Revealed Preference Foundations of Expectations-Based Reference-Dependence

David Freeman*

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Abstract

This paper provides revealed preference foundations for a model of expectations-based reference-dependence à la Kőszegi and Rabin (2006). Novel axioms provide distinguishing features of expectations-based reference-dependence under risk. The analysis completely characterizes the model's testable implications when expectations are unobservable.

*Department of Economics, Simon Fraser University. E-mail: david_freeman@sfu.ca. I am extremely grateful to Yoram Halevy for support and guidance throughout the process of developing and writing this paper. I would also like to thank Faruk Gul, Li Hao, Maibang Khounvongsa, Terri Kneeland, Wei Li, Ryan Oprea, Mike Peters, Matthew Rabin, and conference and seminar participants at UBC, SFU, Toronto, Exeter, Decision Day at Duke, the Canadian Economic Theory Conference at Concordia, RUD at Paris, Maastricht Behavioral and Experimental Economics Symposium, the Econometric Society North American Summer Meetings in Los Angeles and the BRIC Workshop at St. Andrew's for helpful conversations and comments.

1 Introduction

Seminal work by Kahneman and Tversky introduced psychologically and experimentally motivated models of *reference-dependence* to economics. A limitation preventing the adoption of reference-dependent models is that reference points are not a directly observable economic variable. Kahneman and Tversky (1979) acknowledge that while it may be natural to assume that a decision-maker's status quo determines her reference point in their experiments, it is not appropriate in many interesting economic environments. The lack of a generally applicable model of reference point formation in economic environments has hindered applications of reference-dependence to economic settings.

Kőszegi and Rabin (2006) propose a model in which a decision-maker's recently-held expectations determine her reference point. Their solution concept for endogenously determined reference points has made their model convenient in numerous economic applications, including risk-taking and insurance decisions, consumption planning and informational preferences, firm pricing, short-run labour supply, labour market search, contracting under both moral hazard and adverse selection, and domestic violence.[1] In many of these applications, observed behaviour that appears impossible to explain using standard models naturally fits the intuition of expectations-based reference-dependence.

Little is known about the testable implications of expectations-based reference-dependence in more general settings in spite of the large number of applications. It has been suggested that models of expectations-based reference-dependence may have no meaningful revealed preference implications, and that their success comes from adding in an unobservable variable, the reference point, used at the modeller's discretion (Gul and Pesendorfer, 2008). The results here confront this claim: models of expectations-based reference-dependence do have economically meaningful and testable implications for standard economic data. The revealed preference axioms of this paper completely summarize the implications of a widely-applied version of the

[1] Kőszegi and Rabin (2007); Sydnor (2010); Kőszegi and Rabin (2009); Heidhues and Kőszegi (2008, Forthcoming); Karle and Peitz (2012); Crawford and Meng (2011); Abeler et al. (2011); Pope and Schweitzer (2011); Eliaz and Spiegler (2013); Herweg et al. (2010); Carbajal and Ely (2012); Card and Dahl (2011).

model.

The main contribution of this paper is to provide a set of revealed preference axioms that constitute necessary and sufficient conditions for a model of expectations-based reference-dependence. Commonly-used cases of Kőszegi and Rabin's model are special cases of the model studied here. The revealed preference axioms clarify how the model can be tested against both the standard rational model and against alternative behavioural theories.

As in existing models of reference-dependence, behaviour is consistent with maximizing preferences conditional on the decision-maker's reference point. The main challenge of the analysis is that expectations are not observed in standard economic data. Under expectations-based reference-dependence, the interaction between optimality given a reference point and the determination of the reference point as rational expectations can generate behaviour that appears unusual since expectations are not observed. Axioms justified by the logic of expectation-dependent decisions are shown to summarize the testable content of this unusual behaviour.

1.1 Background: expectations-based reference-dependence

The logic of reference-dependence suggests that rather than using a single utility function, a reference-dependent decision-maker has a set of reference-dependent utility functions. The utility function $v(\cdot|r)$ defines the decision-maker's utility function given reference lottery r. When the reference lottery r is observable, as in the case where a decision-maker's status quo is her referent, standard techniques can be applied to study $v(\cdot|r)$. But when the reference lottery is determined endogenously and is unobserved, as in the case where the reference lottery is determined by the decision-maker's recent expectations, an additional modelling assumption is needed. To that end, Kőszegi and Rabin (2006) introduce two solution concepts - personal equilibrium and preferred personal equilibrium - that capture the endogenous determination of the reference lottery for models with expectations as the reference lottery.

In an environment in which a decision-maker faces a fully-anticipated choice set D, rational expectations require that the decision-maker's reference lottery corresponds with her actual choice from D. In such an environment, the set of *personal equilibria* of D provides a natural set of predictions of a decision-maker's choice from a set D:

$$PE_v(D) = \{p \in D : \ v(p|p) \geq v(q|p) \ \forall q \in D\} \tag{1}$$

The personal equilibrium concept has the following interpretation. When choosing from choice set D, a decision-maker uses her reference-dependent preferences $v(\cdot|r)$ given her reference lottery (r) and chooses $\arg\max_{p \in D} v(p|r)$. When forming expectations, the decision-maker recognizes that her expected choice p will determine the reference lottery that applies when she chooses from D. Thus, she would only expect a $p \in D$ if it would be chosen by the reference-dependent utility function $v(\cdot|p)$, that is, if $p \in \arg\max_{q \in D} v(q|p)$. The set of personal equilibria of D in (1) is the set of all such p.

There may be a multiplicity of personal equilibria for a given choice set. Indeed, if reference-dependence tends to bias a decision-maker towards her reference lottery, multiplicity is natural. At the time of forming her expectations, a decision-maker evaluates the lottery p according to $v(p|p)$, which reflects that she will evaluate outcomes of lottery as gains and losses relative to outcomes of p itself. The *preferred personal equilibrium* concept is a natural refinement of the set of personal equilibria based on a decision-maker picking her best personal equilibrium expectation according to $v(p|p)$:

$$PPE_v(D) = \arg\max_{p \in PE(D)} v(p|p) \tag{2}$$

Kőszegi and Rabin (2006) adopt a particular functional form for v. They assume that given probabilistic expectations summarized by the lottery r, a decision-maker ranks a lottery p according to:

$$v^{KR}(p|r) = \sum_k \sum_i p_i m^k(x_i^k) + \sum_k \sum_i \sum_j p_i r_j \mu\left(m^k(x_i^k) - m^k(x_j^k)\right) \tag{3}$$

In (3), m^k is a consumption utility function in "hedonic dimension" k; different hedonic dimensions are akin to different goods in a consumption bundle, but specified based on "psychological principles". The function μ is a gain-loss utility function which captures reference-dependent outcome evaluations.

The Kőszegi-Rabin model with the preferred personal equilibrium concept has been particularly amenable to applications, since the model's predictions are pinned down by (3) and (2). However, little is known about how the Kőszegi-Rabin model behaves except in very specific applications.

This paper focuses on expectations-based reference-dependent preferences with the preferred personal equilibrium concept as in (2). Theorem 1 provides a complete revealed preference characterization of the choice correspondence c that equals the set of all preferred personal equilibria of a choice set, $c(D) = PPE_v(D)$. The model of decision-making equivalent to the axioms does not restrict v to the form in (3) but does require that v be jointly continuous in its arguments, $v(\cdot|r)$ satisfy expected utility, and v satisfy a property related to disliking mixtures of lotteries.

The tight characterization of the PPE model of expectations-based reference-dependence in Theorem 1 may come as a surprise relative to previous work (e.g. Gul and Pesendorfer 2008; Kőszegi 2010).[2] The analysis here also provides additional surprising connections. First, the PPE representation is related to the shortlisting representation of Manzini and Mariotti (2007), a connection clarified in Proposition 1. Second, there is a tight connection between expectations-based reference-dependence and failures of the Mixture Independence Axiom; violations of Independence of Irrelevant Alternatives (IIA) are sufficient but not necessary for expectations-dependent behaviour in the model (Proposition 2).

1.2 Outline

Section 2 provides two examples that motivate expectations-based reference-dependence, and a result that illustrates the limits to the model's testable implications in environments without risk. Section 3 provides axioms and a representation theorem for PPE decision-making, and suggests a way of defining expectations-dependence in terms

[2] Gul and Pesendorfer (2008) show that with the personal equilibrium concept and without using any lottery structure, the reference-dependent preferences of Kőszegi and Rabin (2006) have no testable implications beyond an equivalence with a choice correspondence generated by a binary relation. Kőszegi (2010) initially proposed the personal equilibrium concept studied here but provides only a limited set of testable implications, and suggested that a complete revealed preference may not be possible: "I do not offer a revealed-preference foundation for the enriched preferences—it is not clear to what extent the decisionmaker's utility function can be extracted from her behavior."

of observable behaviour. Section 4 explores special cases of the model, including Kőszegi-Rabin and a new axiomatic model of expectations-based reference lottery bias. Section 5 shows how the analysis can be adapted to study PE decision-making and also to decision-making under Kőszegi and Rabin's (2007) choice-acclimating personal equilibrium (CPE).

2 Two examples and a motivating result

2.1 Formal setup

Let Δ denote the set of all *lotteries* with support on a given finite set X, with typical elements $p, q, r \in \Delta$. Let $\mathcal{D}$ denote the set of all finite subsets of Δ, a typical $D \in \mathcal{D}$ is called a *choice set*. The starting point for analysis is a *choice correspondence*, $c : \mathcal{D} \to \mathcal{D}$, which is taken as the set of elements we might observe a decision-maker choose from a set D. Assume $\emptyset \neq c(D) \subseteq D$, that is, a decision-maker always chooses something from her choice set.

Define the mixture operation $(1-\lambda)D + \lambda D' := \{(1-\lambda)p + \lambda q : \, p \in D, \, q \in D'\}$.

2.2 Mugs, pens, and expectations-based reference-dependence

The classic experimental motivation for loss-aversion in riskless choice comes from the *endowment effect*. An example of an endowment effect comes from the experimental finding that randomly-selected subjects given a mug have a median willingness-to-accept for a mug that is double the median willingness-to-pay of subjects who were not given a mug (Kahneman et al., 1990). This classic experiment provides no separation between status-quo-based and expectations-based theories of reference-dependence since subjects given a mug could expect to be able to keep it at the end of the experiment.

To separate expectations-based theories of reference-dependence from status-quo based theories, Ericson and Fuster (2011) design an experiment in which all subjects are endowed with a mug, and subjects are told that there is a fixed probability (either 10% or 90%) they will receive their choice between a retaining the mug or instead obtaining a pen, and with the remaining probability they will retain the mug;

the conditional choice must be made before uncertainty is resolved.[3] Subjects in a treatment with a 10% chance of receiving their choice must expect to receive a mug with at least a 90% chance, and consistent with expectations-based reference-dependence, 77% of these subjects' conditionally choose the mug. In contrast, only 43% of subjects conditionally choose the mug in the treatment in which subjects received their chosen item with a 90% chance.

The Mixture Independence axiom below adapts of von-Neuman and Morgenstern's axiom to a choice correspondence.

Mixture Independence. $(1-\alpha)c(D) + \alpha c(D') = c((1-\alpha)D + \alpha D')\ \forall \alpha \in (0,1)$

The median choice pattern in Ericson and Fuster's experiment has $\{\langle \text{mug}, 1\rangle\} = c(.9\{\langle \text{mug}, 1\rangle\} + .1\{\langle \text{mug}, 1\rangle, \langle \text{pen}, 1\rangle\})$ but $\{\langle \text{mug}, .1; \text{pen}, .9\rangle\} = c(.1\{\langle \text{mug}, 1\rangle\} + .9\{\langle \text{mug}, 1\rangle, \langle \text{pen}, 1\rangle\})$. This choice pattern suggests an intuitive and empirically supported violation of Mixture Independence that is consistent with expectation-bias.

2.3 IIA violations under Kőszegi-Rabin under PPE

Consider a decision-maker with a Kőszegi-Rabin v as in (3), with linear utility and linear loss aversion:[4][5]

$$m(x) = x, \quad \mu(x) = \begin{cases} x & \text{if } x \geq 0 \\ 3x & \text{if } x < 0 \end{cases}$$

When faced with a set of lotteries, suppose that our decision-maker chooses his preferred personal equilibrium lottery as in (2).

Consider the three lotteries $p = \langle \$1000, 1\rangle$, $q = \langle \$0, .5; \$2900, .5\rangle$, and $r = \langle \$0, .5; \$2000, .25; \$4100, .25\rangle$. As broken down in Table 1, the decision-maker's

[3] This paper interprets the subjects' choice as being between two lotteries, each of which involves the prize of the mug with a fixed probability (10% or 90%) and the prize chosen by the subject with the remaining probability. An alternative interpretation of the experimental setup is that subjects face a lottery over choice sets, one of which is a singleton, and must choose from the non-singleton choice set before the lottery is resolved. For a result on the formal relationship between these choice spaces, see Ortoleva (2013).

[4] I would like to specially thank Matthew Rabin for suggesting this example.

[5] Linear loss aversion is used in most applications of Kőszegi-Rabin, and the chosen parameterization is broadly within the range implied by experimental studies.

Table 1: Example of reference-dependent preferences

	$v(p\|\cdot)$	$v(q\|\cdot)$	$v(r\|\cdot)$
$v(\cdot\|p)$	1000	900	1050
$v(\cdot\|q)$	-1350	0	-75
$v(\cdot\|r)$	-1575	-450	-262.50

choice correspondence, c, is given by $\{p\} = c(\{p, q\})$, $\{q\} = c(\{q, r\})$, $\{r\} = c(\{p, r\})$, and $\{q\} = c(\{p, q, r\})$.

Choice from binary sets reveals an intransitive *cycle*. Because of this, there is no possible choice from $\{p, q, r\}$ is consistent with preference-maximization! Consider the Independence of Irrelevant Alternatives (IIA) axiom below, which Arrow (1959) shows is equivalent to maximization of a complete and transitive preference relation.

IIA. $D' \subset D$ and $c(D) \cap D' \neq \emptyset \implies c(D') = c(D) \cap D'$.

In the Kőszegi-Rabin PPE example, adding the lottery r to the set $\{p, q\}$ generates a violation of IIA, since r is not chosen yet affects choice from the larger set. Given fixed expectations r, our decision-maker's behaviour would be consistent with the standard model: she would maximize $v(\cdot|r)$. The decision-maker exhibits novel behaviour because her expectations, and hence preferences, are determined endogenously in a choice set. However, rational expectations combined with preferred personal equilibrium put quite a bit of structure on the decision-maker's novel behaviour. The axiomatic analysis that follows will clarify the nature of such structure.

2.4 The testable implications of Kőszegi-Rabin under PE: a negative result

The preceding example demonstrates that the Kőszegi-Rabin model with PPE generates choice behaviour that cannot be rationalized by a complete and transitive preference relation. Gul and Pesendorfer (2008) suggest that compared to the standard rational model, this may be the *only* revealed preference implication of the

Kőszegi-Rabin model when paired with the personal equilibrium solution criteria in (1). Gul and Pesendorfer take as a starting point a finite set X of riskless elements, a reference-dependent utility $v : X \times X \to \Re$, and offer the following result:

Proposition 0. (Gul and Pesendorfer 2008). *The following are equivalent: (i) c is induced by a complete binary relation, (ii) there is a v such that $c(D) = PE_v(D)$ for any choice set D, (iii) there is a v that satisfies (3) such that $c(D) = PE_v(D)$ for any choice set D.*

Proof. (partial sketch)

If $c(D) = \{x \in D : xRy \, \forall y \in D\}$ then define v by: $v(x|x) \geq v(y|x)$ if xRy, and $v(y|x) > v(x|x)$ otherwise. Then, $\{xRy \; \forall y \in D\} \iff \{v(x|x) \geq v(y|x) \; \forall y \in D\}$. By reversing the process, we could construct R from v. Thus (i) holds if and only if (ii) holds.

