Digital Transmission through bandlimited AWGN channels

ENSC 428 – Spring 2008
Reference: Lecture 11 of Gallager
Chapter 8 of Proakis & Salehi
Review of AWGN channels

- Use of complete orthonormal set of $L_2$.
  - Projection, correlation, matched filter.
- No channel distortion, no channel bandwidth limitation
  - Separate symbol-by-symbol detection, assuming statistically independent source symbols
Band-limited channel possibly with distortion

\[ s(t) = \sum_{k} a_k p(t - kT), \]
where \( p(t) \) is a stair-well function.

Let us model the channel by an LTI system with known impulse response \( c(t) \). Then,
\[ r(t) = \sum_{k} a_k h(t - kT) + n(t). \]
Received waveform in time duration \([kT, (k+1)T]\] contains information of many symbols, not just one.

Given \( c(t) \), we can cleverly design pulse \( p(t) \) and a receiver filter to cope with it.
Sufficient statistics for optimal detection of $L$ symbols

\[ r(t) = \sum_{n=1}^{L} a_n h(t - nT) + w(t), \text{ real signal. Each } a_n \text{ can be one of } M \text{ symbols.} \]

Signal part $\sum_{n=1}^{L} a_n h(t - nT)$ is one of $M^L$ possible signals.

Use orthonormal functions $\{f_1, f_2, \ldots, f_K\}$.

\[ r_k = \langle r(t), f_k(t) \rangle = \langle \sum_{n=1}^{L} a_n h(t - nT), f_k(t) \rangle + \langle w(t), f_k(t) \rangle, k = 1, 2, \ldots, K \]

are sufficient statistics.

\[ r_k = \langle r(t), \phi_k(t) \rangle = \langle w(t), f_k(t) \rangle, k = K + 1, K + 2, \ldots. \]

\[ p(r|a) = \frac{1}{\sqrt{\pi N_0^K}} \exp \left[ - \frac{\sum_{k=1}^{K} |r_k - \sum_{n=1}^{L} a_n \langle h(t - nT), f_k(t) \rangle|^2}{N_0} \right] \]
For simple implementation

Maximum Likelihood decision

$$\arg\max_a \frac{1}{\sqrt{\pi N_0}} \exp \left[ - \frac{1}{N_0} \sum_{k=1}^{K} \left| r_k - \sum_{n=1}^{L} a_n \langle h(t-nT), f_k(t) \rangle \right|^2 \right]$$

$$= \arg\min_a \sum_{k=1}^{K} \left| r_k - \sum_{n=1}^{L} a_n \langle h(t-nT), f_k(t) \rangle \right|^2$$

$$= \arg\min_a \left\{ \sum_{k=1}^{K} \left| r_k - \sum_{n=1}^{L} a_n \langle h(t-nT), f_k(t) \rangle \right|^2 + \sum_{k=K+1}^{\infty} |r_k - 0|^2 \right\}$$

$$= \arg\min_a \left\{ \int \left| r(t) - \sum_{n=1}^{L} a_n h(t-nT) \right|^2 dt \right\}$$
\[
\begin{align*}
\arg \min_a \int \left| r(t) - \sum_{n=1}^{L} a_n h(t - nT) \right|^2 dt &= \arg \min_a \left[ \int |r(t)|^2 dt - 2 \sum_{n=1}^{L} a_n \int r(t)h(t - nT)dt \right. \\
&\quad\quad\quad\quad\left. + \sum_{n=1}^{L} \sum_{m=1}^{L} a_n a_m \int h(t - nT)h(t - mT) dt \right] \\
&= \arg \min_a \left[ -2 \sum_{n=1}^{L} a_n \int r(t)h(t - nT)dt \right. \\
&\quad\quad\quad\quad\left. + \sum_{n=1}^{L} \sum_{m=1}^{L} a_n a_m \int h(t - nT)h(t - mT) dt \right]
\end{align*}
\]

\[\int_{-\infty}^{\infty} r(t)h(t - nT) dt, \ n = 1, 2, ..., L \text{ are sufficient statistics.}\]
Sufficient statistics for optimal detection of L symbols

- We will later consider the signal processing for optimal detection.
- For now, we conclude that we can design an optimal receiver by using a filter followed by a sampler.
Sufficient statistics for optimal detection of L complex symbols

