Preferred Personal Equilibrium and Simple Choices

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Abstract

This paper studies choices under the Preferred Personal Equilibrium concept introduced by Kőszegi and Rabin (2006) for modeling choice given expectations-based reference-dependent preferences. The main result of this paper is that when expectations are not observed and parametric assumptions on utility are not made, Preferred Personal Equilibrium choice can be characterized by three axioms that together weaken the Weak Axiom of Revealed Preference.

Keywords: reference-dependent preferences, expectations-based reference-dependence, personal equilibrium, rational shortlist method.

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1 Introduction

Models of reference-dependent preferences allow a decision-maker's behavior to be influenced by her reference point. However, reference points are not naturally observable, which means that applications of reference-dependent preferences need to make an assumption about how references points are formed in order to generate predictions. To that end, Kőszegi and Rabin (2006) introduce the Personal Equilibrium (PE) and Preferred Personal Equilibrium (PPE) concepts for modeling choice with reference-dependent preferences when reference points are given by a decision-maker's lagged expectations. In a PE, a potential reference point in a given choice environment is required to be consistent with a decision-maker's rational expectations about her subsequent choice given her reference-dependent preferences. The PPE concept refines the PE concept by assuming that a decision-maker picks the best possible PE, taking into account that her chosen option determines both what she ends up with and also her reference point. In both PE and PPE, the decision-maker's reference point is consistent with rational expectations about her choice. Thus in any PE or PPE, the decision-maker's reference point and her choice coincide. This raises the question: what is the content of reference-dependence in the PPE model in terms of observable choice behavior?

The main result of this paper (Proposition 2) shows that without making parametric assumptions about the functional form taken by reference-dependent preferences, the PPE model of reference-dependent choice can be characterized in terms of three axioms which, taken together, weaken the Weak Axiom of Revealed Preference (WARP) that characterizes rational choice. In contrast, if reference points are not restricted by a model, any choice correspondence is rationalizable when both utility functions and reference points must be elicited from behavior. The characterization here allows the PPE model to be judged solely on the basis of revealed preferences and without commiting to Kőszegi and Rabin's functional form.

The characterization result here relates to an analogous result by Gul and Pesendorfer (2008) for the less restrictive PE concept. Indeed, their result emerges as a corollary to Proposition 1 here, which shows that PPE decision-making is observationally equivalent to a special case of the rational shortlist method studied by Manzini and Mariotti (2007). However, the numerous applications of Kőszegi and Rabin's model of reference-dependent preferences have mostly used the more restrictive PPE concept, motivating the current exercise.[1]

The existing choice theoretic literature on reference-dependence and loss aversion almost exclusively assumes that the reference point is observed. Examples include Masatlioglu and Ok (2005; 2014) and Sagi (2006). Ok, Ortoleva and Riella's (2015) model of the attraction effect is one notable exception. In another notable exception, Apesteguia and Ballester (2013) propose and characterize a model of endogenous reference points that builds on Masatlioglu and Ok (2005). I discuss the relation between the PE and PPE models and some of these models in Section 4.

2 Reference-dependent preferences and preferred personal equilibrium

Let X denote a finite set of alternatives. No structure on X is assumed; for example, elements of X could be lotteries. Let $\mathscr{D}$ denote the set of all non-empty subsets of X; a typical $D \in \mathscr{D}$ is called a choice set.

In models of reference-dependent utility, a decision-maker has a set of reference-dependent utility functions. The reference-dependent utility function is a mapping $v : X \times X \to \mathbb{R}$, where $v(\cdot|x)$ defines the decision-maker's utility function when given the reference point x. A general reference-dependent model that does not place any restrictions on how reference points are determined consists of a reference-dependent utility function v and a reference map $\psi : \mathscr{D} \to \mathscr{D}$, where $\psi(D)$ specifies the set of alternatives that sometimes act as reference points for choice from D. The choice behavior of a general reference-dependent model given v and ψ, $RD_{v,\psi}$, is given by:

$$RD_{v,\psi}(D) = \{x \in D : \exists z \in \psi(D) \text{ s.th. } v(x|z) \geq v(y|z) \; \forall y \in D\}. \tag{1}$$

In the general reference-dependent model, a given option $x \in D$ can be chosen in D if it maximizes the reference-dependent utility function that corresponds to some reference point in the set $\psi(D)$.

[1]Examples include Kőszegi and Rabin (2007, 2009); Heidhues and Kőszegi (2008, 2014); Crawford and Meng (2011); Eliaz and Spiegler (2013); Herweg and Mierendorff (2013); Herweg, Karle and Müller (2014); Karle, Kirchsteiger and Peitz (2015); Pagel (2014)

Kőszegi and Rabin (2006) introduce three solution concepts that restrict the general reference-dependent model, PE, PPE, and choice-acclimating personal equilibrium (CPE), for modeling the endogenous determination of expectations-based reference points for a decision-maker with a reference-dependent utility function v.

In an environment in which a decision-maker faces a fully-anticipated choice set D and has a reference point consistent with her past expectations, rational expectations requires that the decision-maker's reference point corresponds with her actual choice from D. The PE concept imposes this consistency requirement between expectations and choice. Given a reference-dependent utility function v, the set of PE for a given choice set D, denoted $PE_v(D)$, is given by:

$$PE_v(D) = \{x \in D : v(x|x) \geq v(y|x) \; \forall y \in D\}. \tag{2}$$

The PE concept has the following interpretation. When choosing from choice set D, a decision-maker uses her reference-dependent preferences $v(\cdot|r)$, given her reference point (r), and chooses $\arg\max_{x \in D} v(x|r)$. When forming expectations, the decision-maker recognizes that her expected choice p will determine the reference point that applies when she chooses from D. Thus, she would only expect a $x \in D$ if it would be chosen by the reference-dependent utility function $v(\cdot|x)$, that is, if $x \in \arg\max_{y \in D} v(y|x)$. The set of PE of D in (2) is the set of all such x. There may be a multiplicity of PE for a given choice set. Indeed, if reference-dependence tends to bias a decision-maker towards her reference point, multiplicity is natural.[2]

At the time of forming her expectations, a decision-maker might evaluate the lottery x according to $v(x|x)$, which reflects that she will evaluate the x relative to itself as the reference point. The PPE concept is a natural refinement of the set of PE, based on a decision-maker picking her best PE expectation according to $v(x|x)$. Given a reference-dependent utility function v, the set of PPE for a given choice set D, denoted $PPE_v(D)$, is given by:

[2]Kőszegi (2010, Section 3.1) emphasizes the prediction of multiple "stable" personal equilibria as an observable implication of PE decision-making. Note that the opposite – the non-existence of a PE – is possible depending on v. The analysis that follows explicitly assumes that the set of PE is always non-empty – this can be proved to hold for functional forms that are used in applications.

$$PPE_v(D) = \underset{x \in PE_v(D)}{\arg\max}\, v(x|x) \tag{3}$$

In addition to PE and PPE, Kőszegi and Rabin (2007) also introduce the CPE concept:

$$CPE_v(D) = \underset{x \in D}{\arg\max}\, v(x|x) \tag{4}$$

They consider the CPE model most appropriate in environments in which a decision-maker commits to her choice at the time of forming expectations, and thus is not restricted by to choose from among PE alternatives.