Gul and Pesendorfer cite Kőszegi and Rabin's (2006) argument that the set of hedonic dimensions in a given problem should be specified based on "psychological principles". Since X has no assumed structure, Gul and Pesendorfer infer hedonic dimensions from c and the structure imposed by (3). Their construction shows any v has a representation in terms of the functional form in (3).

□

The analysis that follows uses two assumptions that allow for a rich set of testable implications of expectations-based reference-dependence. First, c is defined on a subsets of *lotteries* over a finite set. The structure of lotteries in choice sets places additional observable restrictions on expectations in a choice set and additional information on behaviour relative to expectations. New axioms make particular use of this lottery structure to trace the observable implications of expectations-based reference-dependence.

Second, the main analysis looks for the revealed preference implications of *preferred* personal equilibrium. The sharper predictions of preferred personal equilibrium lead to different testable implications of the PPE based model expectations-based reference-dependence in the absence of risk.

This choice space does not allow the analysis to say anything insightful about the set of hedonic dimensions of the problem. In light of Gul and Pesendorfer's result, the representation here does not seek any particular structure on the v that represents reference-dependent preferences. The analysis considers the particular structure imposed by the functional form (3) as a secondary issue for future work.

3 Revealed preference analysis of PPE

3.1 Technical prelude

Define distance on lotteries using the Euclidean distance metric, $d^E(p,q) := \sqrt{\sum_i (p_i - q_i)^2}$, and the distance between choice sets using the Hausdorff metric,
$d^H(D,D') := \max\left(\max_{p\in D}\left[\min_{q\in D'} d^E(p,q)\right],\ \max_{q\in D'}\left[\min_{p\in D} d^E(p,q)\right]\right)$.

It will be useful to offer a few definitions in advance of the analysis. For any set T with typical element t, let $\{t^\epsilon\}$ denote a *convergent net* indexed by a set $(0,\bar{\epsilon}]$ and with limit point t; t^ϵ will be used to denote the ϵ term in the net.[6] Define $c^U(D)$ as the upper hemicontinuous extension of c; that is, $c^U(D) := \{p \in D : \exists\{p^\epsilon, D^\epsilon\}$ such that $p^\epsilon \in c(D^\epsilon),\ p^\epsilon \to p,\ D^\epsilon \to D\}$. For $p \in \Delta$ and $\epsilon > 0$, let $N_p^\epsilon := \{p^\epsilon \in \Delta :\ d^E(p,p^\epsilon) < \epsilon\}$ denote a ϵ-neighbourhood of p. For any binary relation R, let cl$R$ denote its closure, defined by: $p(\text{cl}R)q$ if $\exists\{p^\epsilon\} \to p$, $\{q^\epsilon\} \to q$ such that for each $\epsilon > 0$, $p^\epsilon R q^\epsilon$. For any finite set D and binary relation R, define $m(D,\ R) := \{p \in D : \nexists q \in D$ such that qRp but not $pRq\}$ as the set of undominated elements in D according to binary relation R.

3.2 Revealed preference analysis without risk

Ignoring restrictions specific to risks, the classic IIA axiom provides the point of departure from standard models. The two axioms below allow for failures of IIA that can arise from the endogenous determination of expectations and preferences in each choice set. For this section, restrict attention to axioms and restrictions on the representation in (2) that do not make use of the particular economic structure of lotteries, except for the continuity of Δ.

The following Expansion axiom is due to Sen (1971).

[6] A *net* in a set T is a function $t : S \to T$ for some directed set S (Aliprantis and Border, 1999).

Expansion. $p \in c(D) \cap c(D') \implies p \in c(D \cup D')$

Expansion says that if a lottery p is chosen in both D and D' then it is chosen in $D \cup D'$. This seems weak as both a normative and a descriptive property, and is an implication of variations on the Weak Axiom of Revealed Preference (see Sen (1971)). Expansion rules out the attraction and compromise effects, in which an agent chooses p over both q and r in pairwise choices, but chooses q from $\{p, q, r\}$.[7] In the attraction effect, r is similar to, but dominated by q and attracts the decision-maker to p in $\{p, q, r\}$; in the compromise effect, q is a compromise between more extreme options p and r in the choice set $\{p, q, r\}$.

The Weak RARP (RARP for Richter's (1966) Axiom of Revealed Preference[8]) is in the spirit of the classic axioms of revealed preference (like WARP, SARP, and GARP) albeit with an embedded continuity requirement. In particular, the axiom weakens (a suitably continuous version of) RARP.

Define $p\tilde{\tilde{R}}q$ if $p \in c(D)$ and $q \in c^U(\bar{D})$ for some $D, \bar{D}$ with $\{p,q\} \subseteq D \subseteq \bar{D}$. The relation $\tilde{\tilde{R}}$ is defined whenever sometimes p is chosen when q is available, and sometimes q is choosable (in the sense that $q \in c^U(\bar{D})$) when p is available. The statement $p\tilde{\tilde{R}}q$ holds when p is weakly chosen over q in a smaller set, but q is weakly choosable over p in a set that is larger in the sense of set inclusion. Define $p\tilde{\tilde{W}}q$ if there exist $p^0 = p,\ p^1, ..., p_{n-1}, p_n = q$ such that $(p^{i-1}, p^i) \in \text{cl}\tilde{\tilde{R}}$ for $i = 1, ..., n$. That is, $\tilde{\tilde{W}}$ is the continuous and transitive extension of $\tilde{\tilde{R}}$.

Weak RARP. $p \in c(D),\ q \in c^U(\bar{D}),\ q \in D \subseteq \bar{D},$ and $q\tilde{\tilde{W}}p \implies q \in c(D)$

The crucial implication of Weak RARP is captured by its main economic implication, *Weak WARP*: if $p = c(\{p,q\})$ and $p \in c(D)$ then $q \notin c(D')$ whenever $p \in D' \subseteq D$.[9] Manzini and Mariotti (2007) offer an interpretation in terms of constraining *reasons*: an agent might choose p over q in a smaller set, like $\{p,q\}$, yet might have

[7] See Simonson (1989) for evidence on attraction and compromise effects. Ok et al. (2012) provide a model of the attraction effect that captures this phenomenon.

[8] Richter refers to his axiom as "Congruence". I use RARP to emphasize the close connection with WARP, SARP, GARP, etc. For more on the connection between these axioms, see Sen (1971).

[9] The following proof that Weak RARP implies Weak WARP may help clarify the connection. Suppose $p \in c(D)$, $p \in D' \subset D$, and $q \in c(D')$. Then $q\tilde{\tilde{W}}p$, and so if $p \in c(\{p,q\})$, Weak RARP implies that $q \in c(\{p,q\})$ as well. Thus Weak RARP implies that if $p = c(\{p,q\})$ and $p \in c(D)$ hold, $q \in c(D')$ could not hold.

a constraining reason against choosing p in a larger set D. However, if we observe p chosen from a large set D, then any D' that is a subset of D contains no constraining reason against choosing p. Thus, her choice in D' should be minimally consistent with her choice in $\{p, q\}$ and she should not choose q.

Weak RARP strengthens the logic of Weak WARP in two ways. Weak WARP allows only WARP violations consistent with the existence of constraining reasons, and takes choices from smaller sets - which can fewer constraining reasons - as the determinant of choice in the absence of constraining reasons. The main way Weak RARP strengthens Weak WARP is by imposing that choice among unconstrained options is determined by a transitive procedure.[10]

Weak RARP as stated also strengthens a transitive version of Weak WARP by imposing continuity in two ways. Taking the topological closure of $\tilde{\tilde{R}}$ and then taking the transitive closure imposes that choice among unconstrained options is determined by a rationale that is both transitive and continuous. This imposes a restriction that is economically natural relative to the topological structure of lotteries. The second continuity aspect of Weak RARP is that if $p \in c^U(D)$, p is seen as chooseable from D. That is, if it is revealed that there is no reason to reject p^ϵ from D^ϵ when p^ϵ and D^ϵ are 'arbitrarily close' to p and D respectively, then Weak RARP assumes that there is no reason revaled to reject p from D (even if p is not chosen at D). These two strengthenings in Weak RARP are natural given the topological structure of the space of lotteries (and many other choice spaces).

Formally, say that a PPE representation in (2) is *continuous* if v is jointly continuous. Proposition 1 (i) $\Longleftrightarrow$ (ii), clarifies the link between the Expansion and Weak RARP axioms on one hand, and the PPE decision-making on the other hand.

Manzini and Mariotti (2007) characterize a *shortlisting representation*, $c(D) = m(m(D, P_1), P_2)$ for two asymmetric binary relations P_1, P_2, in terms of two axioms, Expansion and Weak WARP.[11] If P_2 is transitive and both P_1 and P_2 are continuous, say that P_1, P_2 is a *continuous and transitive shortlisting representation*.[12] Proposition

[10] In this regard, Weak RARP is closely related to the "No Binary Cycle Chains" axiom of Cherepanov et al. (Forthcoming).

[11] Manzini and Mariotti (2007) and follow-up papers assume that c is a single-valued choice function, which simplifies their analysis.

[12] This terminology is different from Au and Kawai (2011) and Horan (2012) who discuss short-

1 (ii) $\iff$ (iii), provides a link between a version of the shortlisting model of Manzini and Mariotti and the PPE representation in (2).

Proposition 1. *(i)-(iii) are equivalent: (i) c satisfies Expansion and Weak RARP, (ii) c has a continuous PPE representation, (iii) c has a continuous and transitive shortlisting representation.*

Proof. (ii) $\iff$ (iii)

Consider the following mapping between a continuous PPE representation v and a continuous and transitive shortlisting representation:

$v(q|p) > v(p|p) \iff qP_1p$

$v(p|p) > v(q|q) \iff pP_2q$

For v and P_1, P_2 that satisfy this mapping, $m(D,\, P_1) = PE_v(D)$, and $m(m(D,\, P_1),\, P_2) = PPE_v(D)$.

It remains to verify that joint continuity in v is equivalent to continuity of P_1 and P_2 - the full argument is in the appendix. □

The v in a PPE representation characterized by Proposition 1 is highly non-unique: *any* $\hat{v}$ that satisfies $\hat{v}(q|p) > \hat{v}(p|p) \iff v(q|p) > v(p|p)$ and has $\hat{v}(p|p) = u(p)$ for *some* u that represents P_2 in the shortlisting representation also represents the same c. Put another way, v includes information about how a decision-maker would choose between any two lotteries p and q given any reference lottery r. However, if the decision-maker's rational expectations determine her reference lottery, as in a PPE representation, choices give us no direct information about a decision-maker would choose between p and q given any reference lottery $r \notin \{p,\, q\}$.

3.3 Revealed preference analysis with risk

The result in Proposition 1 did not consider the possibility of adopting stronger axioms or restrictions on v that are suitable when working with choice among lotteries but may not be economically sensible in other domains. But the evidence supporting expectations-based reference-dependence in Ericson and Fuster (2011) suggests that environments with risk provide a natural environment for studying expectations-based

listing representations in which both P_1 and P_2 are transitive.

reference-dependence. This section explores the possibility of a stronger characterization in environments with risk.

Environments with risk enable a partial separation between expectations and choice. Suppose we view the mixture $(1-\alpha)q + \alpha D$ as arising from a lottery over choice sets that gives the singleton choice set $\{q\}$ with probability $1-\alpha$ and gives choice set D with probability α. Under this interpretation, fraction $1-\alpha$ of expectations are fixed at expecting q and we also observe the decision-maker's conditional choice from D. The three axioms below make use of variations on this interpretation.

The Induced Reference Lottery Bias Axiom uses this partial separation between expectations and choice. The axiom requires that if p is chosen in a choice set D, then p would also be conditionally chosen from D when some of the expectations are fixed at p, as in any mixture of the form $(1-\alpha)p + \alpha D$. This is a natural axiom to adopt under expectations-based reference-dependence: fixing expectations at p at least partially fixes the reference-lottery weakly towards p; if the decision-maker is biased towards her reference-lottery, this should bias her towards choosing p.

Induced Reference Lottery Bias. $p \in c(D)$ implies $p \in c((1-\alpha)p + \alpha D)$ $\forall \alpha \in (0,1)$.

Notice that Induced Reference Lottery Bias allows for the violation of Mixture Independence observed by Ericson and Fuster (2011), but rules out a violation in the opposite direction.

IIA Independence weakens the Mixture Independence Axiom to a variation that only implies a restriction on behaviour in the presence of IIA violations, with an embedded continuity requirement.

IIA Independence. If $p \in c(D)$ and $\exists \alpha \in (0,1]$ such that $p \notin c(D \cup ((1-\alpha)p + \alpha q)) \ni r$ and $p\tilde{\tilde{W}}r$, then $\exists \epsilon > 0$ such that $\forall \alpha' \in (0,1]$, $\forall \hat{p} \in N_p^\epsilon$, $\forall \hat{q} \in N_q^\epsilon$, and $\forall D' \ni (1-\alpha')\hat{p} + \alpha' \hat{q}$, $\hat{p} \notin c(D')$.

The spirit of Weak RARP is the requirement that in the absence of constraining reasons, c is consistent with maximizing $\tilde{\tilde{W}}$, derived from choice from smaller choice sets. The choice pattern $p \in c(D)$, $p \notin c(D \cup q) \ni r$, and $p\tilde{\tilde{W}}r$ then reveals that q blocks p.[13]

[13] In the appendix, it is shown that this choice pattern is ruled out by Weak RARP and Expansion.

Table 2: Two choice correspondences

D	$c(D)$	$\hat{c}(D)$
$\{\langle$apple, 1$\rangle$, $\langle$don't eat, 1$\rangle\}$	$\{\langle$apple, 1$\rangle\}$	$\{\langle$apple, 1$\rangle\}$
$\{\langle$candy, 1$\rangle$, $\langle$apple, 1$\rangle$, $\langle$don't eat, 1$\rangle\}$	$\{\langle$don't eat, 1$\rangle\}$	$\{\langle$don't eat, 1$\rangle\}$
$\{\langle$apple, .9; candy, .1$\rangle$, $\langle$apple, 1$\rangle\}$	$\{\langle$apple, 1$\rangle\}$	$\{\langle$apple, .9; candy, .1$\rangle\}$

This revealed blocking behaviour only appears when the model violates IIA. The IIA Independence axiom requires that in this case, any mixture between q and p also prevents p from being chosen from any choice set. The logic of expectations-dependence then requires that the agent would not choose p when it involves a conditional choice of p over q.

Remark 1. A simple test of IIA Independence that could detect behaviour inconsistent with expectations-dependence would be to find p, q, α, D with $p \in c(D)$, $\{p, q\} \cap c(D \cup q) = \emptyset$ but $p \in c(D \cup ((1-\alpha)p + \alpha q))$. Table 2 shows two possible choice correspondences that describe a decision-maker who finds candy too tempting to turn down for an apple whenever she had been expecting to eat but who can avoid temptation by planning in advance to abstain from snacking. Choice correspondence c captures a decision-maker who can exert limited self-control against the expectations-induced temptation to go for candy, and is inconsistent with the IIA Independence axiom. Choice correspondence $\hat{c}$ cannot exert this limited self-control, and is consistent with the axiom.