For band-pass modulation, we can use complex base-band Representation of signals.
**Representation of Bandpass Signal**

\[
x(t) = s(t) \cos(2\pi f_c t)
\]

Bandpass real signal \(x(t)\) can be written as:

\[
x(t) = \sqrt{2} \text{Re}\left[\tilde{x}(t) e^{j2\pi f_c t}\right] \quad \text{where} \quad \tilde{x}(t) \text{ is complex envelop}
\]

Note that \(\tilde{x}(t) = \tilde{x}_I(t) + j \cdot \tilde{x}_Q(t)\)

\[\uparrow\quad \text{In-phase}\quad \uparrow \quad \text{Quadrature-phase}\]
Representation of Bandpass Signal

\[ x(t) = \sqrt{2} \text{Re} \left[ \tilde{x}(t) e^{j2\pi f_c t} \right] \]
\[ = \sqrt{2} \text{Re} \left[ \tilde{x}_I(t) + j \cdot \tilde{x}_Q(t) \right] \left[ \cos(2\pi f_c t) + j \sin(2\pi f_c t) \right] \]
\[ = \tilde{x}_I(t) \sqrt{2} \cos(2\pi f_c t) + \tilde{x}_Q(t) \left[ -\sqrt{2} \sin(2\pi f_c t) \right] \]

Note that \( \tilde{x}(t) = \left| \tilde{x}(t) \right| e^{j\theta(t)} \)

\[ x(t) = \sqrt{2} \text{Re} \left[ \tilde{x}(t) e^{j2\pi f_c t} \right] = \sqrt{2} \text{Re} \left[ \left| \tilde{x}(t) \right| e^{j\theta(t)} \cdot e^{j2\pi f_c t} \right] \]
\[ = \left| \tilde{x}(t) \right| \sqrt{2} \cos(2\pi f_c t + \theta(t)) \]
Relation between $x(t)$ and $\tilde{x}(t)$

\[ X(f) = \frac{1}{\sqrt{2}} \left[ \tilde{X}(f - f_c) + \tilde{X}^*(-(f + f_c)) \right] \]

\[ X_+(f) = \begin{cases} X(f), & f > 0 \\ 0, & f < 0 \end{cases}, \quad \tilde{X}(f) = X_+(f + f_c) \]
Energy of $s(t)$

$$E = \int_{-\infty}^{\infty} s^2(t) \, dt$$

$$= \int_{-\infty}^{\infty} |S(f)|^2 \, df \quad \text{(Rayleigh's energy theorem)}$$

$$= 2\int_{0}^{\infty} |S(f)|^2 \, df \quad \text{(Conjugate symmetry of real } s(t) \text{ )}$$

$$= \int_{0}^{\infty} |\tilde{S}(f)|^2 \, df$$
Representation of bandpass LTI System

\[ \tilde{r}(t) = \tilde{s}(t) \ast \tilde{h}(t) \]
\[ \tilde{R}(f) = \tilde{S}(f) \tilde{H}(f) \]
\[ = \tilde{S}(f) H(f + f_c) \quad \text{because } s(t) \text{ is band-limited.} \]

\[ H(f) = [\tilde{H}(f - f_c) + \tilde{H}^*(-(f + f_c))] \]
\[ H_+(f) = \begin{cases} H(f), & f > 0 \\ 0, & f < 0 \end{cases} \]
\[ \tilde{H}(f) = H_+(f + f_c) \]
Band-pass pulse modulation

- Band-pass noise
Sufficient statistics for optimal detection of $L$ complex symbols

$$ r(t) = \sum_{n=1}^{L} a_n h(t - nT) + w(t), \text{ base-band complex representation of signal} $$

Each $a_n$ can be one of $M$ symbols. Signal part $\sum_{n=1}^{L} a_n h(t - nT)$ has a finite number of possibilities. Use orthonormal functions $\{f_1, f_2, \cdots f_K\}$.

$$ r_k = \langle r(t), f_k(t) \rangle = \left\langle \sum_{n=1}^{L} a_n h(t - nT), f_k(t) \right\rangle + \langle w(t), f_k(t) \rangle, k = 1, 2, \ldots, K $$

are sufficient statistics.