Kőszegi and Rabin (2006) adopt a particular functional form for v. They assume that a decision-maker ranks lotteries over outcomes, and that given probabilistic expectations summarized by the lottery r, a decision-maker ranks a lottery p according to:

$$v^{KR}(p|r) = \sum_k \sum_i p_i u^k(x_i^k) + \sum_k \sum_i \sum_j p_i r_j \mu \left(u^k(x_i^k) - u^k(x_j^k) \right) \tag{5}$$

where p_i denotes the probability of receiving outcome x_i in lottery p and x_i^k denotes the level of hedonic attribute k possessed by outcome x_i. In (5), u^k is a consumption utility function in "hedonic dimension" k; different hedonic dimensions are akin to different goods in a consumption bundle, but specified based on "psychological principles." The function μ is a gain-loss utility function which captures reference-dependent outcome evaluations.

The Kőszegi-Rabin model with the PPE concept has been particularly amenable to applications since the model's predictions are pinned down by (5), (3), and (2). However, previous studies of the Kőszegi-Rabin model's behavior have mostly been limited to specific applications. The analysis below studies a general behavioral characterization of the choice behavior consistent with the PPE model while assuming that neither expectations, nor reference points, nor reference-dependent preferences are observed.

The case in which elements of X are lotteries is an important special case of the analysis here. Indeed, Kőszegi and Rabin (2006, Proposition 3) show that when alternatives are deterministic and the gain-loss utility function μ is piece-wise linear that the PPE choices under (5) maximize the utility function $\sum_k u^k(x^k)$ and thus reference-dependence plays no role in choice.

3 Reference-Dependent Choice

3.1 Preliminaries

The starting point for analysis is a choice correspondence on X, $c : \mathscr{D} \to \mathscr{D}$, which is taken as the set of elements we observe a decision-maker sometimes choose from a set D. Assume $\emptyset \neq c(D) \subseteq D$, that is, a decision-maker always chooses something from her choice set.

For any finite set D and binary relation P, define $m(D,P) := \{x \in D : \forall y \in D, \text{ if } yPx \text{ then } xPy\}$ as the set of undominated elements in D according to binary relation P.

3.2 Testable implications of the general reference-dependent model

Observation 1 below formalizes the intuition that when reference points are not observed nor restricted, the general reference-dependent model can rationalize any choice correspondence. The intuition behind Observation 1 is that the reference map ψ has the same dimension as c itself; thus when reference points are unobserved, the general reference-dependent model has too many degrees of freedom.[3]

Observation 1. *For any choice correspondence c, there exists a reference map ψ and reference-dependent utility function v such that for each $D \in \mathscr{D}$, $c(D) = RD_{v,\psi}(D)$*

Proof. For each $x \in X$, let $v(\cdot|x)$ be given by $v(x|x) = 1$ and $v(y|x) = 0 \, \forall y \in X \setminus \{x\}$. For any $D \in \mathscr{D}$, let $\psi(D) = c(D)$. Then $c(D) = RD_{v,\psi}(D)$ for any $D \in \mathscr{D}$. □

The implication is that models of reference-dependent preferences have no testable implications without restrictions on the utility function or on how reference points are formed. This confirms commonly-held intuition and motivates the further study of

[3]Kalai, Rubinstein and Spiegler (2002, Theorem 1) is a closely related result. They prove that any single-valued choice function on a finite set X can be rationalized by $|X|-1$ transitive binary relations, where any assignment of binary relations to choice sets is permitted. See also Blow, Crawford and Crawford (2015) for a related result in consumer choice.

the specific implications of how PPE's assumptions about reference point formation discipline choice behavior.

3.3 Testable implications of PPE: an equivalence result

Arrow (1959) shows that c satisfies the version of WARP below if and only if there is a complete and transitive binary relation P such that $c(D) = m(D, P)$ for any D. This provides a point of departure from standard models.

WARP. If $y \in D$, $x \in c(D)$, $x \in D'$, and $y \in c(D')$, then $x \in c(D')$.

Example 1 shows that PE and PPE choices may not satisfy WARP in the Kőszegi-Rabin model.

Example 1. Consider a decision-maker with a Kőszegi-Rabin v, as in (5), with linear utility function u over monetary outcomes, and linear loss aversion:[4]

$$u(z) = z, \quad \mu(z) = \begin{cases} z & \text{if } z \geq 0 \\ 3z & \text{if } z < 0 \end{cases}.$$

Consider the three lotteries $p = \langle \$1000, 1 \rangle$, $q = \langle \$0, .5; \$2900, .5 \rangle$, and $r = \langle \$0, .5; \$2000, .25; \$4100, .25 \rangle$. In this example, $\{p, q\} = PE_v(\{p, q\})$ but $\{q, r\} = PE_v(\{p, q, r\})$; while $\{p\} = PPE_v(\{p, q\})$ and $\{q\} = PPE_v(\{p, q, r\})$, violating WARP in both cases.

The violation of WARP in the Kőszegi-Rabin model arises from a conflict between how v determines the set of PE and the CPE ranking it uses to select from the set of PE.

A distinct model, the rational shortlist method (RSM) representation, is given by:

$$RSM_{P_1,P_2}(D) = m(m(D, P_1), P_2)$$

for two binary relations P_1, P_2. The RSM was introduced by Manzini and Mariotti (2007), who restrict their analysis to single-valued choice functions (a restriction not made here). While PPE decision-making and the RSM are distinct models, Proposition

[4] I would like to thank Matthew Rabin for suggesting this example. Linear loss aversion is used in most applications of Kőszegi-Rabin, and the chosen parameterization is broadly within the range implied by experimental studies.

1 establishes a link between a version of the rational shortlist method and the PPE representation in (3). Specifically, a choice correspondence c has an RSM representation with a transitive P_2 if and only if it also has a PPE representation.