The continuity requirement embedded in IIA Independence slightly strengthens restriction on c when adding q to the choice set prevents p from being conditionally chosen. The IIA Independence axiom requires that in this case, lotteries close to p prevent lotteries close to q from being conditionally chosen as well.

Say that q is a *weak conditional choice over r given p, $q\bar{R}_p r$,* if there exists a net $\{p^\epsilon, q^\epsilon, r^\epsilon\} \to p, q, r$ such that $(1-\epsilon)p^\epsilon + \epsilon q^\epsilon \in c((1-\epsilon)p^\epsilon + \epsilon\{q^\epsilon, r^\epsilon\})$ for each ϵ. A conditional choice involves a choice between q and r for when expectations are close to p.

Transitive Limit. $q\bar{R}_p r$ and $r\bar{R}_p s \implies q\bar{R}_p s$.

If IIA violations are only driven by the behavioural influence of expectations and their endogenous determination, then the agent's behaviour should be consistent with the

Table 3: Two choice correspondences

$D = .9\{\langle \text{mug}, 1\rangle\} + .1$	$c(D)$	$\hat{c}(D)$
$\{\langle \text{pen}, 1\rangle, \langle \text{mug}, 1\rangle\}$	$\{\langle \text{mug}, 1\rangle\}$	$\{\langle \text{mug}, 1\rangle\}$
$\{\langle \text{candy}, 1\rangle, \langle \text{mug}, 1\rangle\}$	$.9\{\langle \text{mug}, 1\rangle\} + .1\{\langle \text{candy}, 1\rangle\}$	$\{\langle \text{mug}, 1\rangle\}$
$\{\langle \text{candy}, 1\rangle, \langle \text{pen}, 1\rangle\}$	$.9\{\langle \text{mug}, 1\rangle\} + .1\{\langle \text{pen}, 1\rangle\}$	$.9\{\langle \text{mug}, 1\rangle\} + .1\{\langle \text{pen}, 1\rangle\}$

standard model when her expectations are fixed. The Transitive Limit axiom says that conditional choice behaviour should look like the standard model when expectations are almost fixed, although the axiom only imposes this restriction on weak conditional choices.

Remark 2. As with continuity axioms, the Transitive Limit axiom is not *exactly* testable. However, the axiom is *approximately* testable. The choice sets in Table 3 provide an approximate test of Transitive Limit; $\hat{c}$ is consistent with what we would expect if the choice correspondence satisfies Transitive Limit. However, the choice pattern displayed by c is approximately inconsistent with Transitive Limit, and suggests that c would violate this axiom.

Formally, say that a PPE representation is an *EU-PPE representation* if $v(\cdot|p)$ takes an expected utility form for any $p \in \Delta$. Say that v *dislikes mixtures* if $v(p|p) \geq v(q|p)$ and $v(q|q) \leq \max[v(p|p), v(p|q)]$ imply that $\forall \alpha \in (0,1)$, $v((1-\alpha)p + \alpha q|(1-\alpha)p + \alpha q) \leq \max[v(p|p), v(p|(1-\alpha)p + \alpha q)]$.

Theorem 1. *c satisfies Weak RARP, Expansion, IIA Independence, Induced Reference Lottery Bias, and Transitive Limit if and only if it has a continuous EU-PPE representation in which v dislikes mixtures.*

The full proof is in the appendix, and is discussed in the next subsection.

Corollary 1. *Given a continuous EU-PPE representation v for c, any other continuous EU-PPE representation $\hat{v}$ for c satisfies $\hat{v}(q|p) \geq \hat{v}(r|p) \iff v(q|p) \geq v(r|p)$ and $\hat{v}(p|p) \geq \hat{v}(q|q)$ whenever $p\tilde{\tilde{W}}q$.*

Corollary 1 clarifies that a continuous EU-PPE is unique in the sense that any v, $\hat{v}$ that represent the same c must represent the same reference-dependent preferences.[14]

[14] A stronger uniqueness result is possible, since (i) each $v(\cdot|p)$ satisfies expected utility and thus has an affinely unique representation, (ii) joint continuity of v in the representation restricts the allowable class of transformations of v.

This definition of uniqueness captures that the underlying reference-dependent preferences are uniquely identified, but says nothing about the cardinal properties of reference-dependent utility functions. In an EBRD, v plays roles in both determining the set of of personal equilibria, and selecting from personal equilibria. The second part of Corollary 1 clarifies that this second role places a restriction that any v representing c must represent the same ranking of personal equilibria, at least when that ranking is revealed from choices.

Remark 3. In the representation in Theorem 1, any p chosen in D is (i) an element of D, and (ii) is in $\arg\max_{q \in D} v(\cdot|p)$. A more general model might allow a decision-maker to randomize among elements of her choice set. An alternative representation might have the decision-maker's reference lottery involve a randomization among elements in D, or perhaps only elements in $c(D)$. However, Theorem 1 proves that if c satisfies the five axioms it has a representation in which it is *as-if* the decision-maker never views herself as randomizing among elements of D.

3.4 Sketch of proof and an intermediate result

The first part of the proof takes $\bar{R}_p$ and characterizes a v such that $v(\cdot|p)$ represents $\bar{R}_p$. By Transitive Limit and because $\bar{R}_p$ is continuous by construction, such a $v(\cdot|p)$ exists. A sequence of lemmas show that the definition of $\bar{R}_p$ and Transitive Limit axiom imply the existence of a jointly continuous v such that $v(\cdot|p)$ represents $\bar{R}_p$ and satisfies expected utility.

Crucial to proof is providing a link between behaviour captured by v and behaviour in arbitrary choice sets. Consider an alternative axiom, Limit Consistency, which was not assumed in Theorem 1 but which would have been a reasonable axiom to adopt. First, define R_p as the asymmetric part of $\bar{R}_p$.

Limit Consistency. qR_pp implies $p \notin c(D)$ whenever $q \in D$.

The statement qR_pp says that q is always conditionally chosen over p when expectations are almost fixed at p. Limit Consistency requires that a decision-maker who always conditionally chooses q over p when her expectations are almost fixed at p would also never choose p when q is available. This is consistent with the logic of expectations-dependence. If instead qR_pp but p were chosen over q in some set D, then the decision-maker would choose p over q when her expectations are p even though she always conditionally chooses q over p when her expectations are almost fixed at p; such behaviour would be inconsistent with expectations-dependence and is ruled out.

The lemma below establishes that the axioms in Theorem 1 imply Limit Consistency.

Lemma. *Expansion, Weak RARP, and Induced Reference Lottery Bias imply Limit Consistency.*

The sufficiency part of the proof of Theorem 1 proceeds by using Expansion, Weak RARP, Limit Consistency, and v constructed from $\bar{R}_p$ to show that $c(D) = PPE_v(D)$. This gives the following intemediate result, a characterization of an EU-PPE representation in terms of Weak RARP, Expansion, IIA Independence, Limit Consistency, and Transitive Limit.

Theorem 2. *c satisfies Weak RARP, Expansion, IIA Independence, Limit Consistency, and Transitive Limit if and only if it has a continuous EU-PPE representation.*

Notice than in any EU-PPE representation, expected utility of $v(\cdot|p)$ and joint continuity of v will imply that $v(q|p) > v(r|p) \implies qR_pr$. With this observation in hand, the necessity of Limit Consistency follows obviously from the representation. The remainder of the proof of the above Theorem follows from the proof of Theorem 1.

3.5 A definition of expectations-dependence and its implications

Say that c exhibits *expectations-dependence* at D, α, p, q, r for $\alpha \in (0, 1)$ and $p, q, r \in \Delta$ if $(1-\alpha)p + \alpha r \in c((1-\alpha)p + \alpha D)$ but $(1-\alpha)q + \alpha r \notin c((1-\alpha)q + \alpha D)$. Interpret $(1-\alpha)p + \alpha r \in c((1-\alpha)p + \alpha D)$ as involving a *conditional choice* of r from D, conditional on fraction $1-\alpha$ of expectations being fixed by p. Say that *c exhibits strict expectations-dependence* at D, α, p, q, r for $D \in \mathcal{D}$, $\alpha \in (0, 1)$, and

Table 4: Two choice correspondences

	c	$\hat{c}$
$.9\{\langle \text{pen}, 1\rangle\} + .1\{\langle \text{pen}, 1\rangle, \langle \text{mug}, 1\rangle\}$	$\{\langle \text{pen}, 1\rangle\}$	$\{\langle \text{pen}, 1\rangle\}$
$.9\{\langle \text{mug}, 1\rangle\} + .1\{\langle \text{pen}, 1\rangle, \langle \text{mug}, 1\rangle\}$	$\{\langle \text{mug}, 1\rangle\}$	$\{\langle \text{mug}, .9;\ \text{pen}, .1\rangle\}$

$p, q, r \in \Delta$ if there is a $\bar{\epsilon} > 0$ such that for all r^ϵ, D^ϵ pairs such that $r^\epsilon \in D^\epsilon$ and $\max\left[d^E(r^\epsilon, r), d^H(D^\epsilon, D)\right] < \epsilon$, $(1-\alpha)p + \alpha r^\epsilon \in c((1-\alpha)p + \alpha D^\epsilon)$ for all $\epsilon < \bar{\epsilon}$ but $(1-\alpha)q + \alpha r^\epsilon \notin c((1-\alpha)q + \alpha D^\epsilon)$ for all $\epsilon < \bar{\epsilon}$. This behavioural definition of expectations-dependence provides a tool for identifying and eliciting expectations-dependence, as illustrated by the example below.

Example (mugs and pens). Fix $\alpha = .1$, let $p = \langle \text{pen}, 1\rangle$; $q = \langle \text{mug}, 1\rangle$, $r = p$, and $D = \{p, q\}$.

Table 4 shows the values that two choice correspondences, c and $\hat{c}$, take on the menus $(1-\alpha)p + \alpha D = \{\langle \text{mug}, 1\rangle, \langle \text{mug}, .9; \text{pen}, .1\rangle\}$ and $(1-\alpha)q + \alpha D = \{\langle \text{mug}, .1;\ \text{pen}, .9\rangle, \langle \text{pen}, 1\rangle\}$. Of these two choice correspondences, c exhibits expectations-dependence given D, α, p, q, r, while $\hat{c}$ does not.

$\diamond$

The definition of exhibiting expectations-dependence bears striking relation to the Mixture Independence axiom. Indeed, expectations-dependence as defined is a type of violation of Mixture Independence. Proposition 2 below clarifies the link between a exhibiting expectations-dependence, properties of a continuous EU-PPE representation, and violations of the IIA axiom.

Proposition 2. *c with a continuous EU-PPE representation strictly exhibits expectations-dependence if and only if $v(\cdot|p)$ is not ordinally equivalent to $v(\cdot|q)$ for some $p, q \in \Delta$. In addition, c with a continuous EU-PPE representation that violates IIA exhibits strict expectations-dependence.*

The first part of Proposition 2 highlights how expectations-dependence in c is captured in a PPE representation. There is a tight tie between expectations-dependence and failures of Mixture Independence in a PPE representation, and the second part of Proposition 2 shows that a failure of IIA implies, but is not necessary for, expectations-dependence.

The mugs and pens example shows how one might study expectations-dependence based on the definition. Ericson and Fuster's (2011) data violate Mixture Independence in a way consistent with expectations-based reference-dependence, and Proposition 2 shows that any PPE representation representing their median subject's behaviour must exhibit expectations-dependence.

3.6 Limited cycle property of a PPE representation

The characterization in Theorem 1 is tight. However, it is possible that some structure already imposed on the problem implies additional structure on v. Proposition 3 shows that this is indeed the case.

Say that a PPE representation satisfies the *limited cycle inequalities* if for any $p^0, p^1, ..., p^n \in \Delta$, $v(p^i|p^{i-1}) > v(p^{i-1}|p^{i-1})$ for $i = 1, ..., n$, then $v(p^n|p^n) \geq v(p^0|p^n)$.

Proposition 3. *Any PPE representation satisfies the limited cycle inequalities. Moreover, if v is jointly continuous, satisfies the limited-cycle inequalities, dislikes mixtures, and $v(\cdot|p)$ is EU for each $p \in \Delta$, then v defines an EU-PPE representation by (2).*

Proof. Take any $p^0, p^1, ..., p^n \in \Delta$, with $v(p^i|p^{i-1}) > v(p^{i-1}|p^{i-1})$. The i^{th} term in this sequence implies by the representation that $p^{i-1} \notin c(\{p^0, ..., p^n\})$; since $c(\{p^0, ..., p^n\}) \neq \emptyset$ by assumption it follows that $p^n = c(\{p^0, ..., p^n\})$. This implies, by the representation, that $v(p^n|p^n) \geq v(p^i|p^n)$ for all $i = 0, 1, ..., n-1$, which implies the desired result.

Conversely, for any v that satisfies the three given restrictions, the limited cycle inequalities imply that $PE(D)$ is non-empty for any $D \in \mathcal{D}$. Thus by Theorem 2, v defines a EU-PPE representation.

□

Munro and Sugden (2003) mention the limited cycle inequalities (their Axiom C7), and defend the limited cycle inequalities based on a money-pump argument. In contrast, the limited cycle inequalities emerge here as a consequence of the assumption that $c(D)$ is always non-empty combined with the reference-dependent preference representation. If one considers a class of choice problems in which the agent always makes a choice, the limited cycle inequalities are a basic consequence of this and the agent's endogenous determination of her reference lottery, regardless of the normative interpretation of the inequalities.

4 Special cases of PPE representations

4.1 Kőszegi-Rabin reference-dependent preferences

It may not be apparent at first glance whether Kőszegi-Rabin preferences in (3) satisfy the limited-cycle inequalities that a PPE representation must satisfy to generate a non-empty choice correspondence. Kőszegi and Rabin (2006) cite a result due to Kőszegi (2010, Theorem 1) that a personal equilibrium exists whenever D is convex, or equivalently, an agent is free to randomize among elements of any non-convex choice set. It is unclear whether or when this restriction is necessary to guarantee the existence of a non-empty choice correspondence.

Kőszegi and Rabin suggest restrictions on (3). In particular, applications of Kőszegi-Rabin have typically assumed *linear loss aversion*, which holds when there are η and λ such that:

$$\mu(x) = \begin{cases} \eta x & \text{if } x \geq 0 \\ \eta\lambda x & \text{if } x < 0 \end{cases} \tag{4}$$

where $\lambda > 1$ captures loss aversion and $\eta \geq 0$ determines the relative weight on gain/loss utility. Proposition 4 shows that under linear loss aversion, Kőszegi-Rabin preferences with the PPE solution concept are a special case of the more general continuous EU-PPE representation.

Proposition 4. *Kőszegi-Rabin preferences that satisfy linear loss aversion satisfy the limited cycle inequalities and dislike mixtures.*

Proposition 4 is alternative result to Kőszegi and Rabin's (2006) Proposition 1.3, and to my knowledge provides the first general proof that a personal equilibrium that does not involve randomization always exists in finite sets for this subclass of Kőszegi-Rabin preferences.

While commonly used versions of Kőszegi-Rabin preferences can provide the v in a PPE representation, there are (pathological?) cases of Kőszegi-Rabin preferences that cannot.

