Calibration without Reduction for Non-Expected Utility

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Abstract

Calibration results in Rabin (2000) and Safra and Segal (2008; 2009) suggest that both expected and non-expected utility theories cannot produce non-negligible risk aversion over small stakes without producing implausible risk aversion over large stakes. This paper provides calibration results for recursive non-expected utility theories that relax the Reduction of Compound Lotteries axiom (as in Segal 1990). These calibration results imply that a broad class of non-expected utility theories can accommodate both small and large stakes risk aversion, even for a decision-maker who faces background risk.

Keywords: risk aversion, calibration, non-expected utility theories, recursive preferences.

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1 Introduction

Rabin's [19] calibration theorem shows that a risk averse expected utility decision-maker who rejects an actuarially favorable gamble over small stakes for a wide range of initial wealth levels must also reject extremely favorable gambles over larger stakes. For example, a decision-maker (DM) who would reject a 50-50 gamble between a loss of $100 and a gain of $110 at any initial wealth level must also reject a 50-50 gamble between a loss of $1000 and an infinite gain. But while we see people rejecting risks like the 50-50 lose $100/gain $110 gamble, we also see people taking risks that are much less favorable than the 50-50 lose $1000/gain infinity gamble. Expected utility's inability to simultaneously accommodate descriptively reasonable levels of risk aversion over different sizes of stakes limits its applicability as a descriptive model.

Rabin [19] suggests that non-expected utility theories are not susceptible to his theorem. However, Safra and Segal [22, 23] show that if a DM faces background risk and evaluates an offered gamble by merging the offered gamble and her background risk into a single-stage lottery by applying the laws of probability, then non-expected utility theories that define utility over final wealth are susceptible to a similar critique.[1]

If we view the offered gamble in the presence of background risk as a two-stage compound lottery, then this assumption used in Safra and Segal [22, 23] is equivalent to the Reduction of Compound Lotteries (ROCL) axiom in their setting. Experimental evidence suggests that many people violate ROCL (e.g. Halevy [14]), which motivates the use of recursive preferences over multi-stage lotteries following Segal [24]. Artstein-Avidan and Dillenberger [1] point out that Safra and Segal's [22, 23] results do not apply when a DM has recursive non-expected utility (RNEU) preferences and views the risk in an offered gamble as resolving at a distinct stage from her background risk. However, they do not explore what alternative results do apply in this case. This paper's contribution is to provide calibration results that clarify the connection between small- and large- stakes risk aversion under RNEU with background risk.

This paper considers a DM with non-expected utility preferences defined over lot-

[1] See also Barberis, Huang, and Thaler [2, Section II], which makes a related argument.

teries over final wealth levels who faces background risk, as in Safra and Segal's [22] analysis of 'non-smooth' non-expected utility theories. This paper assumes that the DM views an offered gamble in the presence of background risk as a two-stage lottery with the offered gamble resolving first, and that the DM evaluates compound lotteries recursively, as in Segal [24]. The main theoretical results of this paper show that under these assumptions, the DM will behave as-if she engages in 'narrow bracketing' – her evaluation of an offered gamble is independent of her background risk – if her single-stage lottery preferences satisfy Constant Absolute Risk Aversion (Theorem 1) or for small stakes if her preferences satisfy a differentiability requirement (Theorems 2, 3). These results elucidate a relationship between RNEU and 'as-if' narrow bracketing conjectured earlier by Dillenberger [10, p. 1976]. Corollary 1 characterizes the calibration implications of small-stakes risk aversion for large-stakes risk aversion for a range of non-expected utility theories. It implies that recursive versions of the major classes of non-expected utility preferences are immune to unreasonable calibration arguments à la Rabin. Section 3 quantitatively calibrates a version of the model and shows it can provide descriptively reasonable risk aversion.

2 Theory: RNEU risk preferences with background wealth risk

2.1 Non-expected utility over single-stage lotteries

Let $W = \mathbb{R}_+$ denote the set of feasible final wealth levels, let $\Delta(W)$ denote the set of all finite-support probability distributions over W, and refer to $\Delta(W)$ as the set of (one-stage) *lotteries* over W. A one-stage lottery over W can be written as $q = [w_1, q_1; ...; w_m, q_m] \in \Delta(W)$, where q_i denotes the probability of receiving prize w_i; for such lotteries, adopt the convention that $w_1 \leq ... \leq w_m$. Given $q \in \Delta(W)$, let F_q denote the cumulative distribution function (CDF) of q, where $F_q(w)$ denotes the probability that q gives an outcome weakly less than w. The notation $\tilde{w} \in \Delta(W)$ will be used to denote a lottery that captures uninsurable wealth risk. Given any two lotteries $p = [w_1, p_1; ...; w_m, p_m]$ and $q = [w_1, q_1; ...; w_m, q_m]$ in $\Delta(W)$ and any $\alpha \in (0, 1)$, the standard mixture operation is denoted $(1 - \alpha)p + \alpha q$ and is given by the lottery $(1-\alpha)p+\alpha q=[w_1,(1-\alpha)p_1+\alpha q_1;...;w_m,(1-\alpha)p_m+\alpha q_m]$. With some abuse of notation, when $q\in\Delta(W)$ and $y\in\mathbb{R}$ let $q+y=[w_1+y,q_1;...;w_m+y,q_m]$.