Proposition 1. *There exist binary relations P_1 and P_2, where P_1 is asymmetric and P_2 is complete and transitive, such that $c = RSM_{P_1,P_2}$ if and only if there exists a v such that $c = PPE_v$.*

Proof. Consider the following mapping between a PPE representation and an RSM representation:

$$v(y|x) > v(x|x) \iff yP_1x \tag{6}$$
$$v(x|x) \geq v(y|y) \iff xP_2y \tag{7}$$

The mapping in (6) and (7) from v to P_1 and P_2 is well-defined and implies that P_1 is asymmetric and P_2 is complete and transitive.

I now argue that the mapping from P_1 and P_2 to v in (6) and (7) is also well-defined. Since X is finite, P_2 is complete and transitive if and only if it has a utility representation – let $u : X \to \mathbb{R}$ denote one such representation. For each $x, y \in X$, define $v(y|x)$ by:

$$v(y|x) = \begin{cases} u(x) + 1 & \text{if } yP_1x \\ u(x) & \text{otherwise} \end{cases}$$

Since P_1 is asymmetric, $v(x|x) = u(x)$ for each $x \in X$. Thus this constructed v is consistent with the mapping in in (6) and (7). For v and P_1, P_2 that satisfy this mapping, $m(D, P_1) = PE_v(D)$, and $m(m(D, P_1), P_2) = PPE_v(D)$. □

An implication of Proposition 1 is that a characterization of the RSM representation with a complete and transitive P_2 also characterizes the choice implications of PPE; I pursue this exercise in the next section.

As an aside, characterizations of PE and CPE are corollaries of Proposition 1; part (i) of Proposition 1 corresponds to Gul and Pesendorfer's (2008) characterization of PE. Say that c is induced by a complete binary relation if there exists a complete binary relation P such that $c(D) = m(D, P)$.[5]

[5]Non-emptiness of c implies that if $c(D) = m(D, P)$ then asymmetric part of P must be acyclic.

Corollary 1. *(i) c is induced by a complete binary relation if and only if there is a v such that $c(D) = PE_v(D)$ for any choice set D. (ii) c is induced by a complete and transitive binary relation if and only if there is a v such that $c(D) = CPE_v(D)$ for any choice set D.*

The insight in Corollary 1 is that from a choice theoretic perspective in which v is not directly observed, the PE model is the special case of PPE in which the CPE ranking is indifferent among all options. Similarly, the CPE model can be viewed as the special case of PPE in which the PE ranking never eliminates any available option.

Observation 2. *(i) $PE_v = PPE_v$ if and only if for all $x, y \in X$, $v(x|x) \geq v(y|x)$ implies $v(x|x) \geq v(y|y)$. (ii) $PPE_v = CPE_v$ if and only if for all $x, y \in X$, $v(x|x) \geq v(y|y)$ implies $v(x|x) \geq v(y|x)$.*

3.4 Axiomatic characterization of PPE and PE

I offer an axiomatization of PPE choices that builds on Manzini and Mariotti's axiomatization of the rational shortlist method; unlike in Manzini and Mariotti (who restrict c to be single-valued), c is allowed to be a multi-valued choice correspondence in the analysis below.[6] I will provide an interpretation of their Expansion and Weak WARP axioms in terms of expectations-based reference-dependence. I then provide the analogous axiomatization for PE choice.

The Expansion axiom requires that if option x is chosen in choice set D and also in choice set D', then it must also be chosen in their union. Under expectations-based reference-dependence, observing $x \in c(D)$ and $x \in c(D')$ implies that x is chosen with x as the reference point over all elements in both D and D'; Expansion requires that the same be true of $D \cup D'$.

Expansion. If $x \in c(D)$ and $x \in c(D')$, then $x \in c(D \cup D')$.

Weak WARP weakens the WARP, and modifies the eponymous axiom of Manzini and Mariotti to be suitable for multivalued choice correspondences. When we observe y chosen in a choice set $\bar{D}$ we can infer that y is chosen with y as the reference point

[6] In the Appendix, Proposition 4 provides an analogue of Manzini and Mariotti's Theorem 1 for choice correspondences using the versions of the Expansion and Weak WARP axioms stated below.

over all other options in $\bar{D}$. Thus if y is not chosen in some subset D of $\bar{D}$ in which y is available, and x is chosen instead, this must be because of the decision-maker chose to form a different expectations when facing D, and thus chose according to a different reference point. Since the smaller set D has fewer constraints on the rational expectations the decision-maker could possibly form, we infer that when forming expectations, the decision-maker would prefer to expect x from D than y. The Weak WARP axiom disciplines this logic. Suppose we have that x is chosen from a smaller set (D) and y a larger one ($\bar{D}$), and we additionally observe that x would be chosen when x is the reference point against all available options in D' (i.e. $x \in c(\bar{D}')$ and $D' \subseteq \bar{D}'$). The preceding logic suggests that if y is chosen in D' that x should be chosen too.

Weak WARP. For any $x, y \in X$ and $D, D', \bar{D}, \bar{D}' \in \mathscr{D}$ for which $\{x,y\} \subseteq D \subseteq \bar{D}$ and $\{x,y\} \subseteq D' \subseteq \bar{D}'$, if $x \in c(D)$, $y \in c(\bar{D})$, $y \in c(D')$, and $x \in c(\bar{D}')$, then $y \in c(D)$.

The Restricted Transitivity axiom restricts attention to pairs of items, denote them x and y, for which the decision-maker is observed to sometimes choose x when y is available, and sometimes y when x is available. Building on the interpretation of Weak WARP, in such a situation, a decision-maker who chooses x from $\{x,y\}$ and chooses y over x in a larger set indicates that she has preference for x over y at the time she forms her expectations. Restricted Transitivity requires this preference at the time of forming expectations be transitive. Define the relation W by xWy if $x \in c(\{x, y\})$ but there exists a $D \supseteq \{x, y\}$ such that $y \in c(D)$.

Restricted Transitivity. If there is a chain $x^0, x^1, \ldots, x^m \in X$ such that $x^i W x^{i+1}$ for $i = 0, ..., m-1$, and there is a $D \supseteq \{x^0, x^m\}$ such that $x^0 \in c(D)$, then $x^0 \in c(\{x^0, x^m\})$.

Expansion, Weak WARP, and Restricted Transitivity collectively characterize choice behavior that is consistent with the PPE model.