Proposition 5. *Not all Kőszegi-Rabin preferences consistent with (3) satisfy the limited cycle inequalities.*

4.2 Reference lottery bias and dynamically consistent non-expected utility

Expectations-based reference-dependence is the central motivation to considering the PPE representation. Now equipped with some understanding of the revealed preference implications of a PPE representation, we might take the preference relations $\succeq_L$ and $\{\succeq_p\}_{p\in\Delta}$ as primitives, where $\succeq_p$ is the preference relation corresponding to $v(\cdot|p)$, and $p \succeq_L q$ corresponds to the ranking $v(p|p) \geq v(q|q)$. With these primitives, we can study axioms that capture reference lottery bias. This is similar to the standard exercise in the axiomatic literature on reference-dependent behaviour (e.g. Tversky and Kahneman (1991; 1992); Masatlioglu and Ok (2005; 2012); Sagi (2006)). In that vein, consider the *Reference Lottery Bias* axiom below, which is closely related to the "Weak Axiom of Status Quo Bias" in Masatlioglu and Ok (2012).

Reference Lottery Bias. $p \succeq_L q \implies p \succeq_p q$

I offer three interpretations of Reference Lottery Bias. The first interprets $\succeq_L$ as representing the preferences that take into account that expecting to choose and then choosing lottery p leads to p being evaluated against itself as the reference lottery. Under this interpretation, if an agent would want to choose p over q, knowing that this choice would also determine the reference-lottery against which they would evaluate outcomes, then the agent would also choose p over q when p is the reference lottery. The second interpretation (along the lines of Masatlioglu and Ok (2012)) is that $\succeq_L$ captures reference-independent preferences; in this second interpretation, if p is preferred to q in a reference-independent comparison, then when p is the reference lottery, p is also preferred to q. According to either interpretation, Reference Lottery Bias imposes that $\succeq_p$ biases an agent towards p relative to $\succeq_L$. This seems like a natural generalization of the endowment effect for expectations-based reference-dependence.

A third interpretation emphasizes $\succeq_L$ as the ranking of lotteries induced by the agent's ex-ante ranking of choice sets when restricted to singleton choice sets. Under this interpretation, an agent who wants to choose a lottery from a choice set according to her ex-ante ranking would also want to choose it from that choice set if she then expected that lottery, and it subsequently acted as her reference point.

What implications does the Reference Lottery Bias axiom have? Kőszegi-Rabin preferences do not satisfy Reference Lottery Bias; recall the example in Section 2.2.2 in which $v(p|p) > v(r|r)$ but $v(r|p) > v(p|p)$. This suggests a conflict between the psychology of reference-dependent loss aversion captured by the Kőszegi-Rabin model and the notion of Reference Lottery Bias defined in the axiom. No experimental evidence to my knowledge sheds light on this matter.

Proposition 6. *A PPE representation satisfies Reference Lottery Bias if and only if* $c(D) = m(D, \succeq_L)$.

Proposition 6 implies (recalling Proposition 2) that under Reference Lottery Bias, reference-dependent behaviour in a PPE representation is tightly connected to non-expected utility behaviour in $\succeq_L$.

The non-expected utility literature has provided numerous models of decision-making under risk based on complete and transitive preferences that, motivated by the Allais paradox, satisfy a relaxed version of the Mixture Independence axiom (e.g. Quiggin (1982); Chew (1983); Dekel (1986); Gul (1991)). The model of expectations-based reference-dependence based on the Reference Lottery Bias axiom is based on a dynamically consistent implementation of non-expected utility preferences (as in Machina (1989)). I offer two examples of PPE representations that satisfy Reference-Lottery Bias and capture expectations-based reference-dependence.

Example (Disappointment Aversion). Suppose $\succeq_L$ satisfies Gul's (1991) disappointment aversion; that is (letting $u(x)$ denote $u(\langle x, 1\rangle)$), $u(p) = \frac{1}{1+\beta} \sum_i p_i \left(u(x_i) + \beta \min[u(x_i), u(p)]\right)$ represents $\succeq_L$ for some $\beta \geq 0$. Then Reference Lottery Bias implies:

$$v^{DA}(p|r) = \frac{1}{1+\beta} \sum_i p_i \left(u(x_i) + \beta \min[u(x_i),\, u(r)]\right) \tag{5}$$

In cases of lotteries over multidimensional choice objects, it is not hard to see how to extend (5) via additive separability across dimensions. The resulting functional form captures loss aversion relative to past expectations (as in Kőszegi-Rabin) but does not generate IIA violations.

◇

Example (Mixture Symmetry). Suppose $\succeq_L$ satisfies Chew et al.'s (1991) mixture symmetric utility; that is, there is a symmetric function ϕ such that $u(p) = \sum_i \sum_j \phi(x_i,\, x_j)$ represents $\succeq_L$. Then Reference Lottery Bias implies:

$$v^{MS}(p|r) = \sum_i \sum_j p_i r_j \phi(x_i,\, x_j) \tag{6}$$

While the functional form for v^{MS} in 6 does capture the Kőszegi-Rabin functional form in (3), but the ϕ function corresponding to v^{KR} is generally not symmetric.

◇

5 Alternative models of expectations-based reference-dependence: analysis of PE and CPE representations

5.1 Characterization of PE

In addition to the PPE representation in (2) which is used in most applications of expectations-based reference-dependence, Kőszegi and Rabin (2006) also discuss the PE as a solution concept as in (1). The analysis below shows that the PE representation can be axiomatized similar to the PPE representation, by replacing Weak RARP with Sen's α, changing the continuity assumptions, and modifying IIA Independence.

Sen's α. $p \in D' \subset D$ and $p \in c(D)$ implies $p \in c(D')$

Sen's α requires that if an item p is choosable in a larger set D, then it is also deemed choosable in any subset D' of D where p is available. Sen's α is strictly weaker than IIA.[15]

The Upper Hemicontinuity axiom is the continuity property satisfied by continuous versions of the standard model, in which choice is determined by a continuous binary relation.

UHC. $c(D) = c^U(D)$

Proposition 7 (i) $\iff$ (ii) provides an axiomatic characterizating of PE decision-making that does not make use of the structure of environments with risk; (ii) $\iff$ (iii) is a continuous version of Gul and Pesendorfer's (2008) result (Proposition 0 in this paper).[16]

Proposition 7. *(i)-(iii) are equivalent: (i) c satisfies Expansion, Sen's a, and UHC, (ii) c has a continuous PE representation, (iii) c is induced by a continuous binary relation.*

IIA Independence 2 modifies the antecedent in the IIA Independence axiom to PE. Under PE, a lottery q is revealed to block p if there is a D such that $p \in c(D)$ but $p \notin c(D \cup q)$. IIA Independence 2 has a different antecedent from IIA Independence that reflects the differences in how constraining lottery pairs are revealed in the two models. IIA Independence 2 also embeds a continuity requirement.

IIA Independence 2. If $p \in c(D)$ and $\exists \alpha \in (0, 1]$ such that $p \notin c(D \cup (1-\alpha)p + \alpha q))$, then $\exists \epsilon > 0$ such that $\forall \alpha' \in (0, 1]$, $\forall \hat{p} \in N_p^\epsilon$, $\forall \hat{q} \in N_q^\epsilon$, and $\forall D' \ni (1-\alpha')\hat{p} + \alpha'\hat{q}$, $\hat{p} \notin c(D')$.

Theorem 3 provides a characterization of a continuous EU-PE representation.

Theorem 3. *c satisfies Expansion, Sen's a, UHC, IIA Independence 2, and Transitive Limit if and only if c has a continuous EU-PE representation. These axioms jointly imply that Induced Reference Lottery Bias holds as well.*

[15] Sen's α and Sen's β are jointly equivalent to IIA; see Sen (1971) and Arrow (1959).

[16] The result (i) $\iff$ (iii) is a continuous version of Theorem 9 in Sen (1971).

5.2 Characterization of CPE

Kőszegi and Rabin (2007) also introduce the choice-acclimating personal equilibrium (CPE) concept:

$$CPE_v(D) = \arg\max_{p \in D} v(p|p) \tag{7}$$

While most applications of expectations-based reference-dependence use the PPE solution concept, many use CPE. Theorem 4 clarifies the revealed preference foundations of CPE decision-making.

Theorem 4. *(i)-(iii) are equivalent. (i) c satisfies IIA and UHC, (ii) c has a continuous EU-CPE representation in which v is continuous, (iii) there is a complete, transitive, and continuous binary relation $\succeq$ such that $c(D) = m(D, \succeq)$ $\forall D$.*

Theorem 4 appears to be a negative result - it suggests that expectations-based reference-dependence combined with CPE has no testable implications beyond the standard model of preference maximization! However, CPE decision-making can fail the Mixture Independence Axiom in ways that are consistent with expectations-based reference-dependent behaviour. This raises the question of what restrictions the Induced Reference Lottery Bias impose on the representation. Say that a binary relation $\succeq$ is *quasiconvex* if $p \succeq q \implies p \succeq (1-\alpha)p + \alpha q \; \forall \alpha \in (0,1)$.

Proposition 8. *Suppose $\exists \succeq, v$ such that $c(D) = m(D, \succeq) = CPE_v(D)$. (i)-(iii) are equivalent: (i) c satisfies Induced Reference Lottery Bias, (ii) $\succeq$ is quasiconvex, (iii) $v(p|p) \geq v(q|q) \implies v(p|p) \geq v((1-\alpha)p + \alpha q|(1-\alpha)p + \alpha q) \; \forall \alpha \in (0,1)$.*

Remark 4. Proposition 6 and Theorem 4 establish that if c has a PPE representation that satisfies the Reference Lottery Bias axiom, then $PPE_v(D) = CPE_v(D)$.

Example (Kőszegi-Rabin and Mixture Symmetry). Under CPE concept, the requirement that ϕ in 6 be symmetric is without loss of generality. Thus the Kőszegi-Rabin functional form in 3 corresponds to a special case of the mixture symmetric utility functional form in 6 under CPE.

◇

Appendix: Proofs

Lemma 1. *For any two sets D, D' and any asymmetric binary relation P, $m(D,\, P) \cup m(D',\, P) \supseteq m(D \cup D',\, P)$.*

Proof. Suppose $p \in m(D \cup D', P) \cap D$.

$\implies \nexists q \in D \cup D'$ s.t. qPp.

$\implies \nexists q \in D$ s.t. qPp

$\implies p \in m(D, P)$.

If $p \in m(D \cup D', P) \cap D'$, an analogous result would follow.

Thus $p \in m(D \cup D', P)$ implies $p \in m(D, P) \cup m(D', P)$.

$\implies m(D,\, P) \cup m(D',\, P) \supseteq m(D \cup D',\, P)$ □

Results on IIA Independence and IIA Independence 2.

Lemma 2. *Suppose Expansion and Weak RARP hold. If $p \in c(D)$, $p \notin c(D \cup q) \ni r$, and $p\tilde{\tilde{W}}r$, then $\nexists D_{pq}$ such that $p \in c(D_{pq})$.*

Proof. If $\exists D_{pq}$ such that $p \in c(D_{pq})$ then by Expansion, $p \in c(D \cup D_{pq})$. Since $D \cup q \subseteq D \cup D_{pq}$ and $r \in c(D \cup q)$ with $p\tilde{\tilde{W}}r$, it follows by Weak RARP that $p \in c(D \cup q)$, a contradiction. Thus no such D_{pq} can exist. □

Lemma 3. *Suppose Expansion and Sen's α hold. If $p \in c(D)$, $p \notin c(D \cup q)$, then $\nexists D_{pq}$ such that $p \in c(D_{pq})$.*

Proof. If $p \in c(D) \cap D_{pq}$ then by Expansion, $p \in c(D \cup D_{pq})$. Then by Sen's α, $p \in c(D \cup q)$. This proves the claim. □

Proof of Proposition 1.

(i) $\iff$ (iii)

Let P_1, P_2 denote the asymmetric part of relations $\bar{P}_1,\, \bar{P}_2$ that form a transitive short-listing representation. By definition, $m(D,\, P_i) = m(D,\, \bar{P}_i)$ for $i = 1, 2$ and for any D.

Necessity of Expansion. $p \in c(D)$ and $p \in c(D')$ implies:

(i) $p \in m(D, P_1)$ and $p \in m(D', P_1)$

$\implies \nexists q \in D$ s.t. qP_1p and $\nexists q \in D'$ s.t. qP_1p

$\implies \nexists q \in D \cup D'$ s.t. qP_1p

$\implies p \in m(D \cup D',\, P_1)$

(ii) $p \in m(m(D, P_1), P_2)$ and $p \in m(m(D', P_1), P_2)$

$\implies \nexists q \in m(D, P_1)$ s.t. qP_2p and $\nexists q \in m(D', P_1)$ s.t. qP_2p

$\implies \nexists q \in m(D, P_1) \cup m(D', P_1)$ s.t. qP_2p

by Lemma 1,

$\implies \nexists q \in m(D \cup D', P_1)$ s.t. qP_2p

By (i),

$\implies p \in m(m(D \cup D', P_1), P_2) = c(D \cup D')$

This implies that Expansion holds.

Necessity of Weak RARP. Suppose $q\tilde{\tilde{W}}p$, and there are D, D' such that: $\{p, q\} \subseteq D \subseteq \bar{D}$ and $p \in c(D)$, $q \in c^U(\bar{D})$.

By definition of $q\tilde{\tilde{W}}p$, there is a chain $q = r^0, r^1, ..., r^{n-1}, r^n = p$ such that for each $i \in \{1, ..., n\}$, there are $D^i, \bar{D}^i$ such that $\{r^{i-1}, r^i\} \subseteq D^i \subseteq \bar{D}^i$, $r^i \in c^U(\bar{D}^i)$ and $r^{i-1} \in c(D^i)$, or (if not) there is a net $\{\bar{D}^{i,\epsilon},\, D^{i,\epsilon}\} \to \bar{D}^i,\, D^i$ for which $r^i \in c^U(\bar{D}^{i,,\epsilon})$ and $r^{i-1} \in c(D^{i,\epsilon})$ $\forall \epsilon > 0$.

For each i, from the representation, it follows that:

$\implies r^i \in m(D^i,\, P_1)$

$\implies$ not $r^iP_2r^{i-1}$.

Since the transitive completion of P_2 is transitive, it follows that not qP_2p.

Since $q \in c^U(\bar{D})$, by continuity of P_1, $q \in m(\bar{D},\, P_1)$.

Since $q \in D \subseteq \bar{D}$ as well, $q \in m(D,\, P_1)$.

Since $p \in m(m(D, P_1), P_2)$, not pP_2q, and P_2 has a transitive completion, it follows that not rP_2q $\forall r \in m(D,\, P_1)$.

Thus, $q \in m(m(D,\, P_1),\, P_2) = c(D)$.

Sufficiency. Part of the idea of the proof follows Manzini and Mariotti (2007). The two rationales constructed here are not unique.

Define P_1 by:

$$qP_1p \text{ if } \nexists D_{pq} \,\text{s.t. } p \in c^U(D_{pq})$$

Define $\bar{P}_2$ by:

$$\bar{P}_2 = \tilde{\bar{W}}$$

Define P_2 as the asymmetric part of P_2.

First, show that P_1 and P_2 are appropriately continuous.