$$ p\left(r^I, r^Q | a\right) = \frac{1}{\sqrt{\pi N_0} 2^K} \exp \left[ \frac{-\sum_{k=1}^{K} r_k - \sum_{n=1}^{L} a_n \left\langle h(t - nT), f_k(t) \right\rangle^2}{N_0} \right] $$
Sufficient statistics for optimal detection of $L$ complex symbols

ML decision:

$$\arg\min_a \sum_{k=1}^{K} \left| r_k - \sum_{n=1}^{L} a_n \langle h(t-nT), f_k(t) \rangle \right|^2$$

$$= \arg\min_a \left\{ \sum_{k=1}^{K} \left| r_k - \sum_{n=1}^{L} a_n \langle h(t-nT), f_k(t) \rangle \right|^2 + \sum_{k=K+1}^{\infty} |r_k - 0|^2 \right\}$$

$$= \arg\min_a \left\{ \int \left| r(t) - \sum_{n=1}^{L} a_n h(t-nT) \right|^2 dt \right\}$$

$$= \arg\min_a \left\{ \int |r(t)|^2 dt + \int r(t) \sum_{n=1}^{L} a_n^* h(t-nT)^* dt + \int r(t)^* \sum_{n=1}^{L} a_n h(t-nT) dt \right\}$$

$$+ \int \left| \sum_{n=1}^{L} a_n h(t-nT) \right|^2 dt$$

$$= \arg\min_a \left\{ \int r(t) \sum_{n=1}^{L} a_n^* h(t-nT)^* dt + \int r(t)^* \sum_{n=1}^{L} a_n h(t-nT) dt \right\}$$

$$+ \int \left| \sum_{n=1}^{L} a_n h(t-nT) \right|^2 dt$$
For rigorous minds only

\[
  r(t) = \sum_{n=1}^{L} a_n h(t - nT) + w(t), \quad \text{base-band complex representation of signal}
\]

\( w(t) \) is a complex base-band representation of band-pass noise, so its power spectral density has a finite support. Yet, we used

\[
  r_k = \langle r(t), f_k(t) \rangle = \left( \sum_{n=1}^{L} a_n h(t - nT), f_k(t) \right) + \langle w(t), f_k(t) \rangle, \quad k = 1, 2, ..., K
\]

Are \( \langle w(t), f_k(t) \rangle, k = 1, 2, ..., K \) I.I.D.?

Yes, instead of the low-pass complex white, we can just use a complex white noise as long as \( f_k(t), k = 1, 2, ..., K \) are low-pass signals. View \( \langle w(t), f_k(t) \rangle \) as a sampled value of the matched filter output. If the matched filter low-pass, then the all-pass white noise input and the low-pass white noise input result in the same output.
Equalization

\[
\arg \min_a \left\{ \int \left| r(t) - \sum_{n=1}^{N} a_n h(t-nT) \right|^2 dt \right\}
\]

\[
= \arg \min_a \left[ \int |r(t)|^2 dt + \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m \int h(t-nT) h(t-mT) dt - 2 \sum_{n=1}^{N} a_n \int r(t) h(t-nT) dt \right]
\]

\[
= \arg \min_a \left[ \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m \int h(t-nT) h(t-mT) dt - 2 \sum_{n=1}^{N} a_n \int r(t) h(t-nT) dt \right]
\]

\[
\int r(t) h(t-nT) dt, \ n = 1, 2, \ldots, N \text{ are sufficient statistics.}
\]

\[
y[n] = y(nT) = \int r(\tau) h^* (\tau - nT) d\tau
\]

**Diagram:**
- **Input:** \(r(t)\)
- **Output:** \(h^*(-t)\)
- **Signal processing:**

**Matched filter**
Matched filter output with colored noise

\[ r(t) = \sum_{m=1}^{L} a_m h(t - mT) + w(t) \]
\[ y[n] = y(nT) = \sum_m a_m \int_{-\infty}^{\infty} h(\tau - mT) h^*(\tau - nT) d\tau + \int_{-\infty}^{\infty} w(\tau) h^*(\tau - nT) d\tau \]
\[ = \sum_{m=1}^{L} a_m x[n - m] + v[n] \]

where \( x[n - m] \equiv \int_{-\infty}^{\infty} h(\tau - mT) h^*(\tau - nT) d\tau \)

DT-signal \( y[n] \) is viewed as a convolution of symbol streams \( a_m \) and \( x[n - m] \) plus colored noise.
Discrete-time model

\[ r(t) = \sum_{m=1}^{L} a_m h(t - mT) + w(t) \]

\[ y[n] = y(nT) = \sum_{m=1}^{L} a_m \int_{-\infty}^{\infty} h(\tau - mT) h^*(\tau - nT) d\tau + \int_{-\infty}^{\infty} w(\tau) h^*(\tau - nT) d\tau \]

\[ = \sum_{m=1}^{L} a_m x[n-m] + v_n \]

where \( x[n-m] \equiv \int h(\tau - mT) h^*(\tau - nT) d\tau \).