Proposition 2. *Suppose c is a choice correspondence on the finite set X. Then the following are equivalent: (i) c satisfies Expansion, Weak WARP, and Restricted Transitivity, (ii), there exist an asymmetric P_1 and a complete and transitive P_2 such that $c = RSM_{P_1,P_2}$, and (iii) there exists a $v : X \times X \to \mathbb{R}$ such that $c = PPE_v$.*

Proof. See the Appendix. □

Remark. Proposition 2 provides an alternative axiomatization of the RSM model with a transitive P_2 for multi-valued choice correspondence to that of Tyson (2013, Corollary 3.19).

The proof to Proposition 2 draws on the following proposition, which also provides a foundation for the RSM model for multi-valued choice correspondences and may be of independent interest.

Proposition 3. *Suppose c is a choice correspondence on the finite set X. Then c satisfies Expansion, and Weak WARP if and only if there exist an asymmetric P_1 and an acyclic P_2 such that $c = RSM_{P_1,P_2}$.*

Remark. To my knowledge, Proposition 3 is new, though there are two previous papers that provide axioms for versions of the RSM model for choice correspondences. Tyson (2013) restricts attention to cases in which P_2 is transitive. García-Sanz and Alcantud (2015, Theorem 1) use a different adaptation of the Weak WARP axiom but restrict to choice correspondences that satisfy their "choosing without dominated elements" property: if x is never chosen when y is available, then for any D with $x, y \in D$, $c(D \backslash \{x\}) = c(D)$. I use a stronger axiom, but do not require this restriction.

To axiomatize the PE model, Sen's α replaces Weak WARP and Restricted Transitivity is dropped. As in our motivation for Weak WARP, when we observe x deemed choosable in a choice set $\bar{D}$, we can infer that x is choosable with x as the reference point over all other options in $\bar{D}$. Sen's α requires that this imples that x is deemed choosable in all subsets of $\bar{D}$ that include x.

Sen's α. If $x \in D \subset \bar{D}$ and $x \in c(\bar{D})$, then $x \in c(D)$.

The class of c that are induced by a complete binary relation has been characterized axiomatically by Sen (1971) in terms of two axioms, Expansion and Sen's α.

Observation 3. *Suppose c is a choice correspondence on the finite set X. Then c satisfies Expansion and Sen's α if and only if there exists a $v : X \times X \to \mathbb{R}$ such that $c = PE_v$.*

Proof. Follows from Corollary 1 and T.9. of Sen (1971). □

4 Implications and Discussion

Testable implications of reference-dependent preferences. Observation 1 shows that the general model of reference-dependent preferences can explain any observed choices. The consequence is that there is no purely choice based test of reference-dependence without making assumptions about how reference points are formed or about the shape of the reference-dependent utility functions (e.g. loss aversion). In practice, most tests of reference-dependence do make an assumption about reference points.

Testable implications of PE and PPE. In a rich setting with both risk and multiple stages of decisions, Kőszegi (2010) identifies three implications of the PE model for choice: multiple stable equilibria, dynamic inconsistency, and informational preferences. By focusing on the PPE model in a static choice setting that abstracts from more specific implications of choice under risk, none of these implications are present in the setting of this paper. Nonetheless, the WARP violations in PPE characterized here parallel the dynamic inconsistency studied by Kőszegi on his richer domain. In spite of the more restrictive domain here, the analysis here can shed light on the WARP violations ruled out by versus those allowed by PPE.

Behavior ruled out by PPE. The axiomatization of the PPE model makes it transparent that some types of behavior cannot be accommodated in the PPE model.

Two choice patterns known as the attraction and compromise effects have been widely documented in experiments; see Simonson (1989) for a review. In each, the decision-maker would choose $\{x\} = c(\{x,y\})$ and $\{x\} = c(\{x,z\})$ but $\{y\} = c(\{x,y,z\})$. In the attraction effect, y and z are similar options but y the clearly better of the two, and adding z to the choice set attracts the decision-maker to y. In the compromise effect, y is a compromise between two disimilar choices x and z. The PPE model was not designed to accommodate these effects. The Expansion axiom clearly rules out these choices, thus the PPE model is unable to capture these effects.

In recent work, Ok, Ortoleva and Riella (2015) provide an axiomatic model of endogenous reference points designed to capture the attraction effect. In their model, a decision-maker has a base utility function $u : X \to \mathbb{R}$, a set of criteria $\mathscr{U}$ consisting of functions mapping $X \to \mathbb{R}$, and a reference map $r : \mathscr{D} \to X \cup \{\diamond\}$ that specifies a reference alternative ($r(D) \in D$) or no reference point ($R(D) = \diamond$) given a choice set, D. They model a reference point as inducing a constraint on the decision-maker to choose from among the available options that dominate the reference point according to all of the criteria functions, with choice represented by

$$c(D) = \operatorname*{arg\,max}_{x \in D \cap \mathscr{U}^{\uparrow}(r(D))} u(x)$$

where $\mathscr{U}^{\uparrow}(x) = \{x \in X : v(x) \geq v(a) \forall v \in \mathscr{U}\}$ if $x \in X$ and $\mathscr{U}^{\uparrow}(\diamond) = X$. Their representation imposes the additional restriction that whenever $D' \subseteq D$, $r(D) \in D'$, and $c(D) \cap D' \neq \emptyset$, we have $r(D') \neq \diamond$ and $c(D') = \operatorname*{arg\,max}_{x \in D' \cap \mathscr{U}^{\uparrow}(r(D))} u(x)$. While the relationship between this model of endogenous reference points and the PE and PPE models is not obvious a priori, the axiomatic approach here facilitates a comparison. Ok, Ortoleva and Riella impose transitivity of pairwise choices, formalized below.

No Cycle Condition. For every $x, y, z \in X$, if $x \in c(\{x, y\})$ and $y \in c(\{y, z\})$, then $x \in c(\{x, z\})$.

This No Cycle Condition is a requirement of their model (Theorem 1 in Ok, Ortoleva and Riella). However, together with Expansion and either Sen's α or Weak WARP, the No Cycle Condition implies WARP. Put differently in light of Proposition 2 and Observation 3, the only intersection between Ok, Ortoleva and Riella's model and either PE or PPE is the standard model.[7]

Proposition 4. *If c satisfies the No Cycle Condition, Expansion, and either Weak WARP or Sen's α, then c is induced by a complete and transitive binary relation.*

Proof. See the Appendix. □

Identifying reference-dependent preferences from choice. Observing a WARP violation provides substantial information about v. For example, if we see that $\{x\} = c(\{x, y\})$ but $\{y\} = c(\{x, y, z\})$ then we can infer that $v(x|x) > v(y|y), x, y \in PE_v(\{x, y\})$, but $v(z|x) > v(x|x)$. More generally, whenever we observe choice from

[7] In light of Proposition 1, the Weak WARP case of the result below extends Theorem 2 in Manzini and Mariotti (2007), which assumes that c is a single-valued choice function.

two nested sets where $x,y \in D \subset \bar{D}$ and we observe $x \in c(D)$ but $y \in c(\bar{D})$, we can infer that x is preferred to y according to the CPE ranking; if $x \notin c(\bar{D})$ we can additionally infer that there is some $z \in \bar{D} \backslash D$ for which $v(z|x) > v(x|x)$.[8]

In general, the binary choices induced by the PPE model will be intransitive. Whenever preferences inferred from binary choices are intransitive, there is necessarily a conflict between the the CPE ranking that select among PE and the reference-dependent preferences that determine the set of PE. An intransive choice pattern $\{x\} = c(\{x, y\})$, $\{y\} = c(\{y, z\})$, and $\{z\} = c(\{x, z\})$ alone does not tell us which choices are determined by the set of PE and which by the CPE ranking – unlike in the example in which a WARP violation is observed.