If pP_1q, $\nexists$ a net $\{D_{p^\epsilon q^\epsilon}\}_\epsilon \to D_{pq}$ with $p^\epsilon \in c(D_{p^\epsilon q^\epsilon})$ and $\max\left[d(p^\epsilon, p), d(q^\epsilon, q)\right] < \epsilon$ for each $\epsilon > 0$, since then we would have $p \in c^U(D_{pq})$ for some D_{pq}. Thus, $\exists \bar{\epsilon} > 0$ such that $\forall p^\epsilon \in N_p^{\bar{\epsilon}}$, $\forall q^\epsilon \in N_q^{\bar{\epsilon}}$, $p^\epsilon P_1 q^\epsilon$. This implies that P_1 has open better and worse than sets.

P_2 is continuous by construction.

Second, show $c(D) \subseteq m(m(D, P_1), P_2)$.

By definition of P_1, $p \in c(D)$ implies $p \in m(D, P_1)$.

Take any $q \in m(D, P_1)$. By the definition of P_1, $\forall r \in D$, $\exists D_{qr}$ such that $q \in c(D_{qr})$. Successively applying Expansion implies that $q \in c(\underset{r\in D}{\cup} D_{qr})$. Since $D \subseteq \underset{r\in D}{\cup} D_{qr}$ and $p \in c(D)$, it follows that $p\tilde{\bar{W}}q$, thus $p\bar{P}_2q$. Since this implies not qP_2p for any arbitrary $q \in m(D, P_1)$, it further follows that $p \in m(m(D, P_1), P_2)$.

Third, show $m(m(D, P_1), P_2) \subseteq c(D)$

Suppose $p \in m(m(D, P_1), P_2)$.

Then, $\forall r \in D$, $\exists D_{pr} : \ p \in c(D_{pr})$. By Expansion, $p \in c(\underset{r\in D}{\cup} D_{pr})$.

Since $p \in m(m(D, P_1), P_2)$, it $p\tilde{\bar{W}}q \ \forall q \in c(D)$ by the definition of $\tilde{\bar{W}}$.

Thus by Weak RARP, $p \in c(D)$.

(ii) $\iff$ **(iii)** Consider a continuous PPE representation v that represents c, and a continuous and transitive shortlisting representation P_1, P_2.

Map between v and P_1 by:

$$qP_1p \iff v(q|p) > v(p|p)$$

Map between v and P_2 by:

$$qP_2p \iff v(q|q) > v(p|p)$$

Joint continuity of v will map to continuity of P_1 and P_2.

Notice that the mapping from P_1 to v only specifies $v(\cdot|p)$ partially; the mapping from P_2 to v imposes an continuous additive normalization on v.

Consider the following construction of v from P_1, P_2:

Let $u : \Delta \to \Re$ be a continuous utility function that represents P_2. Define $v(p|p) = u(p)\ \forall p \in \Delta$. Let $I(p) = \{q \in \Delta :\ (q,p) \in \text{cl}\{(\hat{q},\hat{p}) :\ \hat{q}P_1\hat{p}\} \backslash \{(\hat{q},\hat{p}) :\ \hat{q}P_1\hat{p}\}\}$.

The following definition of v is consistent with the mapping proposed above:

$$v(q|p) = \begin{cases} u(p) + d^H(\{q\}, I(p)) & \text{if } qP_1p \\ u(p) - d^H(\{q\}, I(p)) & \text{otherwise} \end{cases}$$

It can be verified that continuity of P_1 and u imply that v so constructed satisfies joint continuity.

□

Proof of Theorem 1.

Notation.

Let for $p, q \in \Delta$, let $D_{pq} \in \mathcal{D}$ denote an arbitrary choice set that contains p and q.

Sufficiency: Lemmas.

In the lemmas in this section, assume that c satisfies Expansion, Weak RARP, IIA Independence, Induced Reference Lottery Bias, and Transitive Limit.

Lemma 4. *$\bar{R}_p$ is complete, transitive, and if there exists a net $\{p^\epsilon, q^\epsilon, r^\epsilon\} \to p, q, r$ with $q^\epsilon \bar{R}_{p^\epsilon} r^\epsilon$ for each term in the net, then $q\bar{R}_p r$.*

Proof. Transitivity of $\bar{R}_p$ follows by Transitive Limit.

For any net $\{p^\epsilon, q^\epsilon, r^\epsilon\} \to p, q, r$, non-emptiness of c implies that the net either has a convergent subnet $p^\delta, q^\delta, r^\delta$ in which $(1-\delta)p^\delta + \delta q^\delta \in c(\{(1-\delta)p^\delta + \delta q^\delta,\ (1-\delta)p^\delta + \delta r^\delta\}$ or in which $(1-\delta)p^\delta + \delta r^\delta \in c(\{(1-\delta)p^\delta + \delta q^\delta,\ (1-\delta)p^\delta + \delta r^\delta\}$ for each term in the subnet. Thus $\bar{R}_p$ is complete.

Take a net $\{p^\epsilon, q^\epsilon, r^\epsilon\} \to p, q, r$, for which $q^\epsilon \bar{R}_{p^\epsilon} r^\epsilon$ for each term in the net. By the definition of $\bar{R}_{p^\epsilon}$, for each ϵ there is a net $\{p^{\epsilon,\delta}, q^{\epsilon,\delta}, r^{\epsilon,\delta}\}_\delta \to p^\epsilon, q^\epsilon, r^\epsilon$ such that $(1-\delta)p^{\epsilon,\delta} + \delta q^{\epsilon,\delta} \in c((1-\delta)p^{\epsilon,\delta} + \delta\{q^{\epsilon,\delta}, r^{\epsilon,\delta}\})$ for each term in the net. Let $\bar{\delta}_\epsilon$ denote the largest element in the index set for $\{p^{\epsilon,\delta}, q^{\epsilon,\delta}, r^{\epsilon,\delta}\}_\delta$ and $\bar{\epsilon}$ the largest element in the index set for $\{p^\epsilon, q^\epsilon, r^\epsilon\}$. Take $\bar{\delta} := \bar{\delta}_{\bar{\epsilon}}$. For each $\delta < \bar{\delta}$, define ϵ_δ as a decreasing net such that for each $\delta < \bar{\delta}_{\epsilon_\delta}$. Then define $\{\hat{p}^\delta, \hat{q}^\delta, \hat{r}^\delta\} := \{p^{\epsilon_\delta,\delta}, q^{\epsilon_\delta,\delta}, r^{\epsilon_\delta,\delta}\}_\delta$. By construction, $\{\hat{p}^\delta, \hat{q}^\delta, \hat{r}^\delta\}$ establishes that $q\bar{R}_p r$. □

Let R_p denote the strict part of $\bar{R}_p$. Lemma A.5 shows that R_p satisfies the Independence Axiom.

For a binary relation R, say that R *satisfies the Independence axiom* if $qRr \iff (1-\alpha)s + \alpha q R (1-\alpha)s + \alpha r \,\forall \alpha \in (0,1). \forall s \in \Delta$.

Lemma 5. *R_p satisfies the Independence Axiom if $p \in int\Delta$.*

Proof. Part I: suppose qR_pr, and take a $\alpha \in (0,1)$ and $s \in \Delta$.

Then,$\exists \bar{\delta}, \bar{\epsilon} > 0$ such that $\forall \epsilon \in (0, \bar{\epsilon})$, $\hat{p}, \hat{q}, \hat{r} \in N_p^{\bar{\delta}} \times N_q^{\bar{\delta}} \times N_r^{\bar{\delta}}$, $\{(1-\epsilon)\hat{p} + \epsilon\hat{q}\} = c((1-\epsilon)\hat{p} + \epsilon\{\hat{q}, \hat{r}\})$.

Define $\bar{\delta}_\alpha = \min\left[\alpha\bar{\delta}, (1-\alpha)\bar{\delta}\right]$.

Let $\hat{p}, \hat{s} \in N_p^{\bar{\delta}_\alpha} \times N_s^{\bar{\delta}_\alpha}$. Since $d^E(p,q) \leq 1$, it follows that $d^E((1-\beta)\hat{p} + \beta\hat{s}, p) \leq (1-\beta)\bar{\delta}_\alpha + \beta$ by the triangle inequality. Thus if $\beta \leq \bar{\beta}_\alpha := \frac{\bar{\delta} - \bar{\delta}_\alpha}{1 - \bar{\delta}_\alpha}$, then $(1-\beta)\hat{p} + \beta\hat{s} \in N_p^{\bar{\delta}}$.

Then for any $\hat{q}, \hat{r} \in N_q^{\bar{\delta}} \times N_r^{\bar{\delta}}$, $\epsilon \in (0, \bar{\epsilon})$, and $\beta \in (0, \bar{\beta})$, $\{(1-\epsilon)\left((1-\beta)\hat{p} + \beta\hat{s}\right) + \epsilon\hat{q}\} = c((1-\epsilon)\left((1-\beta)\hat{p} + \beta\hat{s}\right) + \epsilon\{\hat{q}, \hat{r}\})$. Define $\hat{\epsilon} := \frac{\epsilon}{\alpha}$ and $\beta^{\epsilon,\alpha} := \frac{\epsilon}{\alpha}\frac{1-\alpha}{1-\epsilon}$. Then $\forall \hat{\epsilon}$ such that $\alpha\hat{\epsilon} \in (0, \bar{\epsilon})$ and $\hat{\epsilon}\frac{1-\alpha}{1-\alpha\hat{\epsilon}} \in (0, \bar{\beta})$, it follows that $\{(1-\hat{\epsilon})\hat{p} + \hat{\epsilon}\left((1-\alpha)\hat{s} + \alpha\hat{q}\right)\} = c((1-\hat{\epsilon})\hat{p} + \hat{\epsilon}\left((1-\alpha)\hat{s} + \alpha\{\hat{q}, \hat{r}\}\right))$. Since $N_{(1-\alpha)s+\alpha q}^{\bar{\delta}_\alpha} \subset (1-\alpha)N_s^{\bar{\delta}} + \alpha N_q^{\bar{\delta}}$ and $N_{(1-\alpha)s+\alpha q}^{\bar{\delta}_\alpha} \subset (1-\alpha)N_s^{\bar{\delta}} + \alpha N_q^{\bar{\delta}}$

It follows that $(1-\alpha)s + \alpha q R_p (1-\alpha)s + \alpha r$.

Part II: suppose $(1-\alpha)s + \alpha q R_p (1-\alpha)s + \alpha r$.

Recall that $N_{(1-\alpha)s+\alpha q}^{\bar{\delta}} \subseteq (1-\alpha)N_s^{\bar{\delta}} + \alpha N_q^{\bar{\delta}}$.

Then,$\exists \bar{\delta}, \bar{\epsilon} > 0$ such that $N_p^{\bar{\delta}} \subset \text{int}\Delta$ and $\forall \epsilon \in (0, \bar{\epsilon})$, $\hat{p}, \hat{q}, \hat{r}, \hat{s} \in N_p^{\bar{\delta}} \times N_q^{\bar{\delta}} \times N_r^{\bar{\delta}} \times N_s^{\bar{\delta}}$, $\{(1-\epsilon)\hat{p} + \epsilon((1-\alpha)\hat{s} + \alpha\hat{q})\} = c((1-\epsilon)\hat{p} + \epsilon((1-\alpha)\hat{s} + \alpha\{\hat{q}, \hat{r}\}))$.

Fix $\kappa \in (0,1)$. Fix $\hat{p}, \hat{q}, \hat{r}, \hat{s} \in N_p^{\kappa\bar{\delta}} \times N_q^{\kappa\bar{\delta}} \times N_r^{\kappa\bar{\delta}} \times N_s^{\kappa\bar{\delta}}$.

Given $\epsilon \in (0, \bar{\epsilon})$, take $\gamma^{\epsilon,\alpha} := \epsilon\frac{1-\alpha}{1-\epsilon}$. If $\gamma < (1-\kappa)\bar{\delta}$, then $\hat{p} + \gamma^{\epsilon,\alpha}(\hat{p} - \hat{s}) \in N_p^{\bar{\delta}} \subseteq \Delta$. Then,

$(1-\epsilon)\left(\hat{p}+\gamma^{\epsilon,\alpha}(\hat{p}-\hat{s})\right)+\epsilon((1-\alpha)\hat{s}+\alpha\hat{q}) = c((1-\epsilon)\left(\hat{p}+\gamma^{\epsilon,\alpha}(\hat{p}-\hat{s})\right)+\epsilon((1-\alpha)\hat{s}+\alpha\{\hat{q},\, \hat{r}\}))$

$\iff (1-\alpha\epsilon)\hat{p}+\alpha\epsilon\{\hat{q}\} = c((1-\alpha\epsilon)\hat{p}+\alpha\epsilon\{\hat{q},\, \hat{r}\})$

Since the above holds $\forall \hat{p}, \hat{q}, \hat{r}, \hat{s}, \epsilon \in N_p^{\kappa\bar{\delta}} \times N_q^{\kappa\bar{\delta}} \times N_r^{\kappa\bar{\delta}} \times N_s^{\kappa\bar{\delta}} \times (0, \bar{\epsilon})$ it follows that qR_pr. □

Lemma 6. *$\bar{R}_p$ satisfies the Independence Axiom if $p \in int\Delta$.*

Proof. I already have a proof that R_p satisfies the Independence Axiom.

Suppose that $q\bar{R}_pr$ and take $(1-\alpha)s+\alpha q$ and $(1-\alpha)s+\alpha r$.

If it is not the case that $(1-\alpha)s+\alpha q\bar{R}_p(1-\alpha)s+\alpha r$, then $(1-\alpha)s+\alpha rR_p(1-\alpha)s+\alpha q$.

Then it follows by Lemma A.5 that rR_pq, which contradicts that $q\bar{R}_pr$. □

Define $q\bar{\bar{R}}_pr$ if either:

(i) $p \in \text{int}\Delta$ and $q\bar{R}_pr$

(ii) $p \notin \text{int}\Delta$, and $\exists \alpha, s \in (0,1) \times \Delta$ such that $(1-\alpha)s+\alpha q\bar{R}_p(1-\alpha)s+\alpha r$

(iii) $\exists \alpha, s, \hat{q}, \hat{r}$ such that $q = (1-\alpha)s+\alpha\hat{q}$, $r = (1-\alpha)s+\alpha\hat{r}$, and $\hat{q}\bar{R}_p\hat{r}$

The relation $\bar{\bar{R}}_p$ is the minimal extension of $\bar{R}_p$ that respects with the Independence Axiom for all $p \in \Delta$.

By construction, $\bar{\bar{R}}_p$ satisfies the joint continuity properties in Lemma A.4 as well.

Lemma 7. *For each $p \in \Delta$, there exists a vector $\hat{u}^p \in \Re^N$ such that $q\bar{\bar{R}}_pr \iff q \cdot \hat{u}^p \geq r \cdot \hat{u}^p$.*

Proof. Lemma A.4. shows $\bar{R}_p$ is complete, and transitive. By construction, $\bar{\bar{R}}_p$ satisfies the Independence axiom. The joint continuity property on $\bar{R}_p$ in Lemma A.4 then implies the notion of mixture continuity required (condition 3) to apply Fishburn's (1970) Theorem 8.2. □

Say that a vector u^p is *flat* if $\max_i u_i^p = \min_i u_i^p$. Let $F := \{p \in \Delta :\ u^p \text{ is flat}\}$.