Consider a utility function over one-stage lotteries $V:\Delta(W)\to\mathbb{R}$. Assume V is increasing in the sense of first-order stochastic dominance and let c denote the associated certainty equivalent function defined implicitly by the relationship $V(q)=V([c(q),1])\ \forall q\in\Delta(W)$. Say that V is *risk averse* if it is averse to mean-preserving spreads.

The following classes of non-expected utility preferences are examples included in this setup.

Rank-dependent utility (RDU). (Quiggin [18], Yaari [26]). V belongs to the RDU class if V can be expressed as $V(q)=\sum\limits_{i=1}^{n}[g(\sum\limits_{j=1}^{i}q_j)-g(\sum\limits_{j=1}^{i-1}q_j)]u(w_i)$. The function $g:[0,1]\to[0,1]$ is called a *probability weighting function*, and is required to be a strictly increasing function that satisfies $g(0)=0$ and $g(1)=1$. The function $u:W\to\mathbb{R}$ is a strictly increasing utility-for-wealth function.

Betweenness. (Dekel [9], Chew [6]). V belongs to the *betweenness* class if V is given implicitly by $V(q)=\sum\limits_{i=1}^{n}q_i v(w_i,V(q))$, where $v:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is increasing in its first argument and continuous in its second argument.

Constant Absolute Risk Aversion (CARA). (Safra and Segal [20]). V exhibits *constant absolute risk aversion* if its corresponding certainty equivalent function c satisfies $c([w_1+a,q_1;...;w_n+a,q_n])=c(q)+a$ for any $a\geq -w_1$.

Constant Risk Aversion. (Safra and Segal [20]). V exhibits *constant risk aversion* if its corresponding certainty equivalent function c satisfies $c([\lambda w_1+a,q_1;...;\lambda w_n+a,q_n])=\lambda c(q)+a$ for any $\lambda>0$ and $a\geq -\lambda w_1$.

Gateaux differentiable utility. (Chew, Karni, and Safra [7]). V is *Gateaux differentiable* if for any $q\in\Delta(W)$, there exists a function $u_q:W\to\mathbb{R}$ such that for each $p\in\Delta(W)$,

$$\lim_{\alpha \to 0^+} \frac{V((1-\alpha)q + \alpha p) - V(q)}{\alpha} = \sum_{i=1}^{m} (p_i - q_i)\, u_q(w_i)\,.$$

V is said to be *continuously Gateaux differentiable* if u_q is continuous as a function of q.

First-order risk averse. (Segal and Spivak [25]). c is *first-order risk averse* at wealth w if for any gamble $\hat{p}^t == (ty_1, p_1; ...; ty_n, p_n)$ with an expected value of zero, $\frac{dc(w+\hat{p}^t)}{dt}|_{t=0^+} < 0$.

Example 1. Let V be a RDU function with probability weighting function $g(p) = \sqrt{p}$ and a linear utility-for-wealth function $u(w) = w$. Given wealth level $w \geq \$10$, the DM evaluates a 50-50 lose \$10/gain \$20 gamble according to:

$$\begin{aligned} V([w-10,\ .5;\ w+20,\ .5]) &= g(.5)\,(w-10) + (1-g(.5))\,(w+20) \\ &= \sqrt{.5}\,(w-10) + \left(1-\sqrt{.5}\right)(w+20) \\ &< w \\ &= V([w,\ 1])\,. \end{aligned}$$

Thus the DM would turn down this gamble.

2.2 Non-expected utility and compound lotteries

Define a (two-stage) *compound lottery* as a finite lottery over lotteries over final wealth levels. A compound lottery can be written as $Q = [q^1, p_1; ...; q^n, p_n]$ where $q^i \in \Delta(W)$ and p_i is the probability of receiving lottery q^i; let $\Delta(\Delta(W))$ denote the set of compound lotteries. Let U denote a utility function over compound lotteries.

