4),x(5),x(6))
      if (active(3)) E(3) = ic3(x(1),x(2),x(3),x(4),x(5),x(6))
      if (active(4)) E(4) = ic4(x(1),x(2),x(3),x(4),x(5),x(6))
      if (active(5)) E(5) = ic5(x(1),x(2),x(3),x(4),x(5),x(6))
      if (active(6)) E(6) = ic6(x(1),x(2),x(3),x(4),x(5),x(6))
!write(*,"(6F10.5,F15.12)") x,-F
end subroutine FCN
!*******************************************************************************************
```

```
!
!      PROGRAM ICC
!
!*****************************************************************************************************
Program icc
use environment
use imslf90
external :: FCN
call ERSET (0, 1, 0)      ! tells ISLM not to stop if there's a fatal error
!open(unit=4,file='solutions.out')
! Check Pi
if(sum(pi)/=1.0_prec) then
     write(*,*) 'Matrix Pi does not sum up to 1.'
     stop
end if
! Derive parameters
alpha = pi(3) / (pi(2) + pi(3))
beta = pi(4) / (pi(3)/3 + pi(4))
gamma1 = pi(4) / (pi(2)/3 + pi(3)*2/3 + pi(4))
gamma2 = (pi(3)/3) / (pi(2)/3 + pi(3)*2/3 + pi(4))
gamma3 = gamma2
gamma4 = (pi(2)/3) / (pi(2)/3 + pi(3)*2/3 + pi(4))
phi      = ((pi(2)+pi(3))/3) / (pi(2)/3 + pi(3)*2/3 + pi(4))
! Set guess
guess = y
!guess = (/ 0.9_prec, 0.933333_prec, 0.857143_prec, 1.0_prec, 0.7_prec, 0.714286_prec
/)
! Solve
call DNCONF (FCN,6,0,6,guess,3,XLB,XUB,xscale,0,10000,sol,fvalue)
write(*,"(A,F10.5)") 'x1=', sol(1)
write(*,"(A,F10.5)") 'x2=', sol(2)
write(*,"(A,F10.5)") 'x3=', sol(3)
write(*,"(A,F10.5)") 'x4=', sol(4)
```

```
write(*,"(A,F10.5)") 'x5=', sol(5)
write(*,"(A,F10.5)") 'x6=', sol(6)
write(*,*)
write(*,"(A,F15.12)") 'IC1=', ic1(sol(1),sol(2),sol(3),sol(4),sol(5),sol(6))
write(*,"(A,F15.12)") 'IC2=', ic2(sol(1),sol(2),sol(3),sol(4),sol(5),sol(6))
write(*,"(A,F15.12)") 'IC3=', ic3(sol(1),sol(2),sol(3),sol(4),sol(5),sol(6))
write(*,"(A,F15.12)") 'IC4=', ic4(sol(1),sol(2),sol(3),sol(4),sol(5),sol(6))
write(*,"(A,F15.12)") 'IC5=', ic5(sol(1),sol(2),sol(3),sol(4),sol(5),sol(6))
write(*,"(A,F15.12)") 'IC6=', ic6(sol(1),sol(2),sol(3),sol(4),sol(5),sol(6))
! Check bank runs
write(*,*)
write(*,*) ' Omega u(x)'
write(*,"(2F10.5)") Omega1(sol(1),sol(2),sol(3),sol(4),sol(5),sol(6)), u(sol(2))
write(*,"(2F10.5)") Omega2(sol(1),sol(2),sol(3),sol(4),sol(5),sol(6)), u(sol(1))
if(Omega1(sol(1),sol(2),sol(3),sol(4),sol(5),sol(6)) < u(sol(2)) .and. Omega2(sol(1),sol(2),sol(3),sol(4),sol(5),sol(6))
< u(sol(1))) then
    write(*,*)
    write(*,*) 'Bank run!'
end if
write(*,*)
write(*,*) R*(3*y-sol(1))/2, R*(3*y-sol(1)-sol(5))
End Program icc
```