Lemma 8. *Suppose u^p is not flat. Then, there is an ϵ neighbourhood N_p^ϵ of p such that $\forall \hat{p} \in N_p^\epsilon$, $u^{\hat{p}}$ is not flat.*

Proof. Suppose there is a net $\hat{p}^\epsilon$ such that $\hat{p}^\epsilon \in N_p^\epsilon$ and $u^{\hat{p}^\epsilon}$ is flat. Since $u^{\hat{p}^\epsilon}$ must represent $\bar{R}_{\hat{p}^\epsilon}$, it follows that $q\bar{R}_{\hat{p}^\epsilon}r\ \forall q, r \in \Delta$ and for each $\hat{p}^\epsilon$. By Lemma A.4, it follows that $q\bar{R}_p r$. It follows that u^p must be flat as well, a contradiction. □

Let $\hat{u}^p$ denote a vector that provides an EU representarion for $\bar{\bar{R}}_p$ (i.e. $q \cdot \hat{u}^p \geq r \cdot \hat{u}^p \iff q\bar{\bar{R}}_p r\ \forall q, r \in \Delta$). For all p such that p is non-flat, define:

$$u^p := \frac{d^H(\{p\}, F)}{\max_i \left[\hat{u}_i^p - \sum_j \hat{u}_j^p\right]} \left(\hat{u}^p - \sum_j \hat{u}_j^p\right) \tag{8}$$

If $\hat{u}^p$ is flat, define u^p as the zero vector.

By Lemma A.8 and the EU theorem, u^p provides an EU representation for $\bar{\bar{R}}_p$.

Lemma 9. *If $p^\epsilon \to p$, then $u^{p^\epsilon} \to u^p$*

Proof. If u^p is flat, then $d^H(\{p^\epsilon\}, F) \to 0$ as $\epsilon \to 0$, thus $u^{p^\epsilon} \to u^p$.

Now suppose that u^p is non-flat. Suppose $p^\epsilon \to p$ but for convergent subnet $\{p^{\epsilon'}\}$ of $\{p^\epsilon\}$, $u^{p^{\epsilon'}} \to \bar{u}^p \neq u^p$. Since $u^{p^{\epsilon'}}$ represents $\bar{R}_{p^{\epsilon'}}$, by the joint continuity property in Lemma A.4., it follows that $\bar{u}^p$ ranks $q \sim r$ if and only if u^p ranks $q \sim r$. Since u^p and $\bar{u}^p$ must satisfy the same normalizations, they must coincide by the uniqueness result of the EU theorem.

□

Define $v : \Delta \times \Delta \to \Re$ by $v(q|p) := q \cdot u^p$

Lemma 10. *v is jointly continuous.*

Proof. $v(q|p) = q \cdot u^p = \sum_i q_i u_i^p$ and u^p is continuous as a function of p, and joint continuity of the sum $\sum_i q_i u_i^p$ in q and u^p is a standard exercise. □

Lemma A.11 shows that Limit Consistency is implied by the axioms assumed in Theorem 1.

Lemma 11. *The axioms in Theorem 1 imply Limit Consistency.*

Proof. Part 1. Suppose $\{q\} = m(D, R_p) \neq \{p\} \subseteq c(D)$.

That is, $qR_p r \; \forall r \in D$.

Then $\forall r \in D \; \exists \bar{\alpha}_r > 0$ such that $\forall \alpha \in (0, \bar{\alpha}_r)$, $(1-\alpha)p + \alpha q = c((1-\alpha) + \alpha\{q, r\})$.

Since D is finite, $\min_{r \in D} \bar{\alpha}_r > 0$.

By Expansion, $\forall \alpha \in (0, \min_{r \in D} \bar{\alpha}_r)$, $(1-\alpha)p + \alpha q \in c((1-\alpha) + \alpha D)$.

By Induced Reference Lottery Bias, $\forall \alpha \in (0, \min_{r \in D} \bar{\alpha}_r)$ $p \in c((1-\alpha) + \alpha D)$. Thus $p\tilde{\bar{W}}(1-\alpha)p + \alpha q$. Weak RARP then implies that $p \in c((1-\alpha)p + \alpha\{p, q\})$ $\forall \alpha \in (0, \min_{r \in D} \bar{\alpha}_r)$. This implies that $p\bar{R}_p q$, a contradiction.

Part 2. Suppose there are elements $q^1, ..., q^l \in D$ such that $q^i R_p p$ for each $i = 1, ..., l$.

Suppose $q^i \in m(D, R_p)$, and let $\hat{D} := D \backslash m(D, R_p)$.

Then by the previous result $\forall i = 1, ..., l$, $\exists \bar{\alpha}_i > 0$ such that $\forall \alpha \in (0, \bar{\alpha}_i)$, $(1-\alpha)p + \alpha q^i \in c((1-\alpha)p + \alpha \left(\hat{D} \cup q^i\right))$.

Since $\{q^1, ..., q^l\}$ is finite and each $\bar{\alpha}_i > 0$, $\min_i \bar{\alpha}_i > 0$.

For each $\alpha \in (0, 1)$, $c((1-\alpha)p + \alpha\{q^1, ..., q^l\})$ is non-empty.

For $\hat{q}$ such that $(1-\alpha)p + \alpha\hat{q} \in c((1-\alpha)p + \alpha\{q^1, ..., q^l\})$, Expansion implies that $(1-\alpha)p + \alpha\hat{q} \in c(\left((1-\alpha)p + \alpha\{q^1, ..., q^l\}\right) \cup \left((1-\alpha)p + \alpha\left(\hat{D} \cup \hat{q}\right)\right))$
$= c((1-\alpha)p + \alpha D)$.

Thus $\forall \alpha \in (0, \min_i \bar{\alpha}_i)$, $((1-\alpha)p + \alpha\{q^1, ..., q^l\}) \cap c((1-\alpha)p + \alpha D) \neq \emptyset$.

It follows that for at least one $\hat{q} \in \{q^1, ..., q^l\}$, $\forall \bar{\alpha} \in (0, \min_i \bar{\alpha}_i)$, $\exists \alpha < (0, \bar{\alpha})$ such that $((1-\alpha)p + \alpha\hat{q} \in c((1-\alpha)p + \alpha D)$.

Since $p \in c((1-\alpha)p + \alpha D)$ $\forall \alpha \in (0, 1)$ by Induced Reference Lottery Bias, it follows that $p\tilde{\bar{W}}(1-\alpha)p + \alpha\hat{q}$ whenever $(1-\alpha)p + \alpha\hat{q} \in c((1-\alpha)p + \alpha D)$. For such α, it further follows by Weak RARP that $p \in c((1-\alpha)p + \alpha\{p, \hat{q}\})$. This contradicts that $\hat{q}R_p p$. □

Define $\hat{PE}(D) = \{p \in D : \, p\bar{R}_p q \, \forall q \in D\}$.

Define $P\hat{P}E(D) = \{p \in \hat{PE}(D) : \, \nexists q \in \hat{PE}(D) \text{ s.t. } q\tilde{\bar{W}}p\}$.

Lemma A.9 establishes that $p \in c(\{p, q\})$ implies $p \in P\hat{P}E(\{p, q\})$.

Lemma 12. *If $q\bar{R}_q p$ and $p \in c(\{p, q\})$, then $p\tilde{\bar{W}}q$.*

Proof. If $\exists D_{pq}$ such that $q \in c^U(D_{pq})$ then the result follows automatically. Similarly if there exists a chain $p = r^0, r^1, ..., r^n = q$ such that $r^{i-1}\tilde{\tilde{W}}r^i$ for $i = 1, ..., n$.

If $\exists p^\epsilon, q^\epsilon$ that establish $q\bar{R}_q p$, then if for some such sequence, $p^\epsilon \in c(\{p^\epsilon,\, q^\epsilon\})$ for a convergent subsequence of p^ϵ, q^ϵ, then $p^\epsilon \tilde{\tilde{W}} q^\epsilon$ for such pairs. Then, continuity of $\tilde{\tilde{W}}$ implies that $p\tilde{\tilde{W}}q$.

So suppose instead that for each sequence p^ϵ, q^ϵ that establishes that $q\bar{R}_q p$, $q^\epsilon = c(\{p^\epsilon,\, q^\epsilon\})$ except on a non-convergent subsequence of p^ϵ, q^ϵ. This implies that $q \in c^U(\{p,\, q\})$. Then by the definition of $\tilde{\tilde{W}}$, $p\tilde{\tilde{W}}q$. ☐

Lemma A.10 establishes that $p \in P\hat{P}E(\{p,\, q\})$ implies $p \in c(\{p,\, q\})$.

Lemma 13. *If $p\bar{R}_p q$ and $p\tilde{\tilde{W}}q$, then $p \in c(\{p,\, q\})$.*

Proof. Since $\{p\} = c(\{p\})$, if $\{q\} = c(\{p,\, q\})$ and $p\tilde{\tilde{W}}q$ it would follow by IIA Independence and the definition of R_p that $qR_p p$. This would contradict the assumption that $p\bar{R}_p q$. Since $c(\{p,\, q\}) \neq \emptyset$, it then follows that $p \in c(\{p,\, q\})$. ☐

Lemmas A.11-A.12 establish that $P\hat{P}E(\{p,\, q,\, r\}) = c(\{p,\, q,\, r\})\ \forall p, q, r \in D$.

Lemma 14. *If $p \in c(\{p,\, q,\, r\})$ and $q\bar{R}_q p$ then $p\tilde{\tilde{W}}q$ or $r\bar{R}_q q$.*

Proof. Suppose $p \in c(\{p,\, q,\, r\})$ and $q\bar{R}_q p$.

If $p \in c(\{p,\, q\})$, then $p\tilde{\tilde{W}}q$ holds.

So suppose instead that $q = c(\{p,\, q\})$.

Then, if $q \in c(\{q,\, r\})$ it would follow by Expansion that $q \in c(\{p,\, q,\, r\})$. Since $p \in c(\{p,\, q,\, r\})$ as well, it follows that $p\tilde{\tilde{W}}q$; by Weak RARP, it follows that $p \in c(\{p,\, q\})$, a contradiction. Thus $r = c(\{q,\, r\})$.

By Lemma A.5, it follows that either $rR_q q$ or $r\tilde{\tilde{W}}q$; in the former case we're done, so suppose $r\tilde{\tilde{W}}q$ and that it is not the case that $rR_q q$.

If $p \in c(\{p,\, r\})$, then it follows that either $pR_r r$ or $p\tilde{\tilde{W}}r$. In the latter case, transitivity of $\tilde{\tilde{W}}$ implies $p\tilde{\tilde{W}}q$ and we're done, so suppose we have that $pR_r r$. Then by Limit Consistency, $p = c(\{p,\, r\})$.

To summarize, we now have that $q = c(\{p,\, q\})$, $p = c(\{p,\, r\}) = c(\{p,\, q,\, r\})$, and $r = c(\{q,\, r\})$. Then, by IIA Independence, it follows that $\exists \epsilon > 0 :\ \forall \alpha \in (0, 1)$, $\forall \hat{q} \in N_q^\epsilon$, $\forall \hat{r} \in N_r^\epsilon$, $\forall \hat{D} \supseteq \{\hat{q},\, (1-\alpha)\hat{q}+\alpha\hat{r}\}$, $\hat{q} \notin c(\hat{D})$. It follows that $rR_q q$, a contradiction.

It follows that either $rR_q q$ or $p\tilde{\tilde{W}}q$. ☐

Lemma 15. *If* $p \in P\hat{P}E(\{p,\, q,\, r\})$ *then* $p \in c(\{p,\, q,\, r\})$.

Proof. Suppose $p \in P\hat{P}E(\{p,\, q,\, r\})$.

We know that $c(\{p,\, q,\, r\}) \neq \emptyset$. So it is sufficient to prove that $q \in c(\{p,\, q,\, r\}) \implies p \in c(\{p,\, q,\, r\})$ and similarly if $r \in c(\{p,\, q,\, r\})$.

Suppose $q \in c(\{p,\, q,\, r\})$; the argument starting from $r \in c(\{p,\, q,\, r\})$ is symmetric.

Then, $q\bar{R}_q p$ and $q\bar{R}_q r$ by Limit Consistency. Since $p \in P\hat{P}E(\{p,\, q,\, r\})$ and $q \in \hat{PE}(\{p,\, q,\, r\})$, it follows that $p\tilde{\bar{\bar{W}}}q$. Then by Lemma A.10, since $p\bar{R}_p q$ as well, $p \in c(\{p,\, q\})$.

If $r \in c(\{p,\, q,\, r\})$ then a similar argument implies $p \in c(\{p,\, r\})$. Then by Expansion, $p \in c(\{p,\, q,\, r\})$.

If instead $r \notin c(\{p,\, q,\, r\})$, we have (recalling Lemma A.6) that either $p \in c(\{p,\, r\})$ or $r = c(\{p,\, r\})$. In the former case, Expansion implies $p \in c(\{p,\, q,\, r\})$. In the latter case, $r = c(\{p,\, r\})$. Recall that $p \in c(\{p,\, q\})$. If $p \notin c(\{p,\, q,\, r\})$ then $q = c(\{p,\, q,\, r\})$; by IIA Independence and the definition of R_p, it follows that $rR_p p$, a contradiction of the assumption that $p \in P\hat{P}E(\{p,\, q,\, r\})$.

It follows that $p \in P\hat{P}E(\{p,\, q,\, r\}) \implies p \in c(\{p,\, q,\, r\})$. □

Remark. $P\hat{P}E(D) = P\hat{P}E(\hat{PE}(D))$

Lemma 16. *Suppose we have established that* $P\hat{P}E(D) = c(D)$ *whenever* $|D| < n$. *If* $\hat{PE}(D) = D$ *and* $|D| \leq n$, *then* $c(D) = P\hat{P}E(D)$.

Proof. First, suppose $\hat{PE}(D) = D$.

Take $p \in P\hat{P}E(D)$. Then $p \in P\hat{P}E(D\backslash r)\ \forall \in D\backslash p$. Take any distinct $r, r' \in D\backslash p$, and then since $|D\backslash r| = |D\backslash r'| = n - 1 < n$, $p \in c(D\backslash r) \cap c(D\backslash r')$. By Expansion, it follows that $p \in c(D)$.

In the reverse, suppose $p \in c(D)$. Then if $q\tilde{\bar{\bar{W}}}r\ \forall r \in D$, since $\hat{PE}(D) = D$, it follows that $q \in c(\{q,\, r\})\ \forall r \in D$. By Expansion, it follows that $q \in c(D)$. Then since $p \in c(D)$ and $q \in c(D)$, $p\tilde{\bar{\bar{W}}}q$ by definition. Thus $p \in P\hat{P}E(D)$. □

Lemma A.14 establishes by induction that $c(D) = P\hat{P}E(D)$ for any $D \in \mathcal{D}$.

Lemma 17. *Suppose $c(D) = P\hat{P}E(D)$ whenever $|D| < n$. Then, $c(D) = P\hat{P}E(D)$ whenever $|D| \leq n$ as well.*

Proof. Consider D with $|D| = n$ and $\hat{PE}(D) \neq D$. Partition D into $\hat{PE}(D)$ and $D \backslash \hat{PE}(D)$. The case where $\hat{PE}(D) = D$ was proven in Lemma A.9.

Since $|\hat{PE}(D)| \leq n - 1 < n$, $c(\hat{PE}(D)) = P\hat{P}E(\hat{PE}(D)) = P\hat{P}E(D)$.

Say that $q^0, q^1, ..., q^m$ form a *chain* if $q^i R_{q^{i-1}} q^{i-1}$ for $i = 1, ..., m$. Notice that if $q^0, ..., q^m$ form a chain, Limit Consistency implies that $q^m = c(\{q^0, ..., q^m\}) = \hat{PE}(D) = P\hat{P}E(D)$. So if the longest chain in D contains all elements of D, then $c(D) = P\hat{P}E(D)$.