Comparison to axiomatic models of status-quo bias. The novelty of behavior in PPE can be seen by comparison with other models of reference-dependence. Masatlioglu and Ok (2005, 2014) take reference-dependent choice correspondences, $\{c(\cdot|x)\}_{x \in X \cup \{\diamond\}}$, where $c(\cdot|x)$ denotes a reference-dependent choice correspondence when x is the exogenously-given reference point, and $c(\cdot|\diamond)$ denotes the choice correspondence in the absence of a reference point. These papers assume that a decision-maker's reference point is given by her (observable) status-quo, unlike in the PPE approach. Both of these models require that WARP hold in the absence of an exogenously-given reference point, unlike PE and PPE models which determine endogenous reference points in such cases and allow WARP to be violated. These papers also consider axiomatic restrictions on reference-dependent which are, in principle, compatible with the PPE model.

An alternative and distinct interpretation of choice without a reference point in these models is that it ought to correspond to the CPE criterion, that is $x \in c(\{x,y\}|\diamond)$ corresponds to $v(x|x) \geq v(y|y)$. Masatlioglu and Ok (2014) propose the Weak Axiom of Status Quo Bias as a restriction on reference-dependent preferences that captures status-quo bias. Their axiom is translated to the setting of this paper (under this particular interpretation of $c(\cdot|\diamond)$) is below.

Weak Axiom of Status Quo Bias. For every $x,y \in X$, if $v(x|x) \geq v(y|y)$, then $v(x|x) \geq v(y|x)$.

[8] The identification exercise here relates to results for the RSM for single-valued choice functions in Dutta and Horan (2015).

This axiom is exactly the condition, stated in Observation 2 (ii), under which $CPE_v = PPE_v$.

Example 1 shows that Kőszegi-Rabin preferences fail the Weak Axiom of Status Quo Bias, indicating a behavioral distinction between their model of expectations-based reference-dependent choice and axiomatic models of status-quo bias as models of choice with reference-dependent preferences.[9]

Comparison from other non-transitive models. The PPE model is not the first to allow choices that violate transitivity. Most previous non-transitive models, like regret theory (Loomes and Sugden, 1982), provide representations of non-transitive binary relations. In contrast, the PPE model also makes predictions for arbitrary choice sets given the reference-dependent utility function. Without functional form assumptions, any comparison between the models is contingent on a particular implementation of regret theory to general choice sets.

Choice under risk and functional form. Kőszegi and Rabin (2006; 2007) emphasize applications that involve risks as do experimental tests of their model (e.g. Abeler et al., 2011; Ericson and Fuster, 2011). In a companion paper, Freeman (2016) explores the implications of a refinement of the PPE model under risk that captures leading cases of the Kőszegi-Rabin form as special cases. In the refinement he considers, expectations-based reference-dependent behavior is tied to failures of the Independence Axiom under risk; considering choice under risk allows reference-dependent preferences to be uniquely identified (unlike in the current paper). A predecessor of this paper (Freeman 2013) and Masatlioglu and Raymond (2014) discuss the link between a special case of the functional form in (5) under CPE and models and axioms in the non-expected utility literature. In the setting of choice among riskless consumption bundles, Blow, Crawford and Crawford (2015) characterize the testable implications of a class of loss averse preferences for both the case in which reference points unobserved and the case in which they are observed.

[9] One could similarly compare the Kőszegi-Rabin functional form to other restrictions on reference-dependent preferences considered in the axiomatic literature by Sagi (2006), Apesteguia and Ballester (2009), and others.

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Appendix

The following result extends Theorem 1 in Manzini and Mariotti (2007) to the case where c is allowed to be a multivalued choice correspondence.

Proposition 5. *Suppose c is a choice correspondence on the finite set X. Then c satisfies Expansion and Weak WARP if and only if there exist binary relations P_1 and P_2 such that $c = RSM_{P_1,P_2}$.*

García-Sanz and Alcantud (2015) offer a related result that extends the RSM model. Their version of Weak WARP is weaker than the version adopted in this paper. However, their representation theorem (their Theorem 1) is restricted to choice correspondences that satisfy their "choosing without dominated elements" property: if x is never chosen when y is available, then for any D with $x, y \in D$, $c(D \backslash \{x\}) = c(D)$. By adopting the (stronger) version of Weak WARP here, my Proposition 5 does not require this restriction.

Since I give a direct proof of Proposition 1, the result above is not an immediately corollary of Propositions 1 and 1. I provide a direct proof here.

Proof of Proposition 2

Necessity. Suppose $c = PPE_v$ for a given v.

Necessity of Expansion. Suppose $c = PPE_v$. If $x \in c(D) \cap c(D')$, then $v(x|x) \geq v(y|x)$ $\forall y \in D$ and $\forall y \in D'$. Thus $x \in PE_v(D \cup D')$. Also, $v(x|x) \geq v(y|y)$ for any y that satisfies $v(y|y) \geq v(z|y) \forall z \in D$ or $v(y|y) \geq v(z|y) \forall z \in D'$. Thus $v(x|x) \geq v(y|y)$ for all y that satisfy both $v(y|y) \geq v(z|y) \forall z \in D$ and $v(y|y) \geq v(z|y) \forall z \in D'$; thus $x \in PPE_v(D \cup D') = c(D \cup D')$.

Necessity of Weak WARP. Suppose there are sets $D, \bar{D}, D', \bar{D}'$ such that $\{x, y\} \subseteq D \subseteq \bar{D}$ and $\{x, y\} \subseteq D' \subseteq \bar{D}'$, and that $x \in c(D)$, $y \in c(\bar{D})$, $y \in c(D')$, $x \in c(\bar{D}')$.