Discrete-time model is convenient, but noise \( v_n \) is not white.

It is natural to assume that \( x[n-m] = 0 \) if \(|k - j| > L_h\) for some \( L_h \).

Auto-correlation of colored noise \( v_n \):

\[ \phi_{vv}(k) \equiv E[v_{n+k} v_n^*] = \begin{cases} N_0 x_k & \text{if} \quad |k| \leq L_h \\ 0 & \text{otherwise} \end{cases} \]
Whiteening filter: review of discrete-time random processes

Wide-sense-stationary zero-mean random processes

\[ f[n] \]

\[ x[n] \rightarrow f[n] \rightarrow y[n] \]

\[ \Gamma_{yy}(z) = F(z)F^*(1/z^*) \Gamma_{xx}(z) \]

where \[ \Gamma_{xx}(z) = \sum_{n} \phi_{xx}(n) z^{-n}, \phi_{xx}(n) = E(x[k+n]x^*[k]), etc. \]
Whitening filter

\[ y[n] = \sum_{m=1}^{L} a_m x[n-m] + v_n \]

where \( x[n-m] = \int h(\tau - mT) h^*(\tau - nT) d\tau. \)

It is natural to assume that \( x[n-m] = 0 \) if \( |k-j| > L_h \) for some \( L_h. \)

Auto-correlation of colored noise \( v_n: \)

\[ \phi_{vv}(k) = E[v_{n+k}v_n^*] = \begin{cases} N_0x_k & \text{if } |k| \leq L_h \\ 0 & \text{otherwise} \end{cases} \]

\( x_k = x_{-k}^*, \) so \( X(z) = X^*(1/z^*) \)

\( X(z) \) has 2\( L \) roots; if \( \rho \) is a root, \( 1/\rho^* \) is also a root.

Factorization \( X(z) = F(z)F^*(1/z^*) \)

so that \( F(z) \) is minimum-phase (i.e., both \( F(z) \) and its inverse filter are causal and stable.)
Equivalent discrete-time white noise filter model

\[ x_n, \quad X(z) = F(z)F^*(1/z^*) \]

\[ q_k = \sum_{n=0}^{L_h} a_{k-n}f_n + \eta_k \]

Noise is white.
Designing a pulse: Condition for no ISI

- For now, let us ignore the additive noise.
- Nyquist criterion for no ISI.
Condition for no ISI
Nyquist’s First Criterion for Zero ISI

To have zero ISI, we need

\[ p(t) = \begin{cases} 
1 & t = 0 \\
0 & t = nT_b 
\end{cases} \]

- if \( p(t) \) is sampled by an impulse train, then

\[ p(t) \sum_n \delta(t - nT_b) = \delta(t) \]

- known as the \textit{zero-forcing} criterion
  - often a goal of equalizers
cont ...

\[ p(t) \sum_n \delta(t - nT_b) = \delta(t) \]

- taking the Fourier transform gives

\[
P(\omega) \ast \frac{1}{T_b} \sum_k \delta \left( \omega - n \frac{2\pi}{T_b} \right) = 1
\]

\[
\sum_k P(\omega - n\omega_b) = T_b \quad \omega_b = \frac{2\pi}{T_b}
\]

\textit{Nyquist’s First Criterion}
cont ... 

- assuming that $P(\omega) = 0$, $|\omega| > \omega_b = 2\pi/T_b$ gives

$$P(\omega) + P(\omega - \omega_b) = T_b \quad 0 < \omega < \omega_b$$

- this happens when

fold over and add
cont ... 