Now suppose $p \in P\hat{P}E(D)$.

First, further suppose the longest chain in D has length $n-1$; denote the chain $q^0, q^1, ..., q^{n-1}$. Then, $q^{n-1} = c(\{q^0, q^1, ..., q^{n-1}\})$ and since $q^0, q^1, ..., q^{n-1}$ is the longest chain in D and $p \in \hat{PE}(D)$, $\{p,\ q^{n-1}\} = \hat{PE}(D)$. Since $p \in P\hat{P}E(D)$, it follows that $p\tilde{\tilde{W}}q^{n-1}$; Lemmas A.8 and A.10, $p \in c(\{p,\ q^{n-1}\})$. Suppose $p \in c(\{p,\ q^k, ..., q^{n-1}\})$ for some $k \leq n-1$. Then, since if $p \notin c(\{p,\ q^{k-1}, ..., q^{n-1}\})$ it follows by IIA Independence and the definition of R_p that $q^{k-1} R_p p$, which contradicts that $p \in \hat{PE}(D)$. Thus it follows by induction that $p \in c(D)$.

Take an arbitrary chain $q^0, ..., q^m$ that cannot be extended further as a chain using elements of D. Since $q^0, ..., q^m$ cannot be extended, $q^m \in \hat{PE}(D)$. Since $p \in P\hat{P}E(D)$, $p\tilde{\tilde{W}}q^m$ and by Lemma A.8, $p \in c(\{p,\ q^m\})$. Suppose $p \in c(\{p,\ q^k, ..., q^m\})$ for some $k \leq m$. Then if $p \notin c(\{p,\ q^{k-1}, ..., q^m\})$ it follows by IIA Independence and the definition of R_p that $q^{k-1} R_p p$; this would which contradicts that $p \in \hat{PE}(D)$. Thus it follows by induction that $p \in c(\{p,\ q^0, ..., q^m\})$.

Notice that any element of $D \backslash \hat{PE}(D)$ is in a chain in D. Let $\hat{D}$ is the choice set formed by taking the union of $\{p\}$ and of the all of the choice sets formed by chains in D. Since for any chain $q^0, ..., q^m$ in D, $p \in c(\{p,\ q^0, ..., q^m\})$, $p \in c(\hat{D})$ follows by Expansion. Since $p \in c(\hat{PE}(D))$ as well follows (because $|\hat{PE}(D)| < n$ or Lemma A.13 applies), it follows by Expansion that $p \in c(D)$. Thus $P\hat{P}E(D) \subseteq c(D)$.

In the reverse direction, now suppose $p \in c(D)$. By Limit Consistency, $p \in \hat{PE}(D)$. Since $c(D) \supseteq P\hat{P}E(D) = P\hat{P}E(\hat{PE}(D)) = c(\hat{PE}(D)) \neq \emptyset$, $\exists q \in c(D) \cap P\hat{P}E(D)$. Since $p, q \in c(D)$, $p\tilde{\tilde{W}}q$. Thus $p \in P\hat{P}E(D)$. $\square$

Lemma A.15 relates the dislike of mixtures property to the Induced Reference Lottery Bias axiom.

Lemma 18. *Induced Reference Lottery Bias implies that v dislikes mixtures.*

Proof. By the representation thus far, $c(D) = P\hat{P}E(D)$.

If $p \in P\hat{P}E(\{p, q\})$ then $v(p|p) \geq v(q|p)$ and either $v(p|p) \geq v(q|q)$ or $v(p|q) > v(q|q)$. Thus $v(p|p) \geq v(q|p)$ and $v(q|q) \leq \max[v(p|p), v(q|p)]$. Then the Induced Reference Lottery Bias axiom implies that then $p \in c((1-\alpha)p + \alpha D) = P\hat{P}E((1-\alpha)p + \alpha D)$, thus $v(p|p) \geq v((1-\alpha)p + \alpha q|p)$ and $v((1-\alpha)p + \alpha q|(1-\alpha)p + \alpha q) \leq \max[v(p|p), v((1-\alpha)p + \alpha q|p)]$. □

Remark. $P\hat{P}E(D) = PPE_v(D)$

Necessity.

Proposition 1 implies that Expansion and Weak RARP are necessary conditions for any PPE representation.

Lemma 19. *Suppose v represents c by a PPE representation. Then $p\tilde{\bar{W}}r$ implies that $v(p|p) \geq v(r|r)$.*

Proof. Suppose $p\tilde{\bar{W}}r$. If $\exists D, \bar{D}$ with $\{p, r\} \subseteq D \subseteq \bar{D}$ and $p \in c(D)$ and $r \in c(\bar{D})$ then it follows that $v(p|p) \geq v(r|r)$ since $r \in PE(\bar{D}) \cap D \subseteq PE(D)$ follows by the representation.

If instead there is a chain such that $p^{i-1}\tilde{\bar{W}}p^i$ for $i = 1, ..., n$ and $p^0 = p$, $p^n = r$, then it follows that $v(p^{i-1}|p^{i-1}) \geq v(p^i|p^i)$ for each i. Chaining these inequalities together, it follows that $v(p|p) \geq v(r|r)$. □

Necessity of IIA Independence. Suppose $p\tilde{\bar{W}}r$. Then by Lemma A.12, $v(p|p) \geq v(r|r)$. If $p \in PPE(D)$ and $p \notin PPE(D \cup q) \ni r$, then it follows that $v(q|p) > v(p|p)$. Since v is jointly continuous, $\exists \epsilon > 0$ such that $\forall \hat{p} \in N_p^\epsilon$, $\forall \hat{q} \in N_q^\epsilon$, $v(\hat{q}|\hat{p}) > v(\hat{q}|\hat{p})$. Since v is expected utility, it follows that for all such $\hat{p}, \hat{q}$ pairs and $\forall \alpha \in [0, 1)$, $v((1-\alpha)\hat{p} + \alpha\hat{q}|\hat{p}) > v(\hat{p}|\hat{p})$. It follows that for all such $\hat{p}, \hat{q}$ pairs and for any such $\alpha \in [0, 1)$, whenever $(1-\alpha)\hat{p} + \alpha\hat{q} \in \hat{D}$ it follows that $\hat{p} \notin PPE(\hat{D}) = c(\hat{D})$. Thus IIA Independence holds.

Necessity of Transitive Limit. First, I show that the antecedent of Transitive Limit has bite in the presence of, and only in the presence of, a strict preference. To be precise, suppose $(1-\epsilon)p^\delta + \epsilon q^\delta = c(\{(1-\epsilon)p^\delta + \epsilon q^\delta,\ (1-\epsilon)p^\delta + \epsilon r^\delta\})$ for all small ϵ, and $p^\delta, q^\delta, r^\delta$ sufficiently close to p, q, r. By the representation, this holds only if for all p^δ close to p, q^δ close to q, r^δ close to r, and ϵ close to zero, $v(q^\delta|(1-\epsilon)p^\delta + \epsilon q^\delta) \geq v(r^\delta|(1-\epsilon)p^\delta + \epsilon q^\delta)$, thus $v(q^\delta|p^\delta) \geq v(r^\delta|p^\delta)$ for all $p^\delta, q^\delta, r^\delta$. If $v(q|p) = v(r|p)$, then for every q^δ near q, $v(q^\delta|p) \geq v(q|p)$ and for every r^δ near r, $v(r|p) \geq v(r^\delta|p)$; this contradicts local strictness of $v(\cdot|p)$ in the representation. Thus when the antecedent of Transitive Limit holds, $v(q|p) > v(r|p)$ must hold.

Now take a continuous EU-PE representation and suppose $v(q|p) > v(r|p)$. Then, joint continuity implies that $v((1-\lambda)s + \lambda q^\delta|p^\delta) > v((1-\lambda)s + \lambda r^\delta|p^\delta)$ for any $s \in \Delta$, $\lambda > 0$, and δ close to zero. It follows that $v((1-\epsilon)p^\delta + \epsilon q^\delta|(1-\epsilon)p^\delta + \epsilon r^\delta) > v((1-\epsilon)p^\delta + \epsilon r^\delta|(1-\epsilon)p^\delta + \epsilon r^\delta)$ for all δ, ϵ sufficiently small. Thus for sufficiently small δ, ϵ, $(1-\epsilon)p^\delta + \epsilon q^\delta = c(\{(1-\epsilon)p^\delta + \epsilon q^\delta,\ (1-\epsilon)p^\delta + \epsilon r^\delta\})$. Thus the antecedent of Transitive Limit holds when $v(q|p) > v(r|p)$.

Since $v(q|p) > v(r|p)$ and $v(r|p) > v(s|p)$ imply $v(q|p) > v(s|p)$, the analysis above implies that qR_pr and rR_ps implies qR_ps, so Transitive Limit must hold.

Necessity of Induced Reference Lottery Bias. In the representation, $v(p|p) \geq v(q|p)$ and $v(q|q) \leq \max[v(p|p),\, v(p|q)]$ imply that $\forall \alpha \in (0,1)$, $v((1-\alpha)p + \alpha q|(1-\alpha)p + \alpha q) \leq \max[v(p|p),\, v(p|(1-\alpha)p + \alpha q)]$.

Suppose $p \in c(D)$. Then, $v(p|p) \geq v(q|p)$ $\forall q \in D$, and $v(p|p) \geq v(q|q)$ $\forall q \in PE(D)$. It follows that $v(p|p) \geq v(q|p)$ and $v(q|q) \leq \max[v(p|p),\, v(p|q)]$. Since $v(\cdot|p)$ satisfies expected utility, $p \in PE((1-\alpha)p + \alpha D)$ $\forall \alpha \in (0,1)$. Since $v((1-\alpha)p + \alpha q|(1-\alpha)p + \alpha q) \leq \max[v(p|p),\, v(p|(1-\alpha)p + \alpha q)]$ $\forall q \in D$, it follows that $v(p|p) \geq v((1-\alpha)p + \alpha q|(1-\alpha)p + \alpha q)$ $\forall q : (1-\alpha)p + \alpha q \in PE((1-\alpha)p + \alpha D)$. Thus $p \in PPE((1-\alpha)p + \alpha D) = c((1-\alpha)p + \alpha D)$ $\forall \alpha \in (0,1)$. Thus Induced Reference Lottery Bias holds.

□

Proof of Proposition 2.

Suppose that $v(\cdot|p)$ and $v(\cdot|q)$ are not ordinally equivalent. Then $\exists \bar{r}, \bar{s} \in \Delta$ such that $v(\bar{r}|p) > v(\bar{s}|p)$ but $v(\bar{r}|q) \leq v(\bar{s}|q)$. By local strictness, $\exists r, s \in \Delta$ that are close to $\bar{r}, \bar{s}$ such that $v(r|p) > v(s|p)$ but $v(r|q) < v(s|q)$. By EU of $v(\cdot|p)$ and continuity of v, this implies that $\exists \bar{\delta}, \bar{\epsilon} > 0$ such that $\forall \epsilon \in (0, \bar{\epsilon})$, $\forall r^\delta \in N_r^\delta$, $\forall s^\delta \in N_s^\delta$, $v((1-\epsilon)p + \epsilon r^\delta | (1-\epsilon)p + \epsilon s^\delta) > v((1-\epsilon)p + \epsilon s^\delta | (1-\epsilon)p + \epsilon s^\delta)$ but $v((1-\epsilon)q + \epsilon s^\delta | (1-\epsilon)q + \epsilon r^\delta) > v((1-\epsilon)q + \epsilon r^\delta | (1-\epsilon)q + \epsilon r^\delta)$. By the representation, this implies that for such $\epsilon, r^\delta, s^\delta$, (a) $(1-\epsilon)p + \epsilon r^\delta = c(\{(1-\epsilon)p + \epsilon r^\delta, (1-\epsilon)p + \epsilon s^\delta\})$ and (b) $(1-\epsilon)q + \epsilon s^\delta = c(\{(1-\epsilon)q + \epsilon r^\delta, (1-\epsilon)q + \epsilon s^\delta\})$. Thus if $v(\cdot|p)$ and $v(\cdot|q)$ are not ordinally equivalent, c strictly exhibits expectations-dependence.

Now suppose that c exhibits expectations-dependence at D, α, p, q, r. That is, $\exists \bar{\epsilon} > 0$ such that $\forall r^\epsilon \in N_r^\epsilon$, $\forall D^\epsilon \ni r^\epsilon$ such that $d^H(D^\epsilon, D) < \epsilon$, $(1-\alpha)p + \alpha r^\epsilon \in c((1-\alpha)p + \alpha D^\epsilon)$ but $(1-\alpha)q + \alpha r^\epsilon \notin c((1-\alpha)q + \alpha D^\epsilon)$. Since $(1-\alpha)q + \alpha r^\epsilon \notin c((1-\alpha)q + \alpha D^\epsilon)$, it follows that for each D^ϵ, $\exists \bar{s}^\epsilon \in D^\epsilon$, $v(\bar{s}^\epsilon | (1-\alpha)p + \alpha \bar{s}^\epsilon) \geq v(r^\epsilon | (1-\alpha)p + \alpha \bar{s}^\epsilon)$. Local strictness then implies that for each such $\bar{s}^\epsilon, r^\epsilon$ pair, there is an arbitrarily close pair $\hat{s}^\epsilon, \hat{r}^\epsilon$ such that $v(\hat{s}^\epsilon | (1-\alpha)p + \alpha \bar{s}^\epsilon) > v(\hat{r}^\epsilon | (1-\alpha)p + \alpha \bar{s}^\epsilon)$. By the representation, $(1-\alpha)p + \alpha r^\epsilon \in c((1-\alpha)p + \alpha D^\epsilon)$ implies that for each r^ϵ, $\forall s^\epsilon \in D^\epsilon$, $v(r^\epsilon | (1-\alpha)p + \alpha r^\epsilon) \geq v(s^\epsilon | (1-\alpha)p + \alpha r^\epsilon)$; thus $v(\hat{r}^\epsilon | (1-\alpha)p + \alpha \hat{r}^\epsilon) \geq v(\hat{s}^\epsilon | (1-\alpha)p + \alpha \hat{r}^\epsilon)$. Thus v exhibits strict expectations-dependence. This proves the first part of the proposition.

Now suppose c violates IIA. Then there are D, D' such that $D' \subset D$ and $c(D) \cap D' \neq \emptyset$ but $c(D') \neq c(D) \cap D'$. This implies that either (a) or (b) holds:

(a) $\exists p \in c(D')$ such that $p \notin c(D)$. Then by the representation, this implies that $v(p|p) = v(q|q)$ for $q \in c(D')$, so for some $r \in D$, $v(r|p) > v(p|p) \geq v(q|p)$ but $v(q|q) \geq v(r|q)$

(b) $\exists p \in c(D) \cap D'$ with $p \notin c(D')$. Since $PE(D) \cap D' \subset PE(D')$, this implies that there is a $q \in c(D')$ with $v(q|q) > v(p|p)$. Thus $q \notin c(D) \implies q \notin PE(D)$, which implies that $\exists r \in D \backslash D'$ such that $v(r|q) > v(q|q) \geq v(p|q)$ but $v(p|p) \geq v(r|p)$.

In either case (a) or (b), by the first part of the proposition, c exhibits strict expectations-dependence.

□

Proof of Proposition 4

First prove that Kőszegi-Rabin preferences with linear loss aversion satisfy the limited-cycle inequalities.