Since $x \in c(\bar{D}')$, it follows that $x \in PE_v(D')$. Since $y \in PPE_v(D')$, it then follows that $v(y|y) \geq v(x|x)$. Since $y \in c(\bar{D})$ it follows that $y \in PE_v(D)$; but since $x \in PPE_v(D)$ and $v(y|y) \geq v(x|x)$, it follows that $y \in PPE_v(D) = c(D)$.

Necessity of Restricted Transitivity. Suppose xWy. Then, there exists a $D \supseteq \{x, y\}$ such that $y \in c(D)$. It follows that $y \in PE_v(\{x, y\})$, thus since $x \in PPE_v(\{x, y\})$, $v(x|x) \geq v(y|y)$.

Now take a chain $x^0, \ldots x^m$ such that $x^i W x^{i+1}$ for $i = 0, \ldots, m$; by the preceding argument, $v(x^0|x^0) \geq v(x^1|x^1) \geq \cdots \geq v(x^m|x^m)$. Thus $v(x^0|x^0) \geq v(x^m|x^m)$. If there exists a $D^{0,m} \supseteq \{x^0, x^m\}$ such that $x^0 \in c(D^{0,m}) = PPE_v(D^{0,m})$, then it follows that $x^0 \in PE_v(\{x^0, x^m\})$ and since $v(x^0|x^0) \geq v(x^m|x^m)$, $x^0 \in PPE_v(\{x^0, x^m\}) = c(\{x^0, x^m\})$ as well.

Sufficiency. Suppose c satisfies Expansion, Weak WARP, and Restricted Transitivity.

Define a candidate v. Define $\bar{W}$ as the transitive closure of W, that is, $x\bar{W}y$ if there exists a chain $x = x^0, \ldots, x^n = y$ with $x^0 W x^1, \ldots, x^{n-1} W x^n$.

For concreteness, for each $x \in X$ define $v(x|x)$ by

$$v(x|x) = |\{y \in X : x\bar{W}y\}|$$

and for each $y \in X \backslash \{x\}$, define $v(y|x)$ by

$$v(y|x) = \begin{cases} v(x|x) - 1 & \text{if } \exists D \in \mathscr{D} \text{ such that } y \in D \text{ and } x \in c(D) \\ v(x|x) + 1 & \text{otherwise} \end{cases}$$

It remains to show that $c(D) = PPE_v(D)$.

Representation when $|D| = 2$. Consider any $x, y \in X$. If $x \in c(\{x,y\})$, it follows that $v(x|x) \geq v(y|x)$, thus $x \in PE_v(\{x,y\})$. If there exists a $D^{x,y} \supseteq \{x, y\}$ such that $y \in c(\{x, y\})$, then by the definition of W, xWy, thus $v(x|x) \geq v(y|y)$, thus $x \in PPE_v(\{x, y\})$.

Conversely, suppose $x \in PPE_v(\{x, y\})$. Either there is no $\hat{D}^{x,y} \supseteq \{x, y\}$ such that $y \in c(\{x, y\})$, in which case $\{x\} = c(\{x, y\})$, or there exists some $\hat{D}^{x,y} \supseteq \{x, y\}$ such that $y \in c(D^{x,y})$; the argument that follows applies for the latter case. By the definition of v, $y \in PE_v(\{x, y\})$. Since $x \in PPE_v(\{x, y\})$, then $x \in PE_v(\{x, y\})$ as well. Since $x \in PPE_v(\{x, y\})$ but $\{x, y\} = PE_v(\{x, y\})$, by the definition of v, it follows that $|\{z \in X : x\bar{W}z\}| \geq |\{z \in X : y\bar{W}z\}|$. By non-emptiness of $c(\{x,y\})$, either $x \in c(\{x,y\})$ or $y \in c(\{x,y\})$ must hold. If $y \in c\{x,y\})$ then it follows that yWx, but then by Restricted Transitivity, $\{z \in X \backslash \{y\} : y\bar{W}z\} \supseteq \{z \in X \backslash \{x\} : x\bar{W}z\}$. Thus for $|\{z \in X : x\bar{W}z\}| \geq |\{z \in X : y\bar{W}z\}|$ to hold, we must have $\{z \in X : x\bar{W}z\} = \{z \in X : y\bar{W}z\}$, which implies $x \in c(\{x,y\})$.

Induction argument. Suppose we have established that $c(D) = PPE_v(D)$ whenever $|D| < n$. Consider an arbitrary D with $|D| = n$.

Induction step: first, show $PPE_v(D) \subseteq c(D)$. Suppose $x \in PPE_v(D)$.

Since $|D| = n$, $c(D \backslash \{z\}) = PPE_v(D \backslash \{z\})$ for each $z \in D$.

Case 1. If there exist distinct $y,z \in D\backslash\{x\}$ for which $x \in c(D\backslash\{y\})$ and $x \in c(D\backslash\{z\})$, then by Expansion, $x \in c(D)$.

Case 2. Suppose Case 1 doesn't hold.

Since $x \in PE_v(D)$, it follows that $x \in PE_v(D\backslash\{z\})$ for all $z \in D\backslash\{x\}$.

Since $c(D\backslash\{z\})$ must be non-empty, there exists some $w_z \in c(D\backslash\{z\})$ for each $z \in D\backslash\{x\}$.

Whenever $z \in D\backslash\{x\}$ but $x \notin c(D\backslash\{z\})$, since $w_z \in PPE_v(D\backslash\{z\})$, it follows that $v(w_z|w_z) > v(x|x)$. Then since $x \in PPE_v(D)$, it must also be the case that $v(z|w_z) > v(w_z|w_z)$.

If there exist distinct $y,z \in D\backslash\{x\}$ for which $c(D\backslash\{y\}) \cap c(D\backslash\{z\}) \neq \emptyset$, then we can take $w \in c(D\backslash\{y\}) \cap c(D\backslash\{z\})$. It follows by the PPE representation that applies to $D\backslash\{y\}$ and $D\backslash\{z\}$ that $v(w|w) > v(x|x)$ and $v(w|w) \geq v(a|w)\, \forall a \in D\backslash\{y\}$ and $\forall a \in D\backslash\{z\}$. Thus $w \in PE_v(D)$. But then since $v(w|w) > v(x|x)$, this contradicts that $x \in PPE_v(D)$. Thus $c(D\backslash\{y\}) \cap c(D\backslash\{z\}) = \emptyset$for any pair of distinct $y,z \in D\backslash\{x\}$.