- form an equivalent pulse by folding over at $\omega_b/2$
  - zero ISI if the result is rectangular

- the minimum bandwidth Nyquist-1 pulse is clearly

\[
p(t) = \text{sinc}\left(\frac{t}{T_b}\right) = \text{sinc}\left[\frac{\omega_b}{2\pi}t\right]
\]
Roll-Off Factor

- Letting the pulse BW grow gives flexibility
  - how does the pulse BW affect the eye?
    - narrower bandwidth → more ringing in the time domain
    - narrower eyes (in general)

- the roll-off factor parameterizes the BW

\[
\beta = \frac{\gamma}{f_b/2} = \frac{2\gamma}{f_b}
\]

also commonly called \( \alpha \)
Raised Cosine Pulses

A family of pulse shapes with different $\beta$’s

invented by Nyquist ~ 1928

$$p(t) = \frac{\cos\left(\frac{\pi \beta t}{T_b}\right)}{1 - (2\beta t / T_b)^2} \operatorname{sinc}(t / T_b)$$
cont ...

- In the frequency domain

\[
P(f) = \begin{cases} 
    T_b & |f| \leq \frac{1-\beta}{2T_b} \\
    \frac{T_b}{2} \left[1 + \cos \left(|f| - \frac{1-\beta}{2T_b}\right)\right] & \frac{1-\beta}{2T_b} < |f| \leq \frac{1+\beta}{2T_b} \\
    0 & |f| \leq \frac{1+\beta}{2T_b}
\end{cases}
\]

\[f_s = 1/T, \quad H(j2\pi f)\]
**Eye Diagram**

- Effectively shows the effects of ISI

- Extract each symbol period
- Plot on top of each other
cont ...
cont ...

- Easy to show on a scope

before and after pre-distortion

NOTES: CABLE LENGTH=1000 FT., DATA RATE=5 MBPS.
cont ... 

- **Interpretation**
  - **slope** indicates sensitivity to timing errors
  - **zero crossing variation**
  - **width**
  - **wasted power**
  - **noise tolerance**
cont ...

- Works for multi-level systems
  - consider a 4-level system

- where should the slicing levels go?
cont ...

- Need pulse shapes that give OPEN eyes
  - the sinc pulse is not one of these!
Examples (1)

- What is the eye diagram

\[ \frac{RC}{1 + j\omega RC} \rightarrow p(t) \]

- we have full width pulses and polar signaling strategy?
  - find the filter pulse response & superimpose
the step response is

\[ s(t) = 1 - e^{-t/RC} = 1 - e^{-t/T_b}, \quad t \geq 0 \]

the pulse response is

\[ p(t) = \left(1 - e^{-t/T_b}\right)u(t) - \left(1 - e^{-(t-T_b)/T_b}\right)u(t - T_b) \]

- assume zero after 3 time constants
cont ...

- the filtered data

- find the resultant

look at all the ISI!
the best sampling time is clearly at $t = nT_b$
cont ... \[ p(t) = (1 - e^{-t/T_b})u(t) - (1 - e^{-(t-T_b)/T_b})u(t-T_b) \]

- compute the eye opening
  - worst case ISI – all destructive

max ISI = \[ (1 - e^{-2T_b/T_b}) - (1 - e^{-T_b/T_b}) + (1 - e^{-3T_b/T_b}) - (1 - e^{-2T_b/T_b}) \]

\[ = e^{-1} = 0.368 \]
The worst case opening is thus

\[
eye \text{ opening} = 2\left[1 - e^{-1} - e^{-1}\right] = 0.53V
\]

+ve and –ve pulses
peak ISI
wanted signal

- probability of error is dominated by the worst case
Examples (2)

- Effect of channel noise
  - quaternary system \((M = 4)\)
  \((M-1) = 3\) eye openings!
  - raised cosine pulse shaping \((\alpha = 0.5)\)
  - symbol time \(T = T_b \log_2 M = 2T_b\)
  - no bandlimiting

No noise

SNR = 20 dB

SNR = 10 dB
cont ... 

- Effect of bandlimiting
  - low-pass Butterworth filter
    \[ |H(f)|^2 = \frac{1}{1 + (f/f_o)^{2N}} \]
    \[(N = 25)\]
  - signal bandwidth
    \[B_T = W(1 + \alpha) = 0.75 \text{ Hz} \]
    \[(W = 0.5 \text{ Hz, } \alpha = 0.5)\]