Start with a finite set X with $|X| = n+1$ and assume (for now) that there is a single hedonic dimension. Without loss of generality, assume $m(x_1) > m(x_2) > ... > m(x_{n+1})$

Define the matrix V according to:

$$[V]_{ij} = m(x_i) + \eta[m(x_i) - m(x_j)] + \eta[\lambda - 1]\min[0,\ m(x_i) - m(x_j)] \tag{9}$$

Observe that $v(p|r) = p^T V r$. Let $\delta, \epsilon \in \Re^{n+1}$ denote vectors with $\sum_{i=1}^{n+1} \delta_i = \sum_{i=1}^{n+1} \epsilon_i = 0$. By matrix multiplication,

$$\begin{gathered}\delta^T V \epsilon = \eta[\lambda - 1]\times \\ [(m(x_1) - m(x_2))\delta_1\epsilon_1 + (m(x_2) - m(x_3))(\delta_1 + \delta_2)(\epsilon_1 + \epsilon_2) + \\ ... + (m(x_n) - m(x_{n+1}))(\sum_{i=1}^{n} \delta_i)(\sum_{i=1}^{n} \epsilon_i)]\end{gathered} \tag{10}$$

Take a cycle $p^{i+1} = p^i + \epsilon^i$ with $v(p^{i+1}|p^i) > v(p^i|p^i)$ for $i = 0, ..., m$. Then:

$v(p^m|p^m) - v(p^0|p^m) = (p + \sum_{l=1}^m \epsilon^l)^T V (p + \sum_{l=1}^m \epsilon^l) - p^T V (p + \sum_{l=1}^m \epsilon^l)$

$= (\sum_{l=1}^m \epsilon^l)^T V (\sum_{l=1}^m \epsilon^l) + (\sum_{l=1}^m \epsilon^l)^T V p$

Rearranging the second term,

$= (\sum_{l=1}^m \epsilon^l)^T V (\sum_{l=1}^m \epsilon^l) + (\sum_{l=1}^{m-1} \epsilon^l)^T V p + (\epsilon^m)^T V (p + \sum_{l=1}^{m-1} \epsilon^l) - (\epsilon^m)^T V (\sum_{l=1}^{m-1} \epsilon^l)$

$= (\sum_{l=1}^m \epsilon^l)^T V (\sum_{l=1}^m \epsilon^l) + (\sum_{l=1}^{m-2} \epsilon^l)^T V p + (\epsilon^{m-1})^T V (p + \sum_{l=1}^{m-2} \epsilon^l) - (\epsilon^{m-1})^T V (\sum_{l=1}^{m-2} \epsilon^l) + (\epsilon^m)^T V (p + \sum_{l=1}^{m-1} \epsilon^l) - (\epsilon^m)^T V (\sum_{l=1}^{m-1} \epsilon^l)$

$= ... = (\sum_{l=1}^m \epsilon^l)^T V (\sum_{l=1}^m \epsilon^l) + \sum_i (\epsilon^i)^T V (p + \sum_{l=1}^{i-1} \epsilon^l) - \sum_{i=2}^m \epsilon^i V (\sum_{l=1}^{i-1} \epsilon^l)$

By the definition of the cycle, $(\epsilon^i)^T V (p + \sum_{l=1}^{i-1} \epsilon^l) > 0$ for each i, thus:

$> (\sum_{l=1}^m \epsilon^l)^T V (\sum_{l=1}^m \epsilon^l) - \sum_{i=2}^m \epsilon^i V (\sum_{l=1}^{i-1} \epsilon^l)$

By symmetry with respect to δ and ϵ in (10), it can be shown that $\sum_{i=2}^m \sum_{l=1}^{i-1} (\epsilon^i)^T V \epsilon^l = \sum_{j=1}^{m-1} \sum_{l=j+1}^m (\epsilon^j)^T V \epsilon^l$. Returning to the previous expression, more algebra establishes:

$= \sum_{l=1}^{m}(\epsilon^l)^T V \epsilon^l + \sum_{i=2}^{m}\sum_{l=1}^{i-1}(\epsilon^i)^T V \epsilon^l$
$= \frac{1}{2}\sum_{l=1}^{m}(\epsilon^l)^T V \epsilon^l + \frac{1}{2}(\sum_{l=1}^{m}\epsilon^l)^T V(\sum_{l=1}^{m}\epsilon^l)$
> 0

This completes the proof for the case with the case of one hedonic dimension.

To extend the argument to $K > 1$, break up a lottery p into marginals $\underset{k}{p}$ in each dimenion k, and define the matrix $\underset{k}{V}$ as the utility matrix corresponding to V in dimension k. we can write $v^{KR}(p|r) = \sum_k \underset{k}{p}^T \underset{k}{V} \underset{k}{r}$. Notice that all of the previously-proven properties of V apply to $\underset{k}{V}$; following through the previous steps yields the desired result.

Second prove that Kőszegi-Rabin preferences with linear loss aversion dislike mixtures.

Suppose $v(p|p) \geq v(q|p)$ and $v(q|q) \leq \max[v(p|p), v(p|q)]$.

Then,

$v((1-\alpha)p + \alpha q|(1-\alpha)p + \alpha q)$

$$= (1-\alpha)^2 v(p|p) + \alpha(1-\alpha)v(p|q) + \alpha(1-\alpha)v(q|p) + \alpha^2 v(q|q) \tag{11}$$

by bilinearity of v under (3) and linear loss aversion.

If $v(p|p) \leq v(p|q)$, then two substitutions to (11) yield
$\leq (1-\alpha)^2 v(p|p) + \alpha(1-\alpha)v(p|q) + \alpha(1-\alpha)v(p|p) + \alpha^2 v(p|q)$
$= v(p|(1-\alpha)p + \alpha q)$ by bilinearity of v
$= \max[v(p|(1-\alpha)p + \alpha q), v(p|p)]$

If instead $v(p|q) \leq v(p|p)$, then two different substitutions to (11) yield
$\leq (1-\alpha)^2 v(p|p) + \alpha(1-\alpha)v(p|p) + \alpha(1-\alpha)v(p|p) + \alpha^2 v(p|p)$
$= v(p|p)$
$= \max[v(p|(1-\alpha)p + \alpha q), v(p|p)]$

This proves that v dislikes mixtures.

□

Proof of Proposition 5

Gul and Pesendorfer (2006) prove that on a finite set X there is an assignment of hedonic dimensions such that any reference-dependent utility function $\hat{v}(x|y)$ can be written as a Kőszegi-Rabin preference as in (3). Extend $\hat{v}(x|y)$ to lotteries by setting $v(p|q) = \sum_i \sum_j p_i q_j \hat{v}(x|y)$. The resulting representation over Δ is thus consistent with (3).

Kőszegi (2010, Example 3 and footnote 6) provides an example of $v : \Delta \times \Delta \to \Re$ in which the only personal equilibrium involves randomization among elements of a choice set. Mapping the v from Kőszegi's example to a Kőszegi-Rabin preference as described provides an example of a Kőszegi-Rabin preference that does not satisfy the limited-cycle inequalities.

□

Proof of Proposition 6

Take a continuous PPE representation corresponding to $\succeq_L, \{\succeq_p\}_{p\in\Delta}$. Take $p \in D$. Reference Lottery Bias implies that if $p \succeq_L q\ \forall q \in D$ then $p \succeq_p q\ \forall q \in D$; thus, $p \in m(D, \succeq_L) \implies p \in PE(D)$, which jointly imply $p \in PPE(D) = c(D)$. Since $\succeq_L$ is continuous and D is finite, it has a maximizer in D, thus there is a $p \in m(D, \succeq_L)$; by the previous argument, for any other $q \in c(D)$ it follows from the representation that $q \succeq_L p$ thus $q \in m(D, \succeq_L)$ as well. It follows that if $\succeq_L, \{\succeq_p\}_{p\in\Delta}$ satisfies Reference Lottery Bias, that $c(D) = m(D, \succeq_L)$.

□

Proof of Proposition 7.

(i) $\iff$ (iii)

Let suppose c is induced by the continuous binary relation P.

Necessity of Expansion. $p \in c(D) \iff \nexists q \in D$ such that qPp.

Thus, $p \in c(D)$ and $p \in c(D')$

$\iff$ both $\nexists q \in D$ such that qPp and $\nexists r \in D'$ such that rPp.

$\iff \nexists q \in D \cup D'$ such that qPp

$\iff p \in m(D \cup D', P)$

$\iff p \in c(D \cup D')$

Necessity of Sen's α. $p \in c(D) = m(D,P) \iff \nexists q \in D$ such that qPp

$\implies$ if $D' \subset D$, then $\nexists q \in D'$ such that qPp

$\iff p \in m(D',P) = c(D')$

Necessity of UHC. By contradiction.

Suppose $p^\epsilon \in c(D^\epsilon) = m(D^\epsilon, P)$ for a sequence $D^\epsilon \to D$ such that $d^H(D^\epsilon, D) < \epsilon$.

If $p \notin c(D)$, then $\exists q \in D$ such that qPp.

Then, since q has open better than and worse than sets, $\exists \epsilon$ such that $\forall p^\epsilon \in N_p^\epsilon$, $\forall q^\epsilon \in N_q^\epsilon$, $q^\epsilon P p^\epsilon$.

Since $d^H(D^\epsilon, D) < \epsilon$, it follows that $\forall D^\epsilon$ in the sequence, $\exists q^\epsilon \in D^\epsilon$ such that $d^E(q^\epsilon, q) < \epsilon$. Thus, $\exists \bar{\epsilon} > 0$ such that $\forall \epsilon < \bar{\epsilon}$, $q^\epsilon P p^\epsilon$. This contradicts that $p^\epsilon \in m(D^\epsilon, P)\ \forall D^\epsilon$. $\Diamond$

Sufficiency. Construct $\bar{P}$ by:

$$p\bar{P}q \text{ if } \exists D_{pq} \text{ such that } p \in c(D_{pq})$$

Define P as the asymmetric part of $\bar{P}$.

(I) show $c(D) \subseteq m(D,P)$

If $p \in c(D)$, then $p \in m(D,P)$ by the definition of P.

(II) show $m(D,P) \subseteq c(D)$

Suppose $p \in m(D,P)$. Then, $\forall r \in D$, $\exists D_{pr} :\ p \in c(D_{pr})$.

By Expansion, $p \in c(\underset{r\in D}{\cup} D_{pr})$.

Since $D \subseteq \underset{r\in D}{\cup} D_{pr}$, by Sen's α, $p \in c(D)$ as well.

(III) show P is continuous.

If $p^\epsilon \bar{P} q^\epsilon$ for a sequence $p^\epsilon, q^\epsilon \to p, q$ then by steps (I) and (II), $p^\epsilon \in c(\{p^\epsilon, q^\epsilon\})$. By UHC, this implies $p \in c(\{p,q\})$ thus $p\bar{P}q$. Thus, $\bar{P}$ has closed better and worse than sets. Thus P has strictly open better and worse than sets.

$\square$

Proof of Theorem 3.

Necessity. Necessity of Expansion, Sen's α, and UHC follows from Proposition 7.

Necessity of IIA Independence 2 and Transitive Limit are similar to Theorem 1.

To prove the necessity of Induced Reference Lottery Bias,

$p \in c(D) = PE(D)$

$\iff v(p|p) \geq v(q|p)\, \forall q \in D$

$\iff v(p|p) \geq v((1-\alpha)p + \alpha q|p)\, \forall q \in D$ since $v(\cdot|p)$ satisfies EU

$\iff p \in PE((1-\alpha)p + \alpha D) = c((1-\alpha)p + \alpha D)$

Thus the representation implies Induced Reference Lottery Bias.

Sufficiency.

Lemma 20. *IIA Independence 2 implies Limit Consistency.*

Proof. Suppose $qR_p p$. Then $\exists \bar{\alpha} > 0$ such that $\forall \alpha \in (0, \bar{\alpha})$, $\{(1-\alpha) + \alpha q\} = c((1-\alpha)p + \alpha\{p,\, q\})$. By IIA Independence 2, it follows that $\forall \alpha \in (0, 1]$, $\forall D_{p,\,(1-\alpha)p+\alpha q}$ that $p \notin c(D_{p,\,(1-\alpha)p+\alpha q})$. Thus Limit Consistency holds. □

Take v from Lemma A.7 (from the proof of Theorem 1).

Define $PE(D) := \{p \in D :\ v(p|p) \geq v(q|p)\, \forall q \in D\}$.

By Lemma A.13, the axioms for Theorem 3 imply Limit Consistency. Since $v(\cdot|p)$ represents $\bar{R}_p$, Limit Consistency implies that $c(D) \subseteq PE(D)$.

Suppose $p \notin c(D)$ - I will show that $p \notin PE(D)$.

If $\forall q \in D$, $\exists D_{pq}$ such that $p \in c(D_{pq})$, then by Expansion, $p \in c(\bigcup_{q \in D} D_{pq})$; by Sen's α, it follows that $p \in c(D)$, a contradiction.

Thus $\exists q \in D$ such that $p \notin c(D_{pq})$ for any $D_{pq} \supseteq \{p,\, q\}$. It follows by IIA Independence 2 that $\exists \epsilon > 0$ such that $\forall \alpha \in (0, 1)$, $D_{p,\,(1-\alpha)p+\alpha q}$, and $\forall (\hat{p}, \hat{q}) \in N_p^\epsilon \times N_q^\epsilon$, $p \notin c(D_{\hat{p},\,(1-\alpha)\hat{p}+\alpha\hat{q}})$. This implies $qR_p p$. Thus $p \notin PE(D)$. It follows that $D \backslash c(D) \subseteq D \backslash PE(D)$, thus $PE(D) \subseteq D$.

This establishes that $PE(D) = c(D)$.

□

Remark. The proof of Theorem 3 makes no use of Induced Reference Lottery Bias. It follows that Induced Reference Lottery Bias is not independent of the remaining axioms.

Proof of Theorem 4.

Ok (2012, Chapters 5 and 9) proves that IIA and UHC hold if and only if c is induced by a continuous preference relation, if and only if c has a utility representation (since Δ is a separable metric space).[17]

For any continuous $u : \Delta \to \Re$, we can take any v that satisfies $v(p|p) = u(p)$; conversely, for any v we can define u by $u(p) := v(p|p)$. Under this mapping $CPE(D) = \max_{p \in D} v(p|p) = \max_{p \in D} u(p)$.

$\square$

Proof of Proposition 8.

(i) $\iff$ (ii)

Assuming IRLB:

$p \in c(\{p, q\})$

$\iff p \succeq q$

$\implies p \in c((1-\alpha)p + \alpha\{p, q\})$ by IRLB

$\iff p \succeq (1-\alpha)p + \alpha q$

which proves that IRLB implies quasiconvexity of $\succeq$

Now assume quasiconvexity of $\succeq$:

$p \in c(D)$

$\iff p \succeq q \, \forall q \in D$

$\implies p \succeq (1-\alpha)p + \alpha q \, \forall q \in D$ by quasiconvexity

$\iff p \in c((1-\alpha)p + \alpha D)$.

(ii) $\iff$ (iii)

comparing the CPE and preference maximization representations, we see that:

$p \succeq q \iff v(p|p) \geq v(q|q)$.

Thus the statement "$p \succeq q \implies p \succeq (1-\alpha)p + \alpha q$" holds if and only if the statement "$v(p|p) \geq v(q|q) \implies v(p|p) \geq v((1-\alpha)p + \alpha q|(1-\alpha)p + \alpha q)$" holds.

$\square$

[17] Arrow (1959) shows that IIA holds if and only if there exists a complete and transitive binary relation R such that c is induced by R.

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