If $x \notin c(D\backslash\{z\})\, \forall z \in D\backslash\{x\}$ then for each $w \in D\backslash\{x\}$, there exists a z such that $v(z|w) > v(w|w)$. But this implies that $PE_v(D\backslash\{x\}) = \emptyset$; since $\emptyset \neq c(D\backslash\{x\}) = PPE_v(D\backslash\{x\}) \subseteq PE_v(D\backslash\{x\}) = \emptyset$, this creates a contradiction. Conclude that there exists one $z \in D\backslash\{x\}$ for which $x \in c(D\backslash\{z\})$.

Take this $z \in D\backslash\{x\}$ for which $x \in c(D\backslash\{z\})$. Since $x \in PE_v(D)$, it follows by the choice of v that there exists a $D^{x,z} \ni \{x,z\}$ for which $x \in c(D^{x,z})$. By Expansion, $x \in c(D^{x,z} \cup (D\backslash\{z\}))$. Since $c(D)$ is non-empty, there exists a $y \in c(D)$. If $y = x$, then $x \in c(D)$; if $y \in D\backslash\{z\}$, then Weak WARP implies that $x \in c(D)$ as well, the desired conclusion. Now consider the case in which $y = z$. In this case, since $c(\{x,z\})$ is non-empty, either xWz or zWx (or both). However, in this case it also is the case that $z \in PE_v(D)$; Then since $x \in PPE_v(D)$, it follows by the definition of v that $|\{w \in X : x\bar{W}w\}| \geq |\{w \in X : z\bar{W}w\}|$. If zWx, then Restricted Transitivity implies that $\{w \in X : z\bar{W}w\} \supseteq \{w \in X : x\bar{W}w\}$, thus it must be the case that $\{w \in X : z\bar{W}w\} = \{w \in X : x\bar{W}w\}$, which implies xWz, which in turn requires $x \in c(\{x,z\})$. But in this case $x \in c((D\backslash\{z\}) \cup \{x,z\}) = c(D)$ follows by Expansion.

Induction step: second, show $c(D) \subseteq PPE_v(D)$. Suppose $x \in c(D)$.

By the definition of v, it follows that $v(x|x) \geq v(y|y)\, \forall y \in D$. Thus $x \in PE_v(D)$.

Now suppose that there exists some $y \in D$ for which $v(y|y) > v(x|x)$. By the definition of v, this is equivalent to the restriction that $|\{w \in X : y\bar{W}w\}| > |\{w \in X : x\bar{W}w\}|$.

If $x \in c(\{x,y\})$, then either there exists a $D^{xy} \supseteq \{x,y\}$ for which $y \in c(D^{xy})$ or no such D^{xy} exists. If such a D^{xy} exists, it follows that xWy. Then by Restricted Transitivity, $x\bar{W}z$ for each z for which $y\bar{W}z$; but then $|\{w \in X : y\bar{W}w\}| \leq |\{w \in X : x\bar{W}w\}|$, a contradiction of our original assumption. If no such D^{xy} exists, then $v(x|y) > v(y|y)$ by the definition of v.

So now suppose $\{y\} = c(\{x,y\})$. If for each $z \in D$ there exists a $D^{y,z} \supseteq \{y,z\}$ for which $y \in c(D^{y,z})$, then $y \in c(\bigcup_{z\in D} D^{y,z})$ follows by repeatedly applying Expansion. Since both (a) $y \in c(\{x, y\})$, $x \in c(D)$, and $\{x, y\} \subseteq D$, and (b) $y \in c(\bigcup_{z\in D} D^{y,z})$, $x \in c(D)$, and $\{x, y\} \subseteq D \subseteq \bigcup_{z\in D} D^{y,z}$, it follows by Weak WARP that $y \in c(D)$. But since both (a) $x \in c(D)$, $y \in c(\bigcup_{z\in D} D^{y,z})$, and $D \subseteq \bigcup_{z\in D} D^{y,z}$, and (b) $x \in c(D)$, $y \in c(\{x, y\})$, and $\{x, y\} \subseteq D$, it follows by Weak WARP that $x \in c(\{x, y\})$; this contradicts that $\{y\} = c(\{x, y\})$. Conclude that for any $y \in D$ for which $v(y|y) > v(x|x)$, there must exist some $w \in D$ such that there is no $D^{w,z} \supseteq \{w, z\}$ for which $z \in c(D^{w,z})$; thus by the definition of v, $v(w|y) > v(y|y)$.

We have shown that for any $y \in D$ for which $v(y|y) > v(x|x)$ that there exists a $z \in D$ for which $v(z|y) > v(y|y)$. It follows that $x \in PPE_v(D)$.

□

Proof of Proposition 3.

Take a c that satisfies the No Cycle Condition and the Expansion axiom. Define the binary relation R by xRy if $x \in c(\{x,y\})$. The relation R is (by construction) complete, and by the No Cycle Condition, is transitive. By completeness and transitivity of R, for each $D \in \mathscr{D}$, $\exists x \in D$ for which $xRy \forall y \in D$. For such an x, it follows by the definition of R and by repeatedly applying Expansion that $x \in c(D)$. Next we consider two cases: first where c is assumed to satisfy Sen's α, second where c is assumed to satisfy Weak WARP.

First, suppose that c also satisfies Sen's α (as well as the No Cycle Condition and Expansion). Then for any $D \in \mathscr{D}$ and for any $x,y \in D$, we have that $x \in c(D)$ implies (by Sen's α) that $x \in c(\{x,y\})$. Thus for any $D \in x \in c(D)$ implies xRy for all $y \in D$. By our preceding result, $c(D) = m(D,R)$ for all $D \in \mathscr{D}$. This proves the result for the case in which c satisfies Sen's α.

Second, suppose c also satisfies Weak WARP (as well as the No Cycle Condition and Expansion). Recall that for any $D \in \mathscr{D}$ there exists an $x \in c(D)$ for which $x \in c(\{x,y\})$ $\forall y \in D$. Then if $y \in c(D)$ as well, Weak WARP implies we have $y \in c(\{x,y\})$ and thus yRx. Then since xRz for all $z \in D$, by transitivity, yRz for all $z \in D$. Thus $c(D) = m(D,R)$ for all $D \in \mathscr{D}$. This proves the result for the case where c satisfies Weak WARP.

□

Proof of Proposition 4.

Necessity. Let P_1, P_2 be relations and suppose $c(D) = m(m(D, P_1), P_2)$ $\forall D \in \mathscr{D}$.

Necessity of Expansion. Suppose $x \in c(D) \cap c(D')$, or equivalently, $x \in m(m(D, P_1), P_2)$ and $x \in m(m(D', P_1), P_2)$.

Then, $\nexists y \in D, D'$ such that yP_1x, thus $x \in m(D \cup D', P_1)$.

Also, $\nexists y \in m(D, P_1), m(D', P_1)$ such that yP_2x.

By the definition of m, $m(D \cup D', P_1) \subseteq m(D, P_1) \cup m(D', P_1)$. Thus, $x \in m(m(D \cup D', P_1), P_2) = c(D \cup D')$.

Necessity of Weak WARP. Suppose there are sets $D \subseteq D'$ and $\bar{D} \subseteq \bar{D}'$, each containing $\{x,y\}$, and $x \in c(D)$, $y \in c(D')$, $y \in c(\bar{D})$ and $x \in c(\bar{D}')$.

Then, since $y \in c(D') \subseteq m(D', P_1)$ and $y \in D \subseteq D'$, it follows that $y \in m(m(D, P_1), P_2)$.

Since $x \in c(D) = m(m(D, P_1), P_2)$, it follows that either xP_2y, or neither xP_2y nor yP_2x.

Since $x \in c(\bar{D}') \subseteq m(\bar{D}', P_1)$ and $x \in \bar{D} \subseteq \bar{D}'$, it follows that $x \in m(\bar{D}, P_1)$.

Thus we have $\{x, y\} \subseteq m(\bar{D}, P_1)$. Since $x \in m(\{x,y\},P_2)$ and the asymmetric part of P_2 must be acyclic when restricted to $m(\bar{D}, P_1)$ in order to have $m(m(\bar{D},P_1),P_2) \neq \emptyset$, it follows that $x \in m(m(\bar{D}, P_1), P_2) = c(\bar{D})$. Thus Weak WARP holds.

Sufficiency. Suppose c satisfies Expansion and Weak WARP.

Define a candidate representation. Define P_1 by xP_1y if $\nexists D_{xy} \supseteq \{x, y\}$ such that $x \in c(D_{xy})$. Define P_2 by xP_2y if $x \in c(\{x, y\})$.

Representation when $|D| = 2$. By the definition of P_1, $\forall D \in \mathscr{D}$, $c(D) \subseteq m(D, P_1)$; this will be invoked repeatedly below.

The definitions of P_1 and P_2 immediately yield that $\forall x, y \in X$, $c(\{x, y\}) = m(m(\{x, y\}, P_1), P_2)$.

Induction argument. Now suppose that we have established that $c(D) = m(m(D, P_1), P_2)$ $\forall D \in \mathscr{D}$ s.th. $|D| < n$.

Take any $D \in \mathscr{D}$ with $|D| = n$. We will show that $m(m(D, P_1), P_2) \subseteq c(D)$, then that $c(D) \subseteq m(m(D, P_1), P_2)$.

The desired result will then follow by induction.

Induction step: show $RSM_{P_1,P_2}(D) \subseteq c(D)$. First suppose that $x \in m(m(D, P_1), P_2)$.

Since $|D| = n$, for each $z \in D$, $c(D \backslash z) = m(m(D \backslash z, P_1), P_2)$.

Thus, for each $z \in D \backslash x$, either (a) $x \in m(m(D \backslash z, P_1), P_2) = c(D \backslash z)$ or (b) $\exists w \in c(D)$ such that zP_1w but wP_2x.

For for all $z \in D \backslash x$ such that (b) holds, we can construct a chain $z = z_1, z_2, ..., z_m$ of (not necessarily distinct) elements in $D \backslash x$ such that $z_iP_1z_{i+1}$ for $i = 1, \ldots, m-1$. If we have any element z' repeated in the chain, then there exist $i < j$ such that $m(\{z_i, \ldots, z_j\}, P_1) = \emptyset$. But then since $c(D) \subseteq m(D,P_1)$, $\emptyset \neq c(\{z_i, \ldots, z_j\}) \subseteq m(\{z_i, \ldots, z_j\}, P_1) = \emptyset$, a contradiction.It thus follows that all of the z_i's must be distinct. If (b) held for all $z \in D \backslash x$, then we would require at least $n > |D \backslash x|$ elements in our chain, which is impossible. It follows that there must exist at least one $z \in D$ such that $x \in c(D \backslash z)$.

Now suppose (a) holds for exactly one $z \in D$. Take the chain $z_1, ..., z_{n-2}, z_{n-1} = z$ contained in $D \backslash x$ constructed as before. By construction, we have $m(D, P_1) \subseteq \{x, z_1\}$.

If $\{x\} \neq c(D)$, then since $\emptyset \neq c(D) \subseteq m(D, P_1) \subseteq \{x, z_1\}$, we must have $z_1 \in c(D)$ and $x \in c(\{x, z_1\})$. Since $x \in m(D, P_1)$, $\exists D_{xz} \ni z$ such that $x \in c(D_{xz})$. Since $x \in c(D \backslash z)$, by Expansion, $x \in c(D_{xz} \cup (D \backslash z))$. By Weak WARP, since $x \in c(\{x, z_1\})$, it follows that $x \in c(D)$ as well, the desired conclusion.

If instead there exists more than one $z \in D \backslash x$ for which (a) holds, then take distinct $z, z' \in D \backslash x$ such that $x \in c(D \backslash z) \cap c(D \backslash z')$. Then $x \in c(D)$ follows by Expansion.

It follows that $x \in c(D)$; thus $m(m(D, P_1), P_2) \subseteq c(D)$.

Induction step: show $c(D) \subseteq RSM_{P_1,P_2}(D)$. Take any $x \in c(D)$.

Since $c(D) \subseteq m(D, P_1)$, we have $x \in m(D, P_1)$.

Suppose now we have some $z \in D$ such that $z = m(\{x,z\}, P_2)$. Suppose further that $\forall y \in D$, $\exists D_{yz} \supseteq \{y,z\}$ such that $z \in c(D_{yz})$.

Then by Expansion, $z \in c(\underset{y\in D}{\cup} D_{yz})$. Then since $\{z\} = m(\{x,z\}, P_2)$, we have $\{z\} = c(\{x, z\})$. But by Weak WARP, since $x \in c(D)$, we have $z \in c(D)$. But by Weak WARP again, since $x \in c(D)$ and $z \in c(D)$ (and $D \subseteq D$), while $z \in c(\{x,z\})$ (with $\{x,z\} \subseteq D$), it follows that $x \in c(\{x,z\}) = \{z\}$, a contradiction. Thus there cannot be any $z \in D$ for which both $\{z\} = m(\{x,z\}, P_2)$ and $z \in m(D, P_1)$. It follows that ;. thus $c(D) \subseteq m(m(D, P_1), P_2)$.

□