Any compound lottery $Q = [q^1, p_1; ...; q^n, p_n]$ can be reduced to a single-stage lottery by multiplying through the first and second stage probabilities of each prize to obtain the one-stage lottery $Q^R = [w_1, \sum_{i=1}^{n} p_i q_1^i; ...; w_K, \sum_{i=1}^{n} p_i q_K^i]$. Say that the DM with single-stage lottery preferences given by c satisfies the *Reduction of Compound*

Lotteries axiom if she evaluates each $Q \in \Delta(\Delta(W))$ according to a U^{ROCL} that satisfies:

$$U^{ROCL}(Q) = c(Q^R)\,. \tag{1}$$

The DM might alternatively apply her single-stage lottery preferences to a compound lottery recursively. That is, the DM could evaluate $Q = [q^1, p_1; ...; q^n, p_n]$ by first applying c to each q^i to obtain the one-stage lottery $[c(q^1), p_1; ...; c(q^n), p_n]$, to which she could apply c a second time. This assumes that the DM applies the same single-stage lottery preferences at each stage of the compound lottery, as in Segal's [24] Time Neutrality axiom; this assumption is not essential for the conclusions.[2] Say that the DM with non-expected utility single-stage lottery preferences given by c evaluates compound lotteries according to *Recursive Non-Expected Utility* if:

$$U(Q) = c([c(q^1), p_1; ...; c(q^n), p_n])\,. \tag{2}$$

Segal [24] introduced RNEU over compound lotteries to provide a consequentialist alternative to prospect theory that captures Kahneman and Tversky's [16] "isolation effect" (a particular example of a failure of ROCL). Halevy [14] finds that 80% of subjects violate ROCL, while 59% of subjects' choices are best explained by RNEU. Previous experimental work also found substantial violations of ROCL that suggest the use of RNEU preferences (e.g. Carlin [4] and Camerer and Ho [3]).

2.3 Risk-taking given pre-existing wealth risk as a compound lottery

Define a *gamble* as a finite lottery over gain and loss prizes (as opposed to wealth levels). Consider a DM who faces background wealth risk described by the one-stage lottery $\tilde{w} = [w_1, q_1; ...; w_m, q_m]$, which is not the subject of choice. The DM is offered the gamble over prizes $\hat{p} = (y_1, p_1; ...; y_n, p_n)$ where $y_i \in \mathbb{R}$ is a monetary prize added to or taken away from her final wealth after gamble $\hat{p}$ resolves.

[2] Alternatively, we could have DM apply a different c_t at stage t of a compound lottery. If each c_t satisfies our assumptions, then versions of the results here will go through with straightforward modifications.

Let $\hat{p} \oplus \tilde{w}$ denote the compound lottery formed by simple gamble over prizes $\hat{p}$, which resolves at the first stage, and independent background risk $\tilde{w}$, which resolves at the second stage. The compound lottery $\hat{p} \oplus \tilde{w}$ is given by:

$$\hat{p} \oplus \tilde{w} = [\tilde{w} + y_1, p_1; ...; \tilde{w} + y_n, p_n] \tag{3}$$

where $\tilde{w} + y_i = [w_1 + y_i, q_1; ...; w_m + y_i, q_m]$ denotes the lottery over final wealth states that the DM faces if prize y_i is won in the gamble $\hat{p}$. The compound lottery $\hat{p} \oplus \tilde{w}$ is well defined whenever $w + y_i \in W$ for each w in the support of $\tilde{w}$ and each y_i in the support of $\hat{p}$.

People face substantial risks in their lives (Guiso, Jappelli, and Pistaferri [12]). For a DM offered a one-time gamble independent of her pre-existing risks, the combination of a one-time gamble (like those offered in lab experiments) and background wealth risk constitutes a compound lottery composed of two distinct and independent sources of risk, as in (3).

Example 2. Take V from Example 1, but suppose now that the DM faces pre-existing wealth-risk in the form of the lottery $\tilde{w} = [w_1,\ .25;\ w_2,\ .25;\ w_3,\ .25;\ w_4,\ .25]$ with $10 < w_i < w_{i+1} - 30$ for $i = 1, 2, 3$. Let $\hat{p} = (-10,\ .5;\ 20,\ .5)$ as in Example 1. If the DM evaluates compound lotteries recursively, she would evaluate $\hat{p}$ given background risk $\tilde{w}$ according to

$$\begin{aligned}
U(\hat{p} \oplus \tilde{w}) &= c([c(\tilde{w} - 10),\ .5;\ c(\tilde{w} + 20),\ .5]) \\
&= g(.5) \sum_{i=1}^{4} [g(.25i) - g(.25i - .25)] (w_i - 10) \\
&\quad + (1 - g(.5)) \sum_{i=1}^{4} [g(.25i) - g(.25i - .25)] (w_i + