Preferred Personal Equilibrium and Choice under Risk

David Freeman*

Current version: July 2016

Abstract

This paper studies the choice behavior of Kőszegi and Rabin's (2006) Preferred Personal Equilibrium (PPE) model of choice with reference-dependent preferences and expectations-based reference points. The choices under risk of leading cases of their model fail to satisfy both the Independence Axiom and the Weak Axiom of Revealed Preference (WARP). I introduce the EU-PPE model as a restriction of the PPE model in which reference-dependent preferences take an expected utility form conditional on the reference point. In a two-period choice setting, I provide a set of necessary and sufficient conditions for behavior to have an EU-PPE representation. I provide a definition of exhibiting expectations-based reference-dependence in choice, and show that in this representation such behavior is tightly tied to violations of the Independence Axiom, while a failure of WARP is sufficient but not necessary to display expectations-based reference-dependence.

Keywords: reference-dependent preferences, expectations-based reference-dependence, personal equilibrium, choice under risk.
JEL Codes: D81, D84, D03.

*Department of Economics, Simon Fraser University, 8888 University Dr., Burnaby, BC, Canada, V5A 1S6. E-mail: david_freeman@sfu.ca. Acknowledgements: I gratefully acknowledge funding from an SFU President's Research Start-up Grant.

1 Introduction

In their 2006 paper, Kőszegi and Rabin introduce the Preferred Personal Equilibrium (PPE) concept for modeling choice with reference-dependent preferences where reference points are given by a decision-maker's (DM) lagged expectations. This model has subsequently been widely applied in environments with risk to model reference-dependence with endogenous expectations-based reference points.[1] In a PPE, the DM's reference point is required to be consistent with rational expectations about her choice, and thus her reference point and her choice coincide. This raises the question: what is the behavioral content of existing specializations of the PPE model to environments with risk?

To study the behavior of the PPE model in environments with risk while abstracting from parametric assumptions about the form of reference-dependent utility functions, I introduce the EU-PPE model. In this model, the DM has a set of reference-dependent utility functions, where $v(\cdot|p)$ denotes her utility function over lotteries when the reference point is p. Choice from a set of lotteries D in the EU-PPE model is given by:

$$PPE_v(D) = \arg\max_{p \in D} \{v(p|p) : \ v(p|p) \geq v(q|p) \ \forall q \in D\}$$

In addition, each reference-dependent utility function, $v(\cdot|p)$, is restricted to be jointly continuous and to take an expected utility form given the reference point p. The contribution of this paper is to provide axiomatic foundations for the EU-PPE model.

The EU-PPE model can fail two standard axioms of choice under risk. Since mixing a choice set with a lottery will change the reference point against which any available lottery is evaluated, the EU-PPE model can fail the Independence Axiom. And since choice is generated by the lexicographic composition of two potentially different criteria, one of which can be incomplete and intransitive, this model can also violate WARP. Thus this model has significant behavioral differences from standard models of choice under risk.

My axioms for the EU-PPE model rely on two forms of choice data. First, I use a DM's *ranking* of lotteries that she applies when she must commit to a choice with delayed resolution – the "choice-acclimating personal equilibrium" (CPE) ranking in the language of Kőszegi and Rabin (2007). The CPE ranking captures situations in which, at the time of choice, a DM will evaluate the lottery's outcomes against the expectation-based reference point that corresponds to the chosen lottery itself. Second, I use a DM's *choices* from situations in which the DM knows in advance of the choice set from which she will choose, but cannot commit to her choice when she is first confronted with the choice set and forms

[1] Examples include Kőszegi and Rabin (2007); Heidhues and Kőszegi (2008, 2014); Eliaz and Spiegler (2013); Herweg and Mierendorff (2013); Herweg et al. (2014); Karle et al. (2015); Pagel (2016)

her expectations.

My characterization of the EU-PPE model draws on four main axioms. The first main axiom, Non-Intrinsic Preference for Commitment, requires that if p and q are available in a small choice set and p is chosen, but then q is chosen but p is not in a larger choice set, then then p must be preferred to q according to the CPE ranking. The other three main axioms draw on the interpretation of choice between two related mixtures of lotteries, $(1-\alpha)p+\alpha q$ and $(1-\alpha)p+\alpha r$, as involving a conditional choice between q and r when expectations are partially given by p. First, Conditional Exclusion Consistency imposes a weakening of the Independence Axiom by restricting that if adding the lottery $(1-\alpha)p+\alpha q$ to D in which p was chosen leads the DM to no longer choose p, but instead choose a lottery she ranks as CPE-worse, then p can never be chosen when any mixture between p and q is available. Second, Weak Reference Bias imposes that if p is ever chosen when q is available, then it must also be (approximately) chosen in pairwise choices between p and $(1-\alpha)p+\alpha q$ for small α – that is, when the reference point is made closer to p. Third, Transitive Limit imposes that as α becomes small, choices between $(1-\alpha)p+\alpha q$ and $(1-\alpha)p+\alpha r$ are made according to a transitive relation, capturing the intuitive notion that if we could observe these conditional choices as $\alpha \to 0$, we would measure the preference between q and r when the reference point is p. Theorem 1 states that a DM has a complete, transitive, and continuous CPE ranking, her choices satisfy a strengthening of Conditional Exclusion Consistency as well as the other three axioms above and a structural axiom if and only if choices are described by an EU-PPE representation in which the criteria used to select from among PE lotteries is consistent with the CPE ranking. Theorem 2 shows the implications for the EU-PPE representation of replacing Weak Reference Bias with a stronger axiom, Strong Reference Bias, that is directly motivated by previous experimental findings.

Following my main results, I give a definition of what it means to exhibit expectations-based reference-dependence in the EU-PPE model based on violations of the Independence Axiom. I show that WARP violations are a sufficient but not necessary condition for choices to exhibit expectations-based reference-dependence in the EU-PPE model.

The exercise in this paper significantly departs from past axiomatic work on reference-dependent choice (e.g. Sugden 2003; Masatlioglu and Ok 2005; Sagi 2006; Apesteguia and Ballester 2009) in that I assume that reference points are not directly observed – I instead infer them from choice. Kőszegi (2010) introduces the less restrictive personal equilibrium (PE) model, of which PPE is a refinement, and provides some observable implications of this model on a rich domain, but does not provide a "revealed-preference foundation".[2] On

[2]Kőszegi's three observable implications the PE model are dynamic consistency, intrinsic informational preferences, and the existence of multiple stable equilibria. My more restrictive domain rules out instrinsic

the simpler domain of choice from arbitrary sets of alternatives, Gul and Pesendorfer (2008) show that if the PE model is not further restricted, it is observationally equivalent to choice that maximizes a (possibly non-transitive) binary relation. A predecessor of this paper (Freeman, 2013) analogously shows that without further restrictions, the PPE model is observationally equivalent to a model of two-stage of maximization in which the second (but not necessarily the first) binary relation applied is complete and transitive. A different exercise by Masatlioglu and Raymond (2015) studies the behavioral implications for choice under risk of a particular class of functional forms used by Kőszegi and Rabin under CPE, but does not discuss behavior under PE or PPE. Relative to other work that is broadly related to endogenous reference-dependence,[3] the current paper is unique in studying the relationship between failures of WARP, failures of the Independence Axiom, and endogenous reference-dependence in the EU-PPE model.

2 The PPE model

2.1 Personal equilibrium for expectations-based reference-dependence

Let X denote a finite set of alternatives, and let Δ denote the set of all lotteries over X. No other structure on X is assumed. Let $\mathcal{D}$ denote the set of all non-empty finite subsets of Δ; a typical $D \in \mathcal{D}$ is called a choice set.

In the models I consider, a DM has a set of reference-dependent utility functions, which can be captured by a mapping $v : \Delta \times \Delta \to \mathbb{R}$, where $v(\cdot|p)$ defines the DM's utility function when given a reference point p. Kőszegi and Rabin (2006) introduce the PPE for modeling the endogenous determination of expectations-based reference points for a DM with a reference-dependent utility function v. Given reference-dependent utility function v, the set of PPE in a given choice set D is given by:

$$PPE_v(D) = \arg\max_{p \in PE_v(D)} v(p|p) \tag{2.1}$$

where

informational preferences, while the existence of multiple stable equilibria is a feature of the less restrictive PE concept that is eliminated in its PPE refinement. Dynamic inconsistency arises in my setting as a deviation of PPE choices from CPE preferences.

[3]Include in this comparison models of disappointment aversion under risk (Loomes and Sugden, 1986; Gul, 1991; Delquié and Cillo, 2006), Masatlioglu and Raymond's (2015) discussion of the Kőszegi-Rabin functional form under CPE, and Ok et al.'s (2015) model of the attraction effect.

$$PE_v(D) = \{p \in D : \ v(p|p) \geq v(q|p) \ \forall q \in D\} \tag{2.2}$$

denotes the set of PE for D.

The PE concept imposes consistency between expectations and choice, and can be interpreted as follows. When choosing from a choice set D, a DM evaluates lotteries by her reference-dependent utility function $v(\cdot|p)$ given her reference point (p), and chooses a lottery in $\arg\max_{q \in D} v(q|p)$. When forming expectations, the DM recognizes that her expected choice p will be the reference point that applies when she chooses from D. Thus, she would only expect a $p \in D$ if it would be chosen by the reference-dependent utility function $v(\cdot|p)$, that is, if $p \in \arg\max_{q \in D} v(q|p)$.

At the time of forming her expectations, a DM might evaluate the lottery p according to $v(p|p)$, which reflects that she will evaluate the p relative to itself as the reference point. The PPE concept is a natural refinement of the set of PE, based on a DM choosing her best PE expectation according to $v(p|p)$. If reference-dependence tends to bias a DM towards her reference point, multiplicity of PE will be natural, motivating the PPE refinement.

In addition to PE and PPE, Kőszegi and Rabin (2007) note that the criteria used in the PPE concept to select among PE might be appropriate for situations in which a DM commits to her choice at the time of forming expectations, and thus is not restricted to choose from among PE alternatives. To that end, they offer CPE, where choices maximize this criterion:

$$CPE_v(D) = \arg\max_{p \in D} v(p|p) \tag{2.3}$$

2.2 Domains for studying PPE under risk

Kőszegi and Rabin (2007) emphasize that the CPE concept is appropriate for decisions in which the DM commits to her choice long before the relevant risk resolves. Based on this interpretation, a DM's CPE ranking is naturally taken as observable from such choices. In contrast, the PPE concept naturally applies when a DM anticipates her choice set, but cannot commit to her choice before making it. Thus a PPE choice correspondence is naturally taken as observable as well, but from a different set of choice situations.

The analysis that follows assumes that I observe a pair of objects: DM's CPE ranking of lotteries, $\succsim^\star$, and DM's choice correspondence $c : \mathcal{D} \rightrightarrows \Delta$ defined on finite sets of lotteries. Conceptually speaking, imagine two types of experiments with two periods, $t = 1$ and $t = 2$, in which a DM chooses among lotteries that resolve and pay out at $t = 2$. The binary relation $\succsim^\star$ on Δ captures how the subject would rank lotteries at $t = 1$ that she will receive at $t = 2$ – i.e. when she commits to her choice. The choice correspondence c captures how the subject chooses among lotteries when she is presented with a choice set at $t = 1$ and chooses from that choice set at $t = 2$, when she has no ability to commit to her choice at $t = 1$.

For notational ease, for any two lotteries p and q and any $\alpha \in [0, 1]$, let $p\alpha q$ denote the lottery $(1 - \alpha)p + \alpha q$, and for any two choice sets D and D', let $D\alpha D' := \{p\alpha q : p \in D,\ q \in D'\}$ to denote mixtures of choice sets. I will occassionally use the notation $\langle x_1, p_1; \ldots; x_n, p_n\rangle$ to denote the lottery in Δ that assigns probabilities $p_1, \ldots, p_n$ to outcomes $x_1, \ldots, x_n$ respectively.

2.3 EU-PPE representation

The definition of the PPE model has no content specific to choice under risk. Below, I define an EU-PPE representation that incorporates additional restrictions. One restriction we might impose under risk is that preferences given a reference point have an expected utility representation.

Definition. A function $f : \Delta \to \mathbb{R}$ is an *expected utility function* (EU function) if there exists a vector $u \in \mathbb{R}^{|X|}$ such that $f(p) = p \cdot u$ for each $p \in \Delta$.

An additional restriction is that there is at least one prize $x^\star \in X$ that is always deemed most desirable at every reference point.

Definition. A representation on Δ with reference-dependent utility function v *is increasing in* $x^\star$ if $v(\langle x^\star, 1\rangle | p) > v(q|p)$ for all $p \in \Delta$ and for all $q \in \Delta \backslash \{\langle x^\star, 1\rangle\}$.

I define an EU-PPE representation as a PPE representation with a jointly continuous v for which each $v(\cdot|p)$ is an EU function.

Definition. A choice correspondence $c : \mathcal{D} \rightrightarrows \Delta$ has an *EU-PPE representation* if there exists a jointly continuous $v : \Delta \times \Delta \to \mathbb{R}$, with $v(\cdot|p)$ being an expected utility function for each $p \in \Delta$, such that $c(D) = PPE_v(D)$, where PPE_v is defined in (2.1). The EU-PPE representation given by v is *consistent with* $\succsim^\star$ if $v(p|p) \geq v(q|q)$ holds if and only if $p \succsim^\star q$.

Kőszegi and Rabin (2006) adopt a particular functional form for v. They assume that a DM ranks lotteries over outcomes, and that given probabilistic expectations summarized by the lottery p, a DM ranks a lottery q according to:

$$v^{KR}(q|p) = \sum_k \sum_i q_i u^k(x_i^k) + \sum_k \sum_i \sum_j q_i p_j \mu\left(u^k(x_i^k) - u^k(x_j^k)\right) \tag{2.4}$$

where q_i denotes the probability of receiving outcome x_i in lottery q and x_i^k denotes the level of hedonic attribute k possessed by outcome x_i. In (2.4), u^k is a consumption utility function in "hedonic dimension" k; different hedonic dimensions are akin to different goods in a consumption bundle, but specified based on "psychological principles." The function μ is a gain-loss utility function which captures reference-dependent outcome evaluations, and in most existing applications has taken the linear loss averse form:

$$\mu(z) = \begin{cases} \eta z & \text{if } z \geq 0 \\ \eta(1+\lambda)z & \text{if } z < 0 \end{cases} \tag{2.5}$$

where $\eta \geq 0$ indicates the relative weight places on placed on gain-loss comparisons, and $\lambda \geq 0$ captures the degree of loss aversion.

Definition. v takes the *linear loss averse Kőszegi-Rabin* functional form if v is given by (2.4) and (2.5) for some set of hedonic dimensions $\{1, \ldots, K\}$, dimension-by-dimension utility functions $u^k : X \to \mathbb{R}$ for each k, and for some $\eta \geq 0$ and $\lambda \geq 0$.

The linear loss averse Kőszegi-Rabin form combined with the PPE concept is a special case of the EU-PPE model which adds only two parameters to a standard model of utility maximization and has been particaularly amenable to applications since the model's predictions are pinned down by (2.4), (2.5), (2.1), and (2.2). However, previous studies of the Kőszegi-Rabin model's behavior have mostly been limited to specific applications using the linear loss averse form. Next, I show that PPE under this linear loss averse Kőszegi-Rabin functional form can generate violations of two standard axioms for choice under risk, the Independence Axiom and WARP.

2.4 Anomalies in the EU-PPE model

2.4.1 Expectations-based endowment effect and the Independence Axiom

I adapt the Independence Axiom to choice correspondences below.

Independence Axiom. For any $\alpha \in (0, 1)$ and $D, D' \in \mathcal{D}$, $c(D\alpha D') = c(D)\alpha c(D')$.

Suppose we interpret a choice from the set $D\alpha D'$ as involving a conditional choice from D and another conditional choice from D'. The Independence Axiom requires that the conditional choice from D does not depend on what lotteries are available in D' or on the DM's conditional choice from D'.

Consider the following specification of the EU-PPE model with a v that takes the linear loss averse Kőszegi-Rabin form. Let $X = \{0, 1\} \times \{0, 1\}$ where (x^1, x^2) denotes an allocation of x^1 mugs and x^2 pens. Use μ from (2.5) with $\eta = 1$ and $\lambda = 2$, and suppose $u^1(x^1) = 5x^1$, $u^2(x^2) = 8x^2$.[4] The EU-PPE model with the v as specified will generate the choice behavior:

$$c(\{\langle(0,1),1\rangle,\langle(1,0),1\rangle\}) = \{\langle(0,1),1\rangle\} \tag{2.6}$$

$$c(\{\langle(1,0),.1;(0,1),.9\rangle,\langle(1,0),1\rangle\}) = \{\langle(1,0),.1;(0,1),.9\rangle\} \tag{2.7}$$

$$c(\{\langle(1,0),.9;(0,1),.1\rangle,\langle(1,0),1\rangle\}) = \{\langle(1,0),1\rangle\} \tag{2.8}$$

The comparison between either (2.6) or (2.7) to (2.8) reveals a violation of the Independence Axiom. This behavior appears to be an intuitive instance of expectations-based reference-dependence. In the choice in (2.8), the person expects to get a mug with at least a 90% chance, regardless of her choice, and thus her reference point ought to incorporate this expectation, and bias her towards mugs. But in the choice in (2.6), a person should expect to end up with a mug if and only if she expects she will subsequently choose it. Since this person's loss aversion makes her biased towards her reference point (and thus both options are PE), when forming expectations, the subject ought to pick her CPE-preferred of the two options – the pen. There is evidence that this reversal occurs: the choices described in (2.7) and (2.8) correspond to median behavior in a between-subject experimental study by Ericson and Fuster (2011).[5]

2.4.2 Expectations-based endowment effect and WARP

A version of WARP, below, is satisfied whenever c can be rationalized by a complete and transitive preference relation.

WARP. If $q \in D$, $p \in c(D)$, $p \in D'$, and $q \in c(D')$, then $p \in c(D')$.

The following example, based on Heidhues and Kőszegi (2014), shows that choices may not satisfy WARP in the EU-PPE model. Consider a DM with a Kőszegi-Rabin linear loss averse form for v, as in (2.4) and (2.5), with linear utility function u^1 over monetary outcomes, who intrinsically values a pair of shoes at 100 (captured by u^2), and who exhibits linear loss aversion with $\eta = 1$, $\lambda = 2$, as in

[4] Linear loss aversion is used in most applications of Kőszegi-Rabin, and the choices of η and λ are broadly within the range implied by experimental studies.

[5] A large number of studies involve both real effort and monetary payments and have been taken as evidence of expectations-based reference-dependence - notably, Abeler et al. (2011); Gill and Prowse (2012); Camerer et al. (1997); Crawford and Meng (2011). The choices reported in these papers fit naturally with the intuition of expectations-based loss aversion, but are difficult to reconcile with standard models.

$$u^1(z)=z,\ u^2(z)=\begin{cases}0 & \text{if } z=0\\ 100 & \text{if } z=1\end{cases},\quad \mu(z)=\begin{cases}z & \text{if } z\geq 0\\ 3z & \text{if } z<0\end{cases}.$$

A shoe store has a regular price for shoes of \$120, but 20% of the time it puts the shoes on sale for \$40, and the timing of sales are random from the perspective of the DM. On average, the price of shoes is \$104, which exceeds the DM's intrinsic valuation.

When forming expectations about when she would buy the shoes, the DM can plan to 'always buy', to 'buy only when on sale', or to 'never buy' (ignoring the possibility to 'only buy at the regular price'). These options respectively correspond to the lotteries of money-shoe outcomes $a=\langle(-40,1),.20;(-120,1),.80\rangle$, $s=\langle(-40,1),.20;(0,0),.80\rangle$, and $n=\langle(0,0),1\rangle$.

The sale price is sufficiently attractive that $v(s|n)>v(n|n)$. Thus n is not a PE. We can further show that $\{a,s\}=PE_v(\{a,s,n\})$ – because of the DM's attachment to her expectations, 'always buy' as well as 'buy only when on sale' are both PE. This DM's loss aversion is sufficiently strong that she would rather expect to pay a high price on average but always get the shoes rather than only ever pay the lower sale price but expect to experience a partial 'loss' of shoes when the regular price prevails. That is, $\{a\}=PPE_v(\{a,s,n\})$ – this DM's PPE plan is to buy the shoes regardless of whether there is a sale. However, were this DM to only have the options of always buying or never buying, she would never buy given the high average price, that is, $\{n\}=PPE_v(\{a,n\})$. This illustrates how an expectations-based attachment effect in the Kőszegi-Rabin model can generate a violation of WARP in choices that involve risk.

3 Axioms for the EU-PPE model

3.1 Preliminaries

Define distance on lotteries using the Euclidean distance metric, $d^E(p,q):=\left[\sum_i(p_i-q_i)^2\right]^{\frac{1}{2}}$, and the distance between choice sets using the Hausdorff metric, $d^H(D,D'):=\max\left(\max_{p\in D}\left[\min_{q\in D'}d^E(p,q)\right],\max_{q\in D'}\left[\min_{p\in D}d^E(p,q)\right]\right)$. For any set T with a typical element t, let $\{t^\epsilon\}$ denote a *convergent net* indexed by a set $(0,\bar{\epsilon}]$ and with limit point t; t^ϵ will be used to denote the ϵ term in the net.[6]

[6] A *net* in a set T is a function $t:S\to T$ for some directed set S (Aliprantis and Border, 1999).

3.2 Axioms for PPE

The first two axioms require that the CPE ranking, $\succsim^\star$, is a complete, transitive, and continuous binary relation.

A1. Weak Order. $\succsim^\star$ is a complete and transitive binary relation.

A2. $\succsim^\star$ **Continuity.** The sets $\{p \in \Delta : p \succsim^\star q\}$ and $\{p \in \Delta : q \succsim^\star p\}$ are closed for each $q \in \Delta$.

Consider next the Non-Intrinsic Preference for Commitment (A3), which relates CPE preferences to choices without commitment.

A3. Non-Intrinsic Preference for Commitment. $p \in c(D)$, $p \in D' \subseteq D$, and $q \in c(D')$ implies that $q \succsim^\star p$.

A3 allows the agent to value smaller choice sets more than larger ones according to her CPE preferences, but only because adding new items to her choice set can affect her choices. The CPE ranking views a lottery as inducing both a choice and an expectations-based reference point. However, when the DM cannot commit to her choice of lottery in advance, an expectations-based reference point will influence the DM's future preferences. This in turn can induce a constraint on the DM's rational expectations when initially evaluating a choice set since her reference-dependent preferences given a reference point may disagree with her CPE ranking. This constraint enters when there exist lotteries in a choice set for which these rankings disagree. There can be more such disagreements in a larger choice set, motivating the axiom below.

3.3 Axioms for Risk

As in my interpretation of the Independence Axiom, environments with risk can enable a partial separation between expectations and choice. Suppose we view the mixture $\{q\}\alpha D$ as arising from a lottery over choice sets that gives the singleton choice set $\{q\}$ with probability $1-\alpha$ and gives choice set D with probability α. Under this interpretation, fraction $1-\alpha$ of expectations are fixed at expecting q and we also observe the DM's conditional choice from D. The axioms below make use of variations on this interpretation. Consider first the Conditional Exclusion Consistency axiom.

A4*. Conditional Exclusion Consistency. If $p \in c(D)$, $p \notin c(D \cup \{p\alpha q\}) \ni r \precsim^\star p$, $\alpha' \in (0,1]$, and $p\alpha' q \in D'$, then $p \notin c(D')$.

A4* is a substantive axiom that disciplines and relates both violations of WARP and violations of the Independence Axiom. As in a standard interpretation of the Independence Axiom, view the choice between $p\alpha p$ and $p\alpha q$ as involving a conditional choice between p and q which is implemented with probability α. Suppose we observe that p is chosen in a choice set D, but when some mixture between p and q is added to D, p is not chosen but an element that is weakly worse than p according to $\succsim^\star$ is chosen. Since the DM would have liked to maximize $\succsim^\star$ when she initially formed expectations about her future choices, then we infer that if were she to expect to choose p and thus have p as her reference point, she would then pick q over p if she faced that conditional choice, which would violate her rational expectations constraint. A4* then imposes that this implies that p is never chosen when any mixture of p and q is available.

I illustrate the content of A4 with the following example. Suppose $p = \langle \$1000,\ 1\rangle$, $q = \langle \$0,\ .5;\ \$2000,\ .25;\ \$4100,\ .25\rangle$, and $r = \langle \$0,\ .5;\ \$2900,\ .5\rangle$, $\{p\} = c(\{p,\ r\})$ and $\{r\} = c(\{p,\ q,\ r\})$. If A3 holds, then $p \succsim^\star r$. Thus the A4 imposes that p can never be chosen whenever any mixture $p\alpha q$ is available.

The following choices that extend the preceding example would violate this restriction. Suppose that a DM is able to exert limited self-control against her temptation to conditionally choose q over p when expecting p. Then, she might then pick $\{p\} = c(\{p,\ (1-\alpha)p + \alpha q\})$ if α is small. However, since $\{p\} = c(\{p,\ r\})$ and $\{r\} = c(\{p,\ q,\ r\})$, this would be inconsistent with A4*.

Consider next the Continuous Conditional Exclusion Consistency axiom, which restricts behavior when c fails to be upper hemicontinuous but, given A4*, places no further restriction on behavior.

A4. Continuous Conditional Exclusion Consistency. If $\{p^\epsilon\}$ and $\{D^\epsilon\}$ are, respectively, nets in Δ and $\mathcal{D}$ with limits p and D, and if $p^\epsilon \in c(D^\epsilon)$ for each ϵ and $p\alpha q \in D$ for some $\alpha \in (0, 1]$, then there is no triple of $D' \in \mathcal{D}$, $\alpha' \in (0, 1]$, and $r \in \Delta$ for which $p \in c(D')$ but $p \notin c(D' \cup \{p\alpha' q\}) \ni r \precsim^\star p$.

A4 imposes a form of continuity on c by restricting when the first two antecedent conditions in A4* ($p \in c(D)$ and $p \notin c(D \cup \{p\alpha q\}) \ni r \precsim^\star p$) can jointly hold. The interpretation is as follows. If, around a choice set D, for any small ϵ, we can find a p^ϵ close to p is conditionally chosen over a q^ϵ in the set D^ϵ (which is close to D), then in the limit D, p must be choosable over q as well if it is desirable according to $\succsim^\star$. Thus, adding any mixture between p and q to an arbitrary choice set in which p is chosen should never prevent p from being chosen over $\succsim^\star$-worse options.

In addition to imposing a notion of continuity on c, A4 also implies A4*. To see this, look at the implication of A4 when we consider $\{p^\epsilon\}$ and $\{D^\epsilon\}$ with $p^\epsilon = p$ and $D^\epsilon = D$,

$p\alpha q \in D$, and $p \in c(D)$ in the antecedent. A4 then implies that that there are no D', r, and α' with $p \in c(D')$ but $p \notin c(D' \cup \{p\alpha' q\}) \ni r \precsim^{\star} p$. This restriction is the contapositive of the restriction in A4*, thus A4 implies A4*.

The Best Prize Dominance axiom imposes a structural assumption on X.

A5. Best Prize Dominance. There exists a $x^{\star} \in X$ such that for all $p \in \Delta$ and all $\alpha \in (0, 1]$, we have $\{p\alpha \langle x^{\star}, 1\rangle\} = c(\{p\}\alpha\{q, \langle x^{\star}, 1\rangle\})$.

A5 requires that the set of alternatives X contains a single best prize $x^{\star}$, and that mixing any lottery p with the lottery giving $x^{\star}$ for sure leads to the mixture always being chosen over the mixture between p and any other lottery in a pairwise choice.

Weak Reference Bias is a substantive but weak axiom.

A6. Weak Reference Bias. $p \in c(D)$ and $q \in D$ implies that $\liminf_{\epsilon \to 0} [\alpha_\epsilon : p\epsilon (p\alpha_\epsilon \langle x^{\star}, 1\rangle) \in c(\{p\}\epsilon\{p\alpha_\epsilon \langle x^{\star}, 1\rangle, q\})] = 0$.

The A6 axiom requires that if $p \in c(D)$ and $q \in D$, then either c chooses p over the mixture $p\epsilon q$ in a pairwise choice for small ϵ, or the α_ϵ required to make p more attractive by mixing it with the best prize $x^{\star}$, as in $p\alpha_\epsilon \langle x^{\star}, 1\rangle$, gets arbitrarily small as ϵ gets small. This is sensible because when c chooses p over q in some choice set, p must be weakly preferred to q when the reference point is p; thus if we manipulate the reference point towards p by looking at a choice set $p\epsilon\{p, q\}$, then at least in the limit ($\epsilon \to 0$) no improvement in p would be needed to make p choosable over q in the ϵ chance the DM receives the chance to make this conditional choice, having previously been expecting to end up with p with a very high probability.

The use of conditional choices under risk to achieve a partial separation between expectations and choices potentially provides us with a way of extracting information about reference-dependent preferences. If we see a DM face the choice set $p\alpha\{q, r\}$ for small α and observe her choosing $p\alpha q$, then we might reasonably infer that she weakly prefers q over r when her expectations are held fixed at p. If reference-dependent preferences are sufficiently continuous and intransitivities in choice only arise due to the endogenous formation of reference points at the level of the choice set, then intransitivies in choice disappear as α gets smaller. Transitive Limit provides an assumption motivated by this argument under which it is possible to elicit reference-dependent utility functions.

Say that q is a *weak conditional choice over r given p, $q\bar{R}_p r$,* if there exist nets $\{p^\epsilon\}, \{q^\epsilon\}, \{r^\epsilon\}$ converging to p, q, r (respectively) such that $p^\epsilon \epsilon q^\epsilon \in c(\{p^\epsilon\}\epsilon\{q^\epsilon, r^\epsilon\})$ for each ϵ. A conditional choice involves a choice between q and r when expectations are close to p.[7]

[7] One might wonder about why I use this particular definition of $\bar{R}_p$ in the axiom. Suppose I had instead defined $\tilde{R}_p$ by $q\tilde{R}_p r$ whenever $p\alpha q \in c(\{p\}\alpha\{q, r\})$ for all small α. Now consider the following example.

Table 1: Examples for Transitive Limit

D	$c(D)$	$\hat{c}(D)$
$.9\{p\} + .1\{p, q\}$	$\{p\}$	$\{p\}$
$.9\{p\} + .1\{p, r\}$	$\{.9p + .1r\}$	$\{.9p + .1r\}$
$.9\{p\} + .1\{q, r\}$	$\{.9p + .1q\}$	$\{.9p + .1r\}$

A7. Transitive Limit. If $q\bar{R}_p r$ and $r\bar{R}_p s$, then $q\bar{R}_p s$.

If WARP violations are only driven by the behavioral influence of expectations and their endogenous determination, then the DM's behavior should be consistent with the standard model when her expectations are fixed. A7 requires that conditional choice behavior looks like the standard model when expectations are almost fixed.

As with continuity axioms, A7 is not exactly testable. However, the axiom is approximately testable. The choice sets in Table 1 provide an approximate test of A7; $\hat{c}$ is consistent with what we would expect if the choice correspondence satisfies Transitive Limit. However, the choice pattern displayed by c is approximately inconsistent with A7, and suggests that c would violate this axiom.

3.4 EU-PPE Representation Theorem

Theorem 1. *The pair $(\succsim^\star, c)$ satisfies A1-A7 if and only if $(\succsim^\star, c)$ has an EU-PPE representation consistent with $\succsim^\star$ which is increasing in $x^\star$.*

The v obtained in Theorem 1 is unique in a meaningful sense. Corollary 1 clarifies that a continuous EU-PPE representation is unique in the sense that any v, $\hat{v}$ that represent the same c must represent the same reference-dependent preferences and the same CPE preference.[8]

Corollary 1. *Given a continuous EU-PPE representation v for $(\succsim^\star, c)$, any other continuous EU-PPE representation $\hat{v}$ for $(\succsim^\star, c)$ satisfies $\hat{v}(q|p) \geq \hat{v}(r|p)$ if and only if $v(q|p) \geq v(r|p)$, and $\hat{v}(p|p) \geq \hat{v}(q|q)$ if and only if $v(p|p) \geq v(q|q)$.*

In the representation in Theorem 1, any p chosen in D is (i) an element of D, and (ii) is in $\arg\max_{q \in D} v(\cdot|p)$. An alternative representation might have the DM's reference lottery

Suppose $|X| = 2$ and v is given by $v(p|r) = p_1 r_1 + p_2 r_3$ for any $p, r \in \Delta$. Now let $p = [.5, .5]$, $q = [1, 0]$, and $r = [.75, .25]$. Then for any $\alpha \in (0, 1)$, $\{p\alpha q\} = c(\{p\}\alpha\{q, r\})$. Then, $q\tilde{R}_p r$ but not $r\tilde{R}_p q$. However, if we define $p^\epsilon = \left[.5 + \frac{2}{\max[\frac{1}{\epsilon}-1, 5]}, .5 - \frac{2}{\max[\frac{1}{\epsilon}-1, 5]}\right]$, we can see that $\{p^\epsilon \epsilon r\} = c(\{p^\epsilon\}\epsilon\{q, r\})$, which establishes that $r\bar{R}_p q$. This example illustrates why the use of $\bar{R}_p$ is appropriate here.

[8] A stronger uniqueness result is possible, since (i) each $v(\cdot|p)$ satisfies expected utility and thus has an affinely unique representation, (ii) joint continuity of v in the representation restricts the allowable class of transformations of v.

involve a randomization among elements in D, or perhaps only elements in $c(D)$. However, Theorem 1 proves that if c satisfies the axioms it has a representation in which it is *as-if* the DM never views herself as randomizing among the elements of D. As such, an EU-PPE representation must have a v for which the set of PE is non-empty – that is, $PE_v(D) \neq \emptyset$ for every $D \in \mathcal{D}$. It is easy to see that this holds if and only if v satisfies the following restriction on v.[9]

Definition. The function $v : \Delta \times \Delta \to \mathbb{R}$ satisfies the *limited cycle inequalities* if for any $p^0, p^1, ..., p^n \in \Delta$, $v(p^i|p^{i-1}) > v(p^{i-1}|p^{i-1})$ for $i = 1, ..., n$, then $v(p^n|p^n) \geq v(p^0|p^n)$.

It can be shown that not all v satisfy the limited cycle inequalities, but linear loss averse Kőszegi-Rabin form for v does.

Proposition 1. *If v takes the Kőszegi-Rabin linear loss averse functional form, then v satisfies the limited cycle inequalities.*

Not all Kőszegi-Rabin preferences consistent with (2.4) satisfy these assumptions. For example, revisit the mugs and pens example in section 2.4.1, but instead assume $\lambda = -1$. This representation violates the loss aversion ($\lambda \geq 0$) assumption in the definition of the linear loss aversion case, but is still a special case of (2.4). Under this specification, the limited cycle inequalities are violated, and as a result, the choice set $\{\langle (0,1), 1\rangle, \langle (1,0), 1\rangle\}$ contains no PE.

3.5 A strong reference bias axiom and representation

The intuition behind expectations-based reference-dependence and supporting evidence ni Ericson and Fuster (2011) motivate a strengthening of A6. Consider a person who would choose a pen over a mug in a straight choice as in (2.6), and who is biased towards an expectations-based reference point in each choice she faces. Were she instead to face a choice in which she receives a mug with a 90% chance and receives her conditional choice between a mug and a pen with a 10% chance, perhaps she might be biased towards the mug and conditionally choose the mug, as in (2.8). Now consider a different person who instead would choose a mug over a pen in a straight choice, and who is also biased towards an expectations-based reference point in each choice. If this person were to receive a mug with a 90% chance and receive her conditional choice between a mug and a pen otherwise,

[9] Closely related properties have been discussed in Sugden (2003), Munro and Sugden (2003), and Apesteguia and Ballester (2009). The limited cycle inequalities impose a significantly weaker restriction on reference-dependent utility functions than Sagi's (2006) "no-regret" condition and than Masatlioglu and Ok's (2014) "Weak Axiom of Status Quo Bias".

she ought to be at least as inclined towards the mug as in a straight choice, and thus ought to choose the mug. This intuition allows the violation of the Independence Axiom captured by (2.6) and (2.8), but it also rules out the opposite reversal. The Strong Reference Bias axiom generalizes this intuition by requiring that if p is chosen over q, then p ought to be chosen in the choice set $\{p\}\alpha\{p,q\}$.

A6'. Strong Reference Bias. If $p \in c(\{p,q\})$ then $p \in c(\{p\}\alpha\{p,q\})$ for any $\alpha \in (0,1)$.

A natural question is what type of restriction on the representation is implied when A6 is replaced by A6'. Consider the following restriction on a reference-dependent utility function.

Definition. A reference-dependent utility function $v : \Delta \times \Delta \to \mathbb{R}$ *PPE-dislikes mixtures* if for any lotteries $p, q \in \Delta$, $v(p|p) \geq v(q|p)$ and either $v(q|q) \leq v(p|p)$ or $v(q|q) < v(p|q)$ implies that either $v(p\alpha q|p\alpha q) \leq v(p|p)$ or $v(p\alpha q|p\alpha q) < v(p|p\alpha q)$ for all $\alpha \in (0,1)$.

Theorem 2 shows that if A6 is replaced by A6', then the conclusions of Theorem 1 hold with the modification that the representation must PPE-dislike mixtures.

Theorem 2. *The pair* $(\succsim^\star, c)$ *satisfies A1-A5, A6', and A7 if and only if* $(\succsim^\star, c)$ *has an EU-PPE representation consistent with* $\succsim^\star$ *which is increasing in* $x^\star$ *and PPE-dislikes mixtures.*

Proposition 2 demonstrates that the linear loss averse Kőszegi-Rabin form given in (2.5) PPE-dislikes mixtures.

Proposition 2. *Kőszegi-Rabin preferences that satisfy linear loss aversion have an EU-PPE representation that PPE-dislikes mixtures.*

4 A behavioral definition exhibiting of expectations-based reference-dependence

I offer a choice-based definition of expectations-based reference-dependence based on violations of the Independence Axiom.

Definition. *c exhibits reference-dependence* at α, p, q, r, s for $\alpha \in (0,1)$ and $p, q, r, s \in \Delta$ if $p\alpha r \in c(\{p\}\alpha\{r,s\})$ but $q\alpha r \notin c(\{q\}\alpha\{r,s\})$. *c strictly exhibits reference-dependence* at α, p, q, r, s for $\alpha \in (0,1)$ and $p, q, r, s \in \Delta$ if $p\alpha r \in c(\{p\}\alpha\{r,s\})$ but, for some $\bar{\epsilon} > 0$ and any $r' = r\alpha' \langle x^\star, 1\rangle$ for $\alpha' \in [0, \bar{\epsilon})$, $q\alpha r' \notin c(\{q\}\alpha\{r',s\})$.

When $p\alpha r \in c(\{p\}\alpha\{r,s\})$ is interpreted as involving a conditional choice of r from $\{r,s\}$, conditional on fraction $1-\alpha$ of expectations being fixed by p, then observing $q\alpha r \notin$

$c(\{q\}\alpha\{r,s\})$ is interpreted as due to the influence of inducing $1-\alpha$ of expectations towards q instead of towards p. Strict reference-dependence requires that this choice pattern also holds for lotteries close to r but assigning higher probabilities of the best prize, $x^\star$. Revisit the choice in (2.8). If in addition, we observed that $c(\{\langle(0,1),.9;(1,0),.1\rangle,\langle(0,1),1\rangle\}) = \{\langle(0,1),1\rangle\}$, then c would directly exhibit reference-dependence by comparing this choice to (2.8).

Proposition 3 clarifies the link between exhibiting expectations-dependence, properties of an EU-PPE representation, and violations of WARP.

Proposition 3. *(i) Suppose c has an EU-PPE representation that is increasing in $x^\star$. Then c exhibits strict reference-dependence if and only if there exist $p,q\in\Delta$ such that $v(\cdot|p)$ and $v(\cdot|q)$ represent different preferences. (ii) Any c with an EU-PPE representation that violates WARP exhibits reference-dependence.*

The first part of Proposition 3 highlights how reference-dependence in c is captured in an EU-PPE representation, as a dependence of $v(\cdot|p)$ on the reference point p.[10] The second part of Proposition 3 shows that a failure of WARP implies, but is not necessary for, reference-dependence in an EU-PPE represenation. Thus, if we were only to observe the choices in subsection 2.4.2, we could infer that any EU-PPE representation of these choices must exhibit reference-dependence and also have a v such that $v(\cdot|p)$ is not the same for all reference lotteries p.

The mugs and pens example shows how one might study reference-dependence based on the definition. Ericson and Fuster's (2011) data violate the Independence Axiom in a way consistent with expectations-based reference-dependence. If we assume that the choices by Ericson and Fuster's subjects reflect strict preferences, then we would conclude from (2.8) that $v(\langle(1,0),1\rangle\,|\,\langle(1,0),1\rangle) > v(\langle(1,0),.9;(0,1),.1\rangle\,|\,\langle(1,0),1\rangle)$ and thus (since $v(\cdot|\,\langle(1,0),1\rangle)$ is an EU function) that $v(\langle(1,0),1\rangle\,|\,\langle(1,0),1\rangle) > v(\langle(0,1),1\rangle\,|\,\langle(1,0),1\rangle)$, and from (2.7) we would similarly conclude that $v(\langle(0,1),1\rangle\,|\,\langle(1,0),.1;(0,1),.9\rangle) > v(\langle(1,0),1\rangle\,|\,\langle(1,0),.1;(0,1),.9\rangle)$. Then Proposition 3 implies that any EU-PPE representation with these rankings would exhibit strict reference-dependence.

Proposition 4. *Consider an EU-PPE representation with linear loss averse Kőszegi-Rabin preferences. If there exist $x,y,z\in X$ for which $u^1(x^1) > u^1(y^1) > u^1(z^1)$ and $u^k(x^k) =$*

[10]To see why exhibiting strict reference-dependence is needed for the EU-PPE model to have reference-dependent preferences that vary with the reference point, consider the following example. Take an EU function u. Pick v so that $v(q|p) = q\cdot u + \hat{u}(p)$, for some $\hat{u}$ that represents the CPE preference. By construction, each $v(\cdot|p)$ has the same ranking of lotteries. If $\hat{u}$ is a non-expected utility function, then the this EU-PPE representation would exhibit reference-dependence since the CPE ranking may determine the choice when u is indifferent among lotteries. However, this representation would not exhibit strict reference-dependence, since indifference according to u can be broken by perturbing the lottery in question.

$u^k(y^k) = u^k(z^k)$ *for all* $k \neq 1$*, then* c *displays strict reference-dependence if and only if* $\eta\lambda > 0$.

5 Discussion

This paper provided axiomatic foundations for the EU-PPE model of choice under risk. I showed that in this model, violations of Independence Axiom are more fundamental to expectations-based reference-dependence than violations of WARP, in the sense that the latter imply the former, but not vice-versa. However, perhaps the most novel feature of the axioms considered here is that they allow the DM to violate WARP. No direct empirical test of this aspect of the model exists, to my knowledge. Experiments are needed to understand whether a descriptive theory of expectations-based reference-dependence should allow for violations of WARP, and whether WARP violations take the forms allowed by the axioms here.

References

Abeler, J., A. Falk, L. Götte, and D. Huffman, "Reference points and effort provision," *American Economic Review*, 2011, *101* (2), 470–492.

Aliprantis, Charalambos D and Kim C Border, *Infinite dimensional analysis: a hitchhiker's guide*, Springer, 1999.

Apesteguia, Jose and Miguel A Ballester, "A theory of reference-dependent behavior," *Economic Theory*, 2009, *40* (3), 427–455.

Camerer, Colin, Linda Babcock, George Loewenstein, and Richard Thaler, "Labor supply of New York City cabdrivers: One day at a time," *The Quarterly Journal of Economics*, 1997, pp. 407–441.

Crawford, V.P. and J. Meng, "New York City Cab Drivers' Labor Supply Revisited: Reference-Dependent Preferences with Rational Expectations Targets for Hours and Income," *American Economic Review*, 2011, *101* (5), 1912–1932.

Delquié, Philippe and Alessandra Cillo, "Disappointment without prior expectation: a unifying perspective on decision under risk," *Journal of Risk and Uncertainty*, 2006, *33* (3), 197–215.

Eliaz, K. and R. Spiegler, "Reference Dependence and Labor-Market Fluctuations," 2013. Working Paper.

Ericson, K.M.M. and A. Fuster, "Expectations as endowments: Evidence on reference-dependent preferences from exchange and valuation experiments," *Quarterly Journal of Economics*, 2011, *126* (4), 1879–1907.

Fishburn, Peter C, *Utility theory for decision making*, New York: Wiley, 1970.

Freeman, David, "Revealed Preference Implications of Expectations-Based Reference-Dependence," 2013. Working Paper.

Gill, D. and V. Prowse, "A structural analysis of disappointment aversion in a real effort competition," *American Economic Review*, 2012, *102* (1), 469–503.

Gul, F., "A theory of disappointment aversion," *Econometrica*, 1991, *59* (3), 667–686.

_ **and W. Pesendorfer**, "The case for mindless economics," in A. Caplin and A. Schotter, eds., *The Foundations of Positive and Normative Economics: A Handbook*, Oxford University Press New York, 2008, pp. 3–39.

Heidhues, P. and B. Kőszegi, "Competition and price variation when consumers are loss averse," *American Economic Review*, 2008, pp. 1245–1268.

_ **and** _ , "Regular prices and sales," *Theoretical Economics*, 2014, *9* (1), 217–251.

Herweg, Fabian and Konrad Mierendorff, "Uncertain demand, consumer loss aversion, and flat-rate tariffs," *Journal of the European Economic Association*, 2013, *11* (2), 399–432.

_ **, Heiko Karle, and Daniel Müller**, "Incomplete Contracting, Renegotiation, and Expectation-Based Loss Aversion," 2014. Working Paper.

Karle, Heiko, Georg Kirchsteiger, and Martin Peitz, "Loss Aversion and Consumption Choice: Theory and Experimental Evidence," *American Economic Journal: Microeconomics*, 2015, *7* (2), 101–120.

Kőszegi, B., "Utility from anticipation and personal equilibrium," *Economic Theory*, 2010, *44* (3), 415–444.

_ **and M. Rabin**, "A model of reference-dependent preferences," *Quarterly Journal of Economics*, 2006, *121* (4), 1133–1165.

_ **and** _ , "Reference-dependent risk attitudes," *American Economic Review*, 2007, *97* (4), 1047–1073.

Loomes, G. and R. Sugden, "Disappointment and dynamic consistency in choice under uncertainty," *Review of Economic Studies*, 1986, *53* (2), 271–282.

Masatlioglu, Y. and E.A. Ok, "Rational choice with status quo bias," *Journal of Economic Theory*, 2005, *121* (1), 1–29.

_ **and** _ , "A Canonical Model of Choice with Initial Endowments," *Review of Economic Studies*, 2014, *81* (2), 851–883.

Masatlioglu, Yusufcan and Collin Raymond, "A Behavioral Analysis of Stochastic Reference Dependence," 2015. Working Paper.

Munro, A. and R. Sugden, "On the theory of reference-dependent preferences," *Journal of Economic Behavior & Organization*, 2003, *50* (4), 407–428.

Ok, E.A., P. Ortoleva, and G. Riella, "Revealed (p)reference theory," *American Economic Review*, 2015, *105* (1), 299–321.

Pagel, Michaela, "Expectations-based Reference-Dependent Preferences and Asset Pricing," *Journal of the European Economic Association*, 2016, *14* (2), 468–514.

Sagi, J.S., "Anchored preference relations," *Journal of Economic Theory*, 2006, *130* (1), 283–295.

Sugden, R., "Reference-dependent subjective expected utility," *Journal of Economic Theory*, 2003, *111* (2), 172–191.

Appendix

Proof of Theorem 1.

Given any $p \in \Delta$ and $\epsilon > 0$, let $N^{\epsilon}(p)$ denote the open ϵ-ball around p.

Sufficiency.

Lemmas 1-9 use A7 as well as A5 to construct a candidate set of EU functions, one for each reference point, continuous in the reference point. Then Lemmas 10-11 use A1 and A2 and A5 and this set of EU functions to provide a reference-dependent utility function v that is increasing in $x^{\star}$ and consistent with $\succsim^{\star}$. Lemmas 12-19 use this constructed v and its relationship to $\{R_p\}_{p \in \Delta}$ from the preceding lemmas with the remaining axioms to establish that $c(D) = PPE_v(D)$ for all D by first establishing the relation for small steps and then providing induction arguments in Lemmas 18-19.

Lemma 1. *If A7 holds, then $\bar{R}_p$ is complete, transitive, and if there exist convergent nets $\{p^{\epsilon}\}, \{q^{\epsilon}\}, \{r^{\epsilon}\}$ with $q^{\epsilon} \bar{R}_{p^{\epsilon}} r^{\epsilon}$ for each term in the net, then $q \bar{R}_p r$.*

Proof. Transitivity of $\bar{R}_p$ follows by A7.

For any nets $\{p^{\epsilon}\}, \{q^{\epsilon}\}, \{r^{\epsilon}\}$ with limits p, q, and r (respectively), non-emptiness of c implies that the net either has a convergent subnet $p^{\delta}, q^{\delta}, r^{\delta}$ in which $p^{\delta} \delta q^{\delta} \in c(\{p^{\delta}\} \delta \{q^{\delta},\, r^{\delta}\})$ or in which $p^{\delta} \delta r^{\delta} \in c(\{p^{\delta}\} \delta \{q^{\delta},\, r^{\delta}\})$ for each term in the subnet. Thus $\bar{R}_p$ is complete.

Take nets $\{p^{\epsilon}\}, \{q^{\epsilon}\}, \{r^{\epsilon}\}$, for which $q^{\epsilon} \bar{R}_{p^{\epsilon}} r^{\epsilon}$ for each term in the net. By the definition of $\bar{R}_{p^{\epsilon}}$, for each ϵ there are nets $\{p^{\epsilon,\delta}\}_{\delta}, \{q^{\epsilon,\delta}\}_{\delta}, \{r^{\epsilon,\delta}\}_{\delta}$ respectively converging to $p^{\epsilon}, q^{\epsilon}, r^{\epsilon}$ such that $p^{\epsilon,\delta} \delta q^{\epsilon,\delta} \in c(\{p^{\epsilon,\delta}\} \delta \{q^{\epsilon,\delta}, r^{\epsilon,\delta}\})$ for each term in the net. Let $\bar{\delta}_{\epsilon}$ denote the largest element in the index set for $\{p^{\epsilon,\delta}, q^{\epsilon,\delta}, r^{\epsilon,\delta}\}_{\delta}$ and $\bar{\epsilon}$ the largest element in the index set for $\{p^{\epsilon}\}, \{q^{\epsilon}\}, \{r^{\epsilon}\}$. Take $\bar{\delta} := \bar{\delta}_{\bar{\epsilon}}$. For each $\delta < \bar{\delta}$, define ϵ_{δ} as a decreasing net such that for each $\delta < \bar{\delta}_{\epsilon_{\delta}}$. Then define $\{\hat{p}^{\delta}\} = \{p^{\epsilon_{\delta},\delta}\}_{\delta}, \{\hat{q}^{\delta}\} = \{q^{\epsilon_{\delta},\delta}\}_{\delta}, \{\hat{r}^{\delta}\} = \{r^{\epsilon_{\delta},\delta}\}_{\delta}$. By construction, $\{\hat{p}^{\delta}\}, \{\hat{q}^{\delta}\}, \{\hat{r}^{\delta}\}$ establish that $q \bar{R}_p r$. □

Lemma 2. *If $q R_p r$, then $\exists \bar{\epsilon} > 0$ such that $\forall p^{\epsilon} \in N^{\epsilon}(p)$, $q^{\epsilon} \in N^{\epsilon}(q)$, $r^{\epsilon} \in N^{\epsilon}(r)$, we have $q^{\epsilon} R_{p^{\epsilon}} r^{\epsilon}$.*

Proof. Suppose $q R_p r$ and no such $\bar{\epsilon}$ exists. Then there exist nets $\{p^{\epsilon}\}, \{q^{\epsilon}\}, \{r^{\epsilon}\}$ with $r^{\epsilon} \bar{R}_{p^{\epsilon}} q^{\epsilon}$. Then by Lemma 1 it follows that $r \bar{R}_p q$, which contradicts that $q R_p r$ by the definition of R_p. □

Lemma 3. *$\langle x^{\star}, 1 \rangle\, R_p q$ for all $p, q \in \Delta$.*

Proof. By A5, for any $p, q \in \Delta$ and nets $\{p^\epsilon\}, \{q^\epsilon\}$ with limits p and q (respectively), we have $\{(1-\epsilon)p^\epsilon + \epsilon \langle x^\star, 1\rangle\} = c(\{(1-\epsilon)p^\epsilon + \epsilon \langle x^\star, 1\rangle, (1-\epsilon)p^\epsilon + \epsilon q^\epsilon\})$ for each ϵ. Thus by the definition of R_p, we have $\langle x^\star, 1\rangle R_p q$. □

In the text, I defined the Independence Axiom as a property of a choice correspondence; now I define the axiom as a property of a binary relation (as is more standard).

Definition. A binary relation R, say that R *satisfies the Independence Axiom* if $qRr \iff s\alpha q R s \alpha r \, \forall \alpha \in (0,1), \forall s \in \Delta$.

Lemma 4 shows that R_p satisfies the Independence Axiom.

Lemma 4. *If A7 holds, R_p satisfies the Independence Axiom if $p \in int\Delta$.*

Proof. Part I: suppose qR_pr, and take a $\alpha \in (0,1)$ and $s \in \Delta$.

Then, $\exists \bar{\delta}, \bar{\epsilon} > 0$ such that $\forall \epsilon \in (0, \bar{\epsilon})$, $\hat{p}, \hat{q}, \hat{r} \in N_p^{\bar{\delta}} \times N_q^{\bar{\delta}} \times N_r^{\bar{\delta}}$, $\{(1-\epsilon)\hat{p} + \epsilon\hat{q}\} = c((1-\epsilon)\hat{p} + \epsilon\{\hat{q}, \hat{r}\})$.

Define $\bar{\delta}_\alpha = \min\left[\alpha\bar{\delta}, (1-\alpha)\bar{\delta}\right]$.

Let $\hat{p}, \hat{s} \in N_p^{\bar{\delta}_\alpha} \times N_s^{\bar{\delta}_\alpha}$. Since $d^E(p,q) \leq 1$, it follows that $d^E((1-\beta)\hat{p} + \beta\hat{s}, p) \leq (1-\beta)\bar{\delta}_\alpha + \beta$ by the triangle inequality. Thus if $\beta \leq \bar{\beta}_\alpha := \frac{\bar{\delta} - \bar{\delta}_\alpha}{1 - \bar{\delta}_\alpha}$, then $(1-\beta)\hat{p} + \beta\hat{s} \in N_p^{\bar{\delta}}$.

Then for any $\hat{q}, \hat{r} \in N_q^{\bar{\delta}} \times N_r^{\bar{\delta}}$, $\epsilon \in (0, \bar{\epsilon})$, and $\beta \in (0, \bar{\beta})$, $\{(1-\epsilon)((1-\beta)\hat{p} + \beta\hat{s}) + \epsilon\hat{q}\} = c((1-\epsilon)((1-\beta)\hat{p} + \beta\hat{s}) + \epsilon\{\hat{q}, \hat{r}\})$. Define $\hat{\epsilon} := \frac{\epsilon}{\alpha}$ and $\beta^{\epsilon,\alpha} := \frac{\epsilon}{\alpha}\frac{1-\alpha}{1-\epsilon}$. Then $\forall \hat{\epsilon}$ such that $\alpha\hat{\epsilon} \in (0, \bar{\epsilon})$ and $\hat{\epsilon}\frac{1-\alpha}{1-\alpha\hat{\epsilon}} \in (0, \bar{\beta})$, it follows that $\{(1-\hat{\epsilon})\hat{p} + \hat{\epsilon}((1-\alpha)\hat{s} + \alpha\hat{q})\} = c((1-\hat{\epsilon})\hat{p} + \hat{\epsilon}((1-\alpha)\hat{s} + \alpha\{\hat{q}, \hat{r}\}))$. Since $N_{(1-\alpha)s+\alpha q}^{\bar{\delta}_\alpha} \subset (1-\alpha)N_s^{\bar{\delta}} + \alpha N_q^{\bar{\delta}}$ and $N_{(1-\alpha)s+\alpha q}^{\bar{\delta}_\alpha} \subset (1-\alpha)N_s^{\bar{\delta}} + \alpha N_q^{\bar{\delta}}$

It follows that $(1-\alpha)s + \alpha q R_p (1-\alpha)s + \alpha r$.

Part II: suppose $(1-\alpha)s + \alpha q R_p (1-\alpha)s + \alpha r$.

Recall that $N_{(1-\alpha)s+\alpha q}^{\bar{\delta}} \subseteq (1-\alpha)N_s^{\bar{\delta}} + \alpha N_q^{\bar{\delta}}$.

Then, $\exists \bar{\delta}, \bar{\epsilon} > 0$ such that $N_p^{\bar{\delta}} \subset \text{int}\Delta$ and $\forall \epsilon \in (0, \bar{\epsilon})$, $\hat{p}, \hat{q}, \hat{r}, \hat{s} \in N_p^{\bar{\delta}} \times N_q^{\bar{\delta}} \times N_r^{\bar{\delta}} \times N_s^{\bar{\delta}}$, $\{(1-\epsilon)\hat{p} + \epsilon((1-\alpha)\hat{s} + \alpha\hat{q})\} = c((1-\epsilon)\hat{p} + \epsilon((1-\alpha)\hat{s} + \alpha\{\hat{q}, \hat{r}\}))$.

Fix $\kappa \in (0,1)$. Fix $\hat{p}, \hat{q}, \hat{r}, \hat{s} \in N_p^{\kappa\bar{\delta}} \times N_q^{\kappa\bar{\delta}} \times N_r^{\kappa\bar{\delta}} \times N_s^{\kappa\bar{\delta}}$.

Given $\epsilon \in (0, \bar{\epsilon})$, take $\gamma^{\epsilon,\alpha} := \epsilon\frac{1-\alpha}{1-\epsilon}$. If $\gamma < (1-\kappa)\bar{\delta}$, then $\hat{p} + \gamma^{\epsilon,\alpha}(\hat{p} - \hat{s}) \in N_p^{\bar{\delta}} \subseteq \Delta$. Then,

$(1-\epsilon)(\hat{p} + \gamma^{\epsilon,\alpha}(\hat{p} - \hat{s})) + \epsilon((1-\alpha)\hat{s} + \alpha\hat{q}) = c((1-\epsilon)(\hat{p} + \gamma^{\epsilon,\alpha}(\hat{p} - \hat{s})) + \epsilon((1-\alpha)\hat{s} + \alpha\{\hat{q}, \hat{r}\}))$

$\iff (1-\alpha\epsilon)\hat{p} + \alpha\epsilon\{\hat{q}\} = c((1-\alpha\epsilon)\hat{p} + \alpha\epsilon\{\hat{q}, \hat{r}\})$

Since the above holds $\forall \hat{p}, \hat{q}, \hat{r}, \hat{s}, \epsilon \in N_p^{\kappa\bar{\delta}} \times N_q^{\kappa\bar{\delta}} \times N_r^{\kappa\bar{\delta}} \times N_s^{\kappa\bar{\delta}} \times (0, \bar{\epsilon})$ it follows that qR_pr. □

Lemma 5. *If A7 holds, R_p satisfies the Independence Axiom if $p \in \Delta$.*

Proof. Lemma 4 covers the case where $p \in \text{int}\Delta$, so suppose $p \in \Delta \backslash \text{int}\Delta$, and take $q, r \in \Delta$ such that qR_pr. By Lemma 2, $\exists \bar{\epsilon} > 0$ such that $\forall \epsilon < \bar{\epsilon}$ and $\forall p^\epsilon \in N^\epsilon(p)$, $q^\epsilon \in N^\epsilon(q)$, $r^\epsilon \in N^\epsilon(r)$, we have $q^\epsilon R_{p^\epsilon} r^\epsilon$.

Now suppose that for $\alpha \in (0,1)$, $s \in \Delta$, and $p^\epsilon \in N^{\bar{\epsilon}}(p) \backslash \text{int}\Delta$ that $(1-\alpha)s + \alpha r \bar{R}_{p^\epsilon}(1-\alpha)s + \alpha q$. By the definition of $\bar{R}_{p^\epsilon}$, it follows that for some convergent net $\{p^{\epsilon,\delta}, s^\delta, r^\delta, q^\delta, \alpha^\delta\} \rightarrow p^\epsilon, s, r, q, \alpha$, we have for all δ sufficiently small that $(1-\delta)p^{\epsilon,\delta} + \delta(1-\alpha^\delta)s^\delta + \delta\alpha^\delta r^\delta \in c((1-\delta)p^{\epsilon,\delta} + \delta(1-\alpha^\delta)s^\delta + \delta\alpha^\delta\{q^\delta, r^\delta\})$. Since the net $\frac{1-\delta}{1-\delta+\delta(1-\alpha^\delta)}p^{\epsilon,\delta} + \frac{\delta(1-\alpha^\delta)}{1-\delta+\delta(1-\alpha^\delta)}s^\delta$ converges to p^ϵ as $\delta \rightarrow 0$, this also establishes that $r\bar{R}_{p^\epsilon}q$. But since $p^\epsilon \in N^{\bar{\epsilon}}(p)$, this contradicts that $qR_{p^\epsilon}r$. Thus no such net can exist that would establish $(1-\alpha)s + \alpha r\bar{R}_p(1-\alpha)s + \alpha q$, so it follows that $(1-\alpha)s + \alpha q R_p (1-\alpha)s + \alpha r$. □

Lemma 6. *If A7 holds, $\bar{R}_p$ satisfies the Independence Axiom.*

Proof. By Lemma 4, R_p satisfies the Independence Axiom.

Suppose that $q\bar{R}_pr$ and take $(1-\alpha)s + \alpha q$ and $(1-\alpha)s + \alpha r$.

If it is not the case that $(1-\alpha)s + \alpha q \bar{R}_p (1-\alpha)s + \alpha r$, then $(1-\alpha)s + \alpha r R_p (1-\alpha)s + \alpha q$.

Then it follows by Lemma 4 that rR_pq, which contradicts that $q\bar{R}_pr$. □

Lemma 7. *For each $p \in \Delta$, there exists a vector $\hat{u}^p \in \mathbb{R}^N$ such that $q\bar{R}_pr \iff q \cdot \hat{u}^p \geq r \cdot \hat{u}^p$.*

Proof. Lemma 1 shows $\bar{R}_p$ is complete, transitive, and satisfies the Independence Axiom. The joint continuity property on $\bar{R}_p$ in Lemma 1 then implies the notion of mixture continuity required to apply Fishburn's (1970) Theorem 8.2. □

Say that a vector u^p is *flat* if $\max_i u_i^p = \min_i u_i^p$. Let $F := \{p \in \Delta : u^p \text{ is flat}\}$.

Lemma 8. *Suppose u^p is not flat. Then, there is an ϵ neighborhood N_p^ϵ of p such that $\forall \hat{p} \in N_p^\epsilon$, $u^{\hat{p}}$ is not flat.*

Proof. Suppose there is a net $\{\hat{p}^\epsilon\}$ such that $\hat{p}^\epsilon \in N_p^\epsilon$ and $u^{\hat{p}^\epsilon}$ is flat. Since $u^{\hat{p}^\epsilon}$ must represent $\bar{R}_{\hat{p}^\epsilon}$, it follows that $q\bar{R}_{\hat{p}^\epsilon}r$ $\forall q, r \in \Delta$ and for each $\hat{p}^\epsilon$. By Lemma 1, it follows that $q\bar{R}_pr$. It follows that u^p must be flat as well, a contradiction. □

Let $\hat{u}^p$ denote a vector that provides an EU representation for $\bar{R}_p$ (i.e. $q \cdot \hat{u}^p \geq r \cdot \hat{u}^p \iff q\bar{R}_pr$ $\forall q, r \in \Delta$). For all p such that p is non-flat, define:

$$u^p := \frac{\min\left[d^H(\{p\}, F), 1\right]}{\max_i \left[\hat{u}_i^p - \sum_j \hat{u}_j^p\right]} \left(\hat{u}^p - \sum_j \hat{u}_j^p\right) \tag{5.1}$$

If $\hat{u}^p$ is flat, define u^p as the zero vector.

By Lemma 7 and the EU theorem, u^p provides an EU representation for $\bar{R}_p$.

Lemma 9. *If $p^\epsilon \to p$, then $u^{p^\epsilon} \to u^p$.*

Proof. If u^p is flat, then $d^H(\{p^\epsilon\}, F) \to 0$ as $\epsilon \to 0$, thus $u^{p^\epsilon} \to u^p$.

Now suppose that u^p is non-flat. Suppose $p^\epsilon \to p$ but for convergent subnet $\{p^{\epsilon'}\}$ of $\{p^\epsilon\}$, $u^{p^{\epsilon'}} \to \bar{u}^p \neq u^p$. Since $u^{p^{\epsilon'}}$ represents $\bar{R}_{p^{\epsilon'}}$, by the joint continuity property in Lemma 1, it follows that $q \cdot \bar{u}^p = r \cdot \bar{u}^p$ if and only if $q \cdot u^p = r \cdot u^p$. Since u^p and $\bar{u}^p$ must satisfy the same normalizations, they must coincide by the uniqueness result of the EU theorem. □

Construct u_p as in Lemmas 1-7. By A1 and A2, $\succsim^\star$ has a continuous utility representation; call it $u_\star$.

Now construct v according to

$$v(q|p) = q \cdot u_p - p \cdot u_p + u_\star(p) \tag{5.2}$$

Lemma 10. *v is jointly continuous and is consistent with $\succsim^\star$.*

Proof. Since $u_\star$ is continuous and $q \cdot u_p$ is jointly continuous in p and q, it follows that v is jointly continuous. Each $v(\cdot|p)$ is also an expected utility function and represents $\bar{R}_p$. By construction, $v(p|p) = u_\star(p)$ for all$p \in \Delta$, and $u_\star$ represents $\succsim^\star$, thus v is consistent with $\succsim^\star$. □

Lemma 11. *v is increasing in $x^\star$.*

Proof. Since each $v(\cdot|p)$ represents $\bar{R}_p$, the result follows from Lemma 3. □

Lemma 12. *If $p \in c(D)$ and $p \notin c(D \cup \{p\alpha q\}) \ni r \precsim^\star p$, then $v(q|p) < v(p|p)$.*

Proof. Suppose $p \in c(D)$ but $p \notin c(D \cup \{p\alpha q\}) \ni r \precsim^\star p$.

Then by A4, there cannot exist nets $\{p^\epsilon\}$ and $\{q^\epsilon\}$ with limits p and q respectively such that for some $\alpha \in (0, 1]$, $p^\epsilon \in c(\{p^\epsilon, p^\epsilon \alpha q^\epsilon\})$ for all small ϵ. It follows that there exists some $\bar{\epsilon} > 0$ such that for all $p' \in N^\epsilon(p)$ and $q' \in N^\epsilon(q)$ and all $\alpha \in (0, 1]$, $\{p'\alpha q'\} = c(\{p', p'\alpha q'\})$. It follows that there can be no nets $\{p^\epsilon\}$ and $\{q^\epsilon\}$ with limits p and q respectively such that $p^\epsilon \in c(\{p^\epsilon, p^\epsilon \epsilon q^\epsilon\})$ for all small ϵ. Thus qR_pp. But then by the definition of $v(\cdot|p)$, $v(q|p) < v(p|p)$. □

Lemma 13. *If $p \in c(D)$, then $v(p|p) \geq v(q|p)$ $\forall q \in D$.*

Proof. Suppose $p \in c(D)$ and $q \in D$.

By A6, there exists a net $(\epsilon, \alpha_\epsilon)$ with $\alpha_\epsilon \to 0$ as $\epsilon \to 0$, for which $p\epsilon(p\alpha_\epsilon\delta_{x^\star}) \in c(\{p\}\epsilon\{p\alpha_\epsilon\delta_{x^\star}, q\})$ for each $\epsilon > 0$, where $x^\star$ denotes the "best prize" from the A5 axiom. But then by the definition of $\bar{R}_p$, we have $p\bar{R}_p q$. Thus by our choice of v, $v(p|p) \geq v(q|p)$. □

Consider the following Limit Consistency property.

Limit Consistency. $qR_p p$ implies $p \notin c(D)$ whenever $q \in D$.

Since $v(\cdot|p)$ was constructed to represent $\bar{R}_p$, Limit Consistency is an implication of Lemma 13.

Lemma 14. *If* $p \in c(\{p, q\})$ *then* $p \in PPE_v(\{p, q\})$.

Proof. Suppose $p \in c(\{p, q\})$. By Lemma 13, $p \in PE_v(\{p, q\})$. If $\{p\} = PE_v(\{p, q\})$ or if $p \succsim^\star q$ then the result follows directly.

If $q \succ^\star p$, then by A3, $\{p\} = c(\{p, q\})$ must hold. But then pBq. Then by Lemma 12, $v(p|q) > v(q|q)$, thus $p = PE_v(\{p, q\}) = PPE_v(\{p, q\})$. □

Lemma 15. *If* $p \in PPE_v(\{p, q\})$ *then* $p \in c(\{p, q\})$.

Proof. By Lemma 13, if $\{p\} = PE_v(\{p, q\})$ then the result is immediate. So suppose $p \in PPE_v(\{p, q\})$ and $q \in PE_v(\{p, q\})$; then, $p \succsim^\star q$. If $\{q\} = c(\{p, q\})$, then by Lemma 12, $v(q|p) > v(p|p)$. This implies that $p \notin PE_v(\{p, q\})$, which contradicts our initial assumption that $p \in PPE_v(\{p, q\})$. □

I will frequently use the relation between the constructed v and $\{\bar{R}_p\}_{p \in \Delta}$ and $\succsim^\star$in the lemmas below.

Lemma 16. *For all* $p, q, r \in \Delta$, $p \in c(\{p, q, r\}) \implies p \in PPE_v(\{p, q, r\})$.

Proof. Suppose $p \in c(\{p,\, q,\, r\})$. Then by Limit Consistency, $p \in PE_v(\{p, q, r\})$ and thus $p \in PE_v(\{p, q\}) \cap PE_v(\{p, r\})$.

Suppose $q \succ^\star p$. Then, by A3, either (a) $\{p\} = c(\{p, q\}) = PPE_v(\{p, q\})$, or (b) $\{q\} = c(\{p, q\}) = PPE_v(\{p, q\})$. In case (a) $c(\{q\}) \ni q \prec^\star p \in c(\{p, q\})$ and in case (b) $c(\{p, q\}) \ni q \prec^\star p \in c(\{p, q, r\})$; thus in either case, A4 implies that $\{(1 - \alpha)q + \alpha r\} = c((1 - \alpha)\{q\} + \alpha\{q, r\})$ for all $\alpha \in (0, 1]$. By Lemma 12, it follows that $v(r|q) > v(q|q)$, thus $q \notin PE_v(\{p, q, r\})$.

The argument above also applies if $r \succ^\star p$. Thus, we've shown that $p \in PE_v(\{p, q, r\})$ and if $q \succ^\star p$ that $q \notin PE_v(\{p, q, r\})$ and similarly if $r \succ^\star p$ that $r \notin PE_v(\{p, q, r\})$. Thus $p \in PPE_v(\{p, q, r\})$. □

Lemma 17. *If $p \in PPE_v(\{p,\, q,\, r\})$ then $p \in c(\{p,\, q,\, r\})$.*

Proof. Suppose $p \in PPE_v(\{p,\, q,\, r\})$.

We know that $c(\{p,\, q,\, r\}) \neq \emptyset$. It is thus sufficient to prove that $q \in c(\{p, q, r\}) \implies p \in c(\{p,\, q,\, r\})$, since the argument starting with $r \in c(\{p,\, q,\, r\})$ is the same, and the result is immediate if $\{p\} = c(\{p,\, q,\, r\})$.

Suppose $q \in c(\{p,\, q,\, r\})$. By Lemma 16, we have $q \in PPE_v(\{p, q, r\})$. Since $p \in PPE_v(\{p, q, r\})$ as well, we have both $\{p, q\} = PPE_v(\{p, q\})$ and thus, by Lemma 15, $\{p, q\} = c(\{p, q\})$. Since $v(p|p) = v(q|q)$ and v is consistent with $\succsim^\star$, we have $p \sim^\star q$. But then by Lemma 12 and since $p \sim^\star q \in c(\{p, q, r\})$, if $p \notin c(\{p, q, r\})$ we would have $v(r|p) > v(p|p)$, which would contradict that $p \in PPE_v(\{p, q, r\})$. Conclude that $p \in c(\{p, q, r\})$. □

Remark. $PPE_v(D) = PPE_v(PE_v(D))$.

Lemma 18. *Suppose that for some $n \geq 3$ we have established that $PPE_v(D) = c(D)$ whenever $|D| < n$. If $PE_v(D) = D$ and $|D| \leq n$, then $c(D) = PPE_v(D)$.*

Proof. First, suppose $PE_v(D) = D$.

Take $p \in PPE_v(D)$. Then $p \in PPE_v(D \backslash \{r\})\ \forall r \in D \backslash \{p\}$. Take any such $r \in D \backslash \{p\}$. By the representation for sets of size $n-1$, $p \in c(D \backslash \{r\})$. Since $p \succsim^\star q\ \forall q \in D$, if $p \notin c(D)$ we must have some $q \in c(D)$ with $q \succsim^\star p$. But then by Lemma 12, $v(r|p) > v(p|p)$, which would contradict that $p \in PPE_v(D)$. Conclude that $p \in c(D)$.

In the reverse, suppose $p \in c(D)$. By A1, there exists some $q \in D$ such that $q \succsim^\star r$ $\forall r \in D$. If $q = p$, it follows that $p \in PPE_v(D)$ directly since v is consistent with $\succsim^\star$. So suppose instead that $q \neq p$. Since $PE_v(D) = D$, we have $q \in PE_v(D \backslash \{r\})$ for any $r \in D \backslash \{q\}$; fix some such r. Since v is consistent with $\succsim^\star$ and $q \succsim^\star s\ \forall s \in D$, we also have $v(q|q) \geq v(s|s)\ \forall s \in D \backslash \{r\}$. Thus $q \in PPE_v(D \backslash \{r\})$. Then by the representation on sets of size $n-1$, $q \in c(D \backslash \{r\})$. Then since $q \succsim^\star p \in c(D)$, if $q \notin c(D)$, by Lemma 12 we would have $v(r|q) > v(q|q)$ which would contradict that $PE_v(D) = D$. Thus we must have $q \in c(D)$. But then since $p \in c(D)$ and $q \in c(D)$, by A3, $p \succsim^\star q$. Thus since v is consistent with $\succsim^\star$, $p \in PPE_v(D)$. □

Lemma 19 establishes by induction that $c(D) = PPE_v(D)$ for any $D \in \mathcal{D}$.

Lemma 19. *Suppose $c(D) = PPE_v(D)$ whenever $|D| < n$. Then, $c(D) = PPE_v(D)$ whenever $|D| \leq n$ as well.*

Proof. Consider D with $|D| = n$ and $PE_v(D) \neq D$. Partition D into $PE_v(D)$ and $D \backslash PE_v(D)$. The case where $PE_v(D) = D$ was proven in Lemma 18.

Since $|PE_v(D)| \leq n - 1 < n$, $c(PE_v(D)) = PPE_v(PE_v(D)) = PPE_v(D)$.

Say that $q^1, ..., q^m$ form a *chain* if $q^i R_{q^{i-1}} q^{i-1}$ for $i = 2, ..., m$. By the relation between each R_q and $v(\cdot|q)$, we can see that if $q^1, ..., q^m$ form a chain, Lemma 13 and non-emptiness of c imply that $\{q^m\} = PE_v(\{q^1, ..., q^m\}) = c(\{q^1, ..., q^m\})$. So if the longest chain in D contains all elements of D, then $c(D) = PPE_v(D)$.

Now suppose $p \in PPE_v(D)$.

Suppose instead we take an arbitrary chain $q^1, ..., q^m$ that cannot be extended further as a chain using elements of D. Since $q^1, ..., q^m$ cannot be extended, $q^m \in PE_v(D)$. Since $p \in PPE_v(D)$, $p \succsim^\star q^m$ and since v is consistent with $\succsim^\star$, by Lemma 15 we have $p \in c(\{p, q^m\})$.

Notice that any element of $D \backslash PE_v(D)$ is in a chain in D. Since D is finite, we can take a finite enumeration $j = 1, \ldots, n$ of non-extendable chains in D (taking one such chain for each element in D would work), such that $q^{1,j}, \ldots, q^{m_j,j}$ denotes the enumeration of chain j, and such that $D = PE_v(D) \cup \{q^{1,1}, \ldots, q^{m_1,1}\} \cup \cdots \cup \{q^{1,n}, \ldots, q^{m_n,n}\}$. By a previous argument, $p \in c(\{p, q^{m_j,j}\})$ and $p \succsim^\star q^{m_j,j}$ for each chain j. I will form the choice set D by starting with $PE_v(D)$, and adding elements one at a time to it from these chains. Since $p \in PPE_v(D)$ and $PPE_v(D) = PPE_v(PE_v(D)) = c(PE_v(D))$, we have $p \in c(PE_v(D))$.

Let $\hat{D}_0 = PE_v(D)$. Now, for $j = 1, \ldots, n$, define $\hat{D}_j = \hat{D}_{j-1} \cup \{q^{1,j}, \ldots, q^{m_j,j}\}$. Suppose $p \in c(\hat{D}_{j-1})$. Since $q^{m_j,j} \in PE_v(D) \subseteq \hat{D}_{j-1}$, we have $p \in c(\hat{D}_{j-1} \cup \{q^{m_j,j}\})$. So now suppose that $p \in c(\hat{D}_{j-1} \cup \{q^{k+1,j}, \ldots, q^{m_j,j}\})$ for some $k > 0$. By the definition of a chain, $q^{k,j} \notin PE_v(\hat{D}_{j-1} \cup \{q^{k,j}, \ldots, q^{m_j,j}\})$. Thus by the definition of PE, $p \in PE_v(\hat{D}_{j-1} \cup \{q^{k+1,j}, \ldots, q^{m_j,j}\}) \subseteq PE_v(\hat{D}_{j-1} \cup \{q^{k,j}, \ldots, q^{m_j,j}\}) \subseteq PE_v(D)$. Thus if $p \notin c(\hat{D}_{j-1} \cup \{q^{k,j}, \ldots, q^{m_j,j}\})$, then by Lemma 13 and non-emptiness of c, there is some $q \in PE_v(\hat{D}_{j-1} \cup \{q^{k+1,j}, \ldots, q^{m_j,j}\})$ with $q \in c(\hat{D}_{j-1} \cup \{q^{k+1,j}, \ldots, q^{m_j,j}\})$. But since $p \in PPE_v(D)$ and v is consistent with $\succsim^\star$, we have $p \succsim^\star r$ for all $r \in PE_v(D) = PE_v(\hat{D}_{j-1} \cup \{q^{k+1,j}, \ldots, q^{m_j,j}\})$, we have $p \succsim^\star q$. Then by $p \in c(\hat{D}_{j-1} \cup \{q^{k+1,j}, \ldots, q^{m_j,j}\})$, $p \notin c(\hat{D}_{j-1} \cup \{q^{k,j}, \ldots, q^{m_j,j}\})$, and Lemma 12, $v(q^{k,j}|p) > v(p|p)$, which contradicts $p \in PPE_v(D)$. Conclude that $p \in c(\hat{D}_{j-1} \cup \{q^{k,j}, \ldots, q^{m_j,j}\})$. By induction on k, we obtain that $p \in c(\hat{D}_j)$. By induction on j, $c \in c(\hat{D}_n)$. But since $\hat{D}_n = D$ by construction, we obtain $p \in c(D)$.

In the reverse direction, now suppose $p \in c(D)$. By Lemma 13, $p \in PE_v(D)$. Since $c(D) \supseteq PPE_v(D) = PPE_v(PE_v(D)) = c(PE_v(D)) \neq \emptyset$, $\exists q \in c(D) \cap PPE_v(D)$. Since $p, q \in c(D)$, $p \succsim^\star q$ follows by Non-Intrinsic Preference for Commitment. Thus, since v is consistent with v and $q \in PPE_v(D)$, it follows that $p \in PPE_v(D)$. $\square$

Necessity.

Axioms on $\succsim^\star$ Since $v(p|p) \geq v(q|q)$ if and only if $p \succsim^\star q$ and v is jointly continuous, it follows that $\succsim^\star$ must be complete, transitive, and continuous (from A1 and A2).

Lemma 20. *The representation implies A3.*

Proof. Notice that $p \in D \subseteq D'$ and $p \in PPE_v(D')$ implies $p \in PE_v(D)$. Thus, if $p \notin PPE_v(D)$ and $q \in PPE_v(D)$, we must have $v(q|q) > v(p|p)$. If instead $p \in PPE_v(D)$, then $v(q|q) = v(p|p)$ for any $q \in PPE_v(D)$ by the representation.. If v is consistent with $\succsim^\star$ and choices are given by $c = PPE_v$, then if $p \in c(D')$, $p \in D \subseteq D'$, and $q \in c(D)$, it follows that $q \succsim^\star p$. □

Lemma 21. *The representation implies A4.*

Proof. Take nets $\{p^\epsilon\}$ and $\{D^\epsilon\}$ with limits p and D respectively and for which $p^\epsilon \in c(D^\epsilon)$ for each ϵ. Then by the EU-PPE representation, for each ϵ, $v(p^\epsilon|p^\epsilon) \geq v(q^\epsilon|p^\epsilon)$ for each $q^\epsilon \in D^\epsilon$. Now fix an arbitrary $q \in D$. Since $D^\epsilon \to D$, there exists a net $\{q^\epsilon\}$ with limit q and with $q^\epsilon \in D^\epsilon$ for each ϵ. Since $v(p^\epsilon|p^\epsilon) \geq v(q^\epsilon|p^\epsilon)$ for each ϵ, $p^\epsilon \to p$, $q^\epsilon \to q$, and v is jointly continuous, it follows that $v(p|p) \geq v(q|p)$. Since $v(\cdot|p)$ is an EU function, it follows that for all $\alpha \in (0, 1]$ that $v(p|p) \geq v(p\alpha q|p)$ as well.

Now take an arbitrary D' with $p \in c(D')$, an $\alpha \in (0, 1]$, and an $r \in c(D \cup \{p\alpha q\})$. Since $p \in c(D')$, by the EU-PPE representation, $v(p|p) \geq v(s|p)$ for all $s \in D'$. Since we showed that $v(p|p) \geq v(p\alpha q|p)$ as well, it follows that $p \in PE_v(D' \cup \{p\alpha q\})$. Thus by the EU-PPE representation, either $p \in PPE_v(D' \cup \{p\alpha q\})$ or $v(p|p) < v(r|r)$; in the latter case, since v is consistent with $\succ^\star$, $r \succ^\star p$. This proves that A4 holds. □

Lemma 22. *If each $v(\langle x^\star, 1\rangle |p) > v(q|p)$ for all $q \in \Delta \backslash \langle x^\star, 1\rangle$, A5 holds.*

Proof. Since $v(\langle x^\star, 1\rangle |p) > v(q|p)$ for all $q \in \Delta \backslash \langle x^\star, 1\rangle$ and for all $p \in \Delta$, for any $\alpha > 0$ we have that $v((1-\alpha)p + \alpha \langle x^\star, 1\rangle |p) > v((1-\alpha)p + \alpha a|p)$. Thus for any $q \in \Delta \backslash \langle x^\star, 1\rangle$, $p \in \Delta$, $\alpha \in (0, 1]$, we have $\{(1-\alpha)p + \alpha \langle x^\star, 1\rangle\} = PE_v(\{p\}\alpha\{q, \langle x^\star, 1\rangle\}) = PPE_v(\{p\}\alpha\{q, \langle x^\star, 1\rangle\}) = c(\{p\}\alpha\{q, \langle x^\star, 1\rangle\})$. Thus A5 holds. □

Lemma 23. *The representation implies A6.*

Proof. Suppose $p \in c(D)$ and $q \in D$ with $q \neq \langle x^\star, 1\rangle$. Then $v(p|p) \geq v(q|p)$. Thus, for any $\alpha \in (0, 1)$, $v((1-\alpha)p + \alpha \langle x^\star, 1\rangle |p) > v(q|p)$. Thus for each $\alpha \in (0, 1)$, by joint continuity and the EU-PPE representation, $v((1-\alpha)p + \alpha \langle x^\star, 1\rangle |(1-\epsilon)p + \epsilon q) > v(q|(1-\epsilon)p + \epsilon q)$ for ϵ sufficiently small, thus $\{(1-\epsilon)p + \epsilon(1-\alpha)p + \epsilon\alpha \langle x^\star, 1\rangle\} = c((1-\epsilon)p + \epsilon \{(1-\alpha)p + \alpha \langle x^\star, 1\rangle, q\}$ for all ϵ sufficiently small. Since for α arbitrarily small we can find ϵ arbitrarily small to make this hold, it follows that A6 must hold. □

Lemma 24. *The representation implies A7.*

Proof. Suppose $q\bar{R}_p r$. Then by the definition of $\bar{R}_p$, there exists nets $\{p^\epsilon\}, \{q^\epsilon\}, \{r^\epsilon\}$ such that $(1-\epsilon)p^\epsilon + \epsilon q^\epsilon \in c((1-\epsilon)\{p^\epsilon\} + \epsilon\{q^\epsilon, r^\epsilon\})$. By the representation, $v((1-\epsilon)p^\epsilon + \epsilon q^\epsilon|(1-\epsilon)p^\epsilon + \epsilon q^\epsilon) \geq v((1-\epsilon)p^\epsilon + \epsilon r^\epsilon|(1-\epsilon)p^\epsilon + \epsilon q^\epsilon)$ for each ϵ. But since $v(\cdot|(1-\epsilon)p^\epsilon + \epsilon q^\epsilon)$ is an expected utility function for each ϵ, it follows that $v(q^\epsilon|(1-\epsilon)p^\epsilon + \epsilon q^\epsilon) \geq v(r^\epsilon|(1-\epsilon)p^\epsilon + \epsilon q^\epsilon)$ for each ϵ. By joint continuity of v, it follows that $v(q|p) \geq v(r|p)$. Now suppose $r\bar{R}_p s$. By a similar argument, $v(r|p) \geq v(s|p)$. Thus $v(q|p) \geq v(r|p) \geq v(s|p)$.

If $q = \langle x^\star, 1\rangle$ then $q\bar{R}_p s$ will follow by increasingness in $x^\star$; so suppose $q \neq \langle x^\star, 1\rangle$. Define nets by $q^\delta = (1-\delta)q + \delta\langle x^\star, 1\rangle$ and $r^\delta = (1-\frac{\delta}{2})q + \frac{\delta}{2}\langle x^\star, 1\rangle$. Since $v(q|p) \geq v(r|p) \geq v(s|p)$ and $v(\cdot|p)$ is an expected utility function and is increasing in $x^\star$, we have $v(q^\delta|p) > v(r^\delta|p) > v(s|p)$ for all δ. Given any $\epsilon > 0$, by joint continuity of v, we can find $\delta_\epsilon > 0$ such that $v(q^\delta|(1-\epsilon)p + \epsilon q^\delta) > v(q|(1-\epsilon)p + \epsilon q^\delta) > v(s|(1-\epsilon)p + \epsilon q^\delta)$ for all $\delta \in (0, \delta_\epsilon)$. Pick $q^\epsilon = (1-\delta)q + \delta\langle x^\star, 1\rangle$ for some $\delta \leq \min[\delta_{\epsilon'} : \epsilon' \geq \epsilon]$ such that $\delta \to 0$ as $\epsilon \to 0$. Then by construction, $(1-\epsilon)p^\epsilon + \epsilon q^\epsilon \in c((1-\epsilon)\{p^\epsilon\} + \epsilon\{q^\epsilon, s\})$ for all ϵ, and thus $q\bar{R}_p s$. □

Proof of Theorem 2

Since A6' implies A6, all that remains to prove is the set of additional implications that A6' places on v.

Lemma 25. *(i) The axioms of Theorem 2 imply v PPE-dislikes mixtures. (ii) In an EU-PPE representation, if v PPE-dislikes mixtures, then c satisfies A6'.*

Proof. Consider a EU-PPE representation with reference-dependent utility function v.

(i) If $p \in PPE_v(\{p, q\})$ then $v(p|p) \geq v(q|p)$ and either $v(p|p) \geq v(q|q)$ or $v(p|q) > v(q|q)$. Thus $v(p|p) \geq v(q|p)$ and $v(q|q) \leq \max[v(p|p), v(q|p)]$. Then the A6' axiom implies that then $p \in c(\{p\}\alpha\{p, q\}) = PPE(p\alpha\{p, q\})$, thus $v(p|p) \geq v(p\alpha q|p)$ and either $v(p\alpha q|p\alpha q) \leq v(p|p)$ or $v(p\alpha q|p\alpha q) < v(p|p\alpha q)$ (or both).

(ii) If $p \in c(\{p, q\})$, then $v(p|p) \geq v(q|p)$ and either $v(p|q) > v(q|q)$ or $v(p|p) \geq v(q|q)$. Fix $\alpha \in (0, 1)$. Since $v(p|p) \geq v(q|p)$ and $v(\cdot|p)$ is an EU function, $v(p|p) \geq v(p\alpha q|p)$, thus $p \in PE_v(\{p, p\alpha q\})$. If v PPE-dislikes mixtures, then $v(p|p\alpha q) > v(q|q)$ or $v(p|p) \geq v(p\alpha q|p\alpha q)$. Thus, $p \in PPE_v(\{p, q\})$, establishing A6'. □

Proposition 1

Proof. First prove that Kőszegi-Rabin preferences with linear loss aversion satisfy the limited-cycle inequalities.

Start with a finite set X with $|X| = n$ and assume (for now) that there is a single hedonic dimension. Without loss of generality, assume $u(x_1) > u(x_2) > ... > u(x_n)$.

Define the matrix V according to:

$$[V]_{ij} = u(x_i) + \mu(u(x_i) - u(x_j)) \tag{5.3}$$

Observe that $v(p|r) = p^T V r$. Let $\delta, \epsilon \in \mathbb{R}^{n+1}$ denote vectors with $\sum_{i=1}^{n+1} \delta_i = \sum_{i=1}^{n+1} \epsilon_i = 0$. By matrix multiplication, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} z^T V \delta &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left[u(x_i) + \mu(u(x_i) - u(x_j))\right] z_i \delta_j \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \mu(u(x_i) - u(x_j)) z_i \delta_j \end{aligned} \tag{5.4}$$

Take a cycle $p^{i+1} = p^i + \epsilon^i$ with $v(p^{i+1}|p^i) > v(p^i|p^i)$ for $i = 0, ..., m$. Then:

$$\begin{aligned} v(p^m|p^m) - v(p^0|p^m) &= (p + \sum_{l=0}^{m} \epsilon^l)^T V (p + \sum_{l=0}^{m} \epsilon^l) - p^T V (p + \sum_{l=0}^{m} \epsilon^l) \\ &= (\sum_{l=0}^{m} \epsilon^l)^T V (p + \sum_{l=0}^{m} \epsilon^l) \\ &= (\epsilon^0)^T V p + (\epsilon^0)^T V (\sum_{l=0}^{m} \epsilon^l) + (\sum_{l=1}^{m} \epsilon^l)^T V (p + \sum_{l=1}^{m} \epsilon^l) + (\sum_{l=1}^{m} \epsilon^l)^T V \epsilon^0 \end{aligned}$$

Rearranging the second term,

$$
\begin{aligned}
&= (\sum_{l=1}^{m}\epsilon^l)^T V(\sum_{l=1}^{m}\epsilon^l) + (\sum_{l=1}^{m-1}\epsilon^l)^T Vp + (\epsilon^m)^T V(p+\sum_{l=1}^{m-1}\epsilon^l) - (\epsilon^m)^T V(\sum_{l=1}^{m-1}\epsilon^l) \\
&= (\sum_{l=1}^{m}\epsilon^l)^T V(\sum_{l=1}^{m}\epsilon^l) + (\sum_{l=1}^{m-2}\epsilon^l)^T Vp + (\epsilon^{m-1})^T V(p+\sum_{l=1}^{m-2}\epsilon^l) \\
&\quad -(\epsilon^{m-1})^T V(\sum_{l=1}^{m-2}\epsilon^l) + (\epsilon^m)^T V(p+\sum_{l=1}^{m-1}\epsilon^l) - (\epsilon^m)^T V(\sum_{l=1}^{m-1}\epsilon^l) \\
&\vdots \\
&= (\sum_{l=1}^{m}\epsilon^l)^T V(\sum_{l=1}^{m}\epsilon^l) + \sum_{i=1}^{m}(\epsilon^i)^T V(p+\sum_{l=1}^{i-1}\epsilon^l) - \sum_{i=2}^{m}\epsilon^i V(\sum_{l=1}^{i-1}\epsilon^l)
\end{aligned}
$$

By the definition of the cycle, $(\epsilon^i)^T V(p+\sum_{l=1}^{i-1}\epsilon^l) > 0$ for each i, thus, the above is:

$$
> (\sum_{l=1}^{m}\epsilon^l)^T V(\sum_{l=1}^{m}\epsilon^l) - \sum_{i=2}^{m}\epsilon^i V(\sum_{l=1}^{i-1}\epsilon^l)
$$

By symmetry with respect to δ and ϵ in (5.4), it can be shown that $\sum_{i=2}^{m}\sum_{l=1}^{i-1}(\epsilon^i)^T V\epsilon^l = \sum_{j=1}^{m-1}\sum_{l=j+1}^{m}(\epsilon^j)^T V\epsilon^l$. Returning to the previous expression, more algebra establishes that the running expression is:

$$
\begin{aligned}
&= \sum_{l=1}^{m}(\epsilon^l)^T V\epsilon^l + \sum_{i=2}^{m}\sum_{l=1}^{i-1}(\epsilon^i)^T V\epsilon^l \\
&= \frac{1}{2}\sum_{l=1}^{m}(\epsilon^l)^T V\epsilon^l + \frac{1}{2}(\sum_{l=1}^{m}\epsilon^l)^T V(\sum_{l=1}^{m}\epsilon^l) \\
&> 0
\end{aligned}
$$

where the last line follows from (5.4). This completes the proof for the case with the case of one hedonic dimension.

To extend the argument to the case with more than one hedonic dimension, break up a lottery p into marginals $\underset{k}{p}$ in each dimenion k, and define the matrix $\underset{k}{V}$ as the utility matrix corresponding to V in dimension k. we can write $v^{KR}(p|r) = \sum_k \underset{k}{p}^T \underset{k}{V}\underset{k}{r}$. Notice that all of the previously-proven properties of V apply to $\underset{k}{V}$; following through the previous steps yields the desired result.

□

Proposition 2.

Proof. Now prove that Kőszegi-Rabin preferences with linear loss aversion PPE-dislike mixtures.

Suppose $v(p|p) \geq v(q|p)$ and $v(q|q) \leq \max\,[v(p|p),\, v(p|q)]$.

Then,

$v((1-\alpha)p + \alpha q|(1-\alpha)p + \alpha q)$

$$= (1-\alpha)^2 v(p|p) + \alpha(1-\alpha)v(p|q) + \alpha(1-\alpha)v(q|p) + \alpha^2 v(q|q) \tag{5.5}$$

by bilinearity of v under (2.4) and linear loss aversion.

If $v(p|p) \leq v(p|q)$, then two substitutions to (5.5) yield

$\leq (1-\alpha)^2 v(p|p) + \alpha(1-\alpha)v(p|q) + \alpha(1-\alpha)v(p|p) + \alpha^2 v(p|q)$

$= v(p|(1-\alpha)p + \alpha q)$ by bilinearity of v

$= \max\,[v(p|(1-\alpha)p + \alpha q),\, v(p|p)]$

If instead $v(p|q) \leq v(p|p)$, then two different substitutions to (5.5) yield

$\leq (1-\alpha)^2 v(p|p) + \alpha(1-\alpha)v(p|p) + \alpha(1-\alpha)v(p|p) + \alpha^2 v(p|p)$

$= v(p|p)$

$= \max\,[v(p|(1-\alpha)p + \alpha q),\, v(p|p)]$

This proves that v PPE-dislikes mixtures. □

Proof of Proposition 3.

Proof. (i) Suppose c has an EU-PPE representation and strictly exhibits reference-dependence. Pick $p, q, r, s, \alpha, \bar{\epsilon}$ that demonstrate this. Then by the representation, we have $p\alpha r \in PPE_v(\{p\}\alpha\{r,s\})$ and $q\alpha r'_\epsilon \notin PPE_v(\{q\}\alpha\{r'_\epsilon, s\})$ for any $r'_\epsilon = r\epsilon\,\langle x^\star, 1\rangle$ for $\epsilon \in [0, \bar{\epsilon})$. Thus, $v(p\alpha r|p\alpha r) \geq v(p\alpha s|p\alpha r)$ but $v(q\alpha s|q\alpha s) \geq v(q\alpha r'_\epsilon|q\alpha s) > v(q\alpha r|q\alpha s)$ for all such $\epsilon \in (0, \bar{\epsilon})$. Since $v(\cdot|q\alpha s)$ and $v(\cdot|p\alpha r)$ are EU functions, it follows that $v(r|p\alpha r) \geq v(s|p\alpha r)$ but $v(s|q\alpha s) > v(r|q\alpha s)$, the desired result.

In the opposite direction, suppose there exist p, q, r, s such that $v(r|p) > v(s|p)$ and $v(s|q) \geq v(r|q)$. Since v is increasing in $x^\star$, we can pick r and s so that this second inequality is strict, that is, $v(s|q) > v(r|q)$. Then by joint continuity of v, there exists a small $\alpha > 0$ for which $v(r|p\alpha s) > v(s|p\alpha s)$ and $v(s|q\alpha r) > v(r|q\alpha r)$. But since $v(\cdot|p\alpha s)$ and $v(\cdot|q\alpha r)$

are EU functions, we have $v(p\alpha r|p\alpha s) > v(p\alpha s|p\alpha s)$ and $v(q\alpha s|q\alpha r) > v(q\alpha r|q\alpha r)$. Thus by the EU-PE representation, $\{p\alpha r\} = PPE_v(\{p\alpha r, p\alpha s\})$ but $\{q\alpha s\} = PPE_v(\{q\alpha r, q\alpha s\})$. Thus PPE_v exhibits reference-dependence. By joint continuity of v and the fact that we have strict inequalities, $v(q\alpha s|q\alpha r') > v(q\alpha r|q\alpha r')$ for any $r' = r\epsilon \langle x^\star, 1\rangle$ for ϵ small enough, thus $\{q\alpha s\} = PE_v(\{q\alpha r', q\alpha s\}) = PPE_v(\{q\alpha r', q\alpha s\})$ for such r'; thus c strictly exhibits reference-dependence as well.

(ii) EU-PPE case. If c violates WARP, then there must exist D,D' with $D' \subset D$ such that $\emptyset \neq c(D) \cap D' \neq c(D')$. Since $c(D) \cap D' \neq \emptyset$, take $q \in c(D) \cap D'$. It follows that $q \in PE_v(D')$. Thus, there exists a $r \in PE_v(D') \backslash PE_v(D)$ with $v(q|q) \geq v(r|r)$, in which case there exists an $s \in D \backslash D'$ for which $v(s|r) > v(r|r)$ but $v(r|r) \geq v(q|r)$ and $v(q|q) \geq v(s|q)$. But then $v(s|r) > v(q|r)$ but $v(q|q) \geq v(s|r)$, thus by part (i), c exhibits reference-dependence. □

Proof of Proposition 4.

Proof. Take x, y, z from the Proposition. If there is more than one hedonic dimension, then since $u^k(x) = u^k(y) = u^k(z)$ for all $k > 1$, we have that for any lotteries with support only in $\{x, y, z\}$ will yield the exact same intrinsic and gain-loss utilities in dimensions $k > 1$. Thus, it is sufficient to prove the result for the case of one hedonic dimension. The proof proceeds by straightforward calculation.

If $\eta = 0$, then each $v(\cdot|p)$ is the same expected utility function, thus c must satisfy the Independence Axiom, and thus cannot exhibit reference-dependence. Next, suppose $\eta > 0$ and $\lambda = 0$. Then,

$$
\begin{aligned}
v(p|r) &= \sum_i \sum_j \sum_k p_i \left[u^k(x_i) + \eta r_j \left[u^k(x_i) - u^k(x_j)\right]\right] \\
&= \sum_k \left[\sum_i p_i u^k(x_i) + \eta \sum_i \sum_j p_i r_j \left[u^k(x_i) - u^k(x_j)\right]\right] \\
&= \sum_k \left[\sum_i p_i u^k(x_i) + \eta \sum_i p_i \sum_j r_j u^k(x_i) - \eta \sum_j r_j \sum_i p_i u^k(x_j)\right] \\
&= \sum_k \left[\sum_i p_i u^k(x_i) + \eta \sum_i p_i u^k(x_i) - \eta \sum_j r_j u^k(x_j)\right] \\
&= \sum_k \sum_i (1+\eta) p_i u^k(x_i) - \eta \sum_k \sum_j r_j u^k(x_j)
\end{aligned}
$$

Thus, the utilities function given any given two reference lotteries are affine transformations of each other. Thus for any $p, q, r, s \in \Delta$, $v(p|r) \geq v(q|r)$ if and only if $v(p|s) \geq v(q|s)$.

Thus, $p\alpha r \in PPE_v(\{p\}\alpha\{r,s\})$ implies $q\alpha r \in PE_v(\{q\}\alpha\{r,s\})$. But $p\alpha r \in PPE_v(\{p\}\alpha\{r,s\})$ implies that either $p\alpha s \notin PE_v(\{p\}\alpha\{r,s\})$, or $v(p\alpha r|p\alpha r) \geq v(p\alpha s|p\alpha s)$. In the former case, it follows that $q\alpha s \notin PE_v(\{q\}\alpha\{r,s\})$, thus $\{q\alpha r\} = PPE_v(\{q\}\alpha\{r,s\})$. In the latter case, notice that:

$$\begin{aligned} v(p\alpha r|p\alpha r) &= \sum_k\sum_i (1+\eta)\left((1-\alpha)p_i + \alpha r_i\right) u^k(x_i) - \eta\sum_k\sum_j \left((1-\alpha)p_j + \alpha r_j\right) u^k(x_j) \\ &= (1-\alpha)\left[\sum_k\sum_i (1+\eta)\, p_i u^k(x_i) - \eta\sum_k\sum_j p_j u^k(x_j)\right] \\ &\quad +\alpha\left[\sum_k\sum_i (1+\eta)\, r_i u^k(x_i) - \eta\sum_k\sum_j r_j u^k(x_j)\right] \\ &= (1-\alpha)v(p|p) + \alpha v(r|r) \end{aligned}$$

A similar simplification applies to $v(p\alpha s|p\alpha s)$, $v(q\alpha r|q\alpha r)$, and $v(q\alpha s|q\alpha s)$. It follows that: $v(p\alpha r|p\alpha r) \geq v(p\alpha s|p\alpha s)$ if and only if $v(r|r) \geq v(s|s)$ if and only if $v(q\alpha r|q\alpha r) \geq v(q\alpha s|q\alpha s)$. Thus $q\alpha r \in PPE_v(\{q\}\alpha\{r,s\})$ in the latter case as well.

It follows that if $\lambda = 0$, then the model does not display reference-dependence.

Now to show that if $\eta > 0$ and $\lambda > 0$ it does display reference-dependence.

First, suppose $u(x) > u(y) > u(z)$; without loss of generality, normalize $1 = u(x) > \bar{u} = u(y) > 0 = u(z)$. Let $p = \langle x, \beta; z, 1-\beta\rangle$ be chosen such that $\frac{(1+\eta)\beta}{1+\eta+(1-\beta)\lambda\eta} < \bar{u} < \frac{(1+\eta)\beta+\lambda\beta^2\eta}{1+\eta+\lambda\beta\eta}$.

Then,

$$\begin{aligned} v(p|p) &= \beta + \eta\beta(1-\beta) + \eta\beta(1-\beta)(1+\lambda)\left[-1\right] \\ &= \beta - \eta\beta(1-\beta)\lambda \end{aligned}$$

$$\begin{aligned} v(\langle y,1\rangle\,|p) &= \bar{u} + \eta(1-\beta)\bar{u} + \eta\beta(1+\lambda)(\bar{u}-1) \\ &= (1+\eta+\lambda\beta\eta)\bar{u} - \eta\beta(1+\lambda) \end{aligned}$$

$$v(\langle y,1\rangle\,|\,\langle y,1\rangle) = \bar{u}$$

$$
\begin{aligned}
v(p|\langle y,1\rangle) &= \beta+\eta\beta(1-\bar{u})+\eta(1-\beta)(1+\lambda)(-\bar{u}) \\
&= (1+\eta)\beta-\eta\beta\bar{u}
\end{aligned}
$$

Working with these equations, if we pick β so that $\frac{(1+\eta)\beta}{1+\eta+(1-\beta)\lambda\eta} < \bar{u} < \frac{(1+\eta)\beta+\lambda\beta^2\eta}{1+\eta+\lambda\beta\eta}$, then $v(\langle y,1\rangle \,|\, \langle y,1\rangle) > v(p|\langle y,1\rangle)$ but $v(p|p) > |v(\langle y,1\rangle \,|p)$. Such a β exists whenever $\eta,\lambda > 0$. By Proposition 3, v displays strict reference-dependence in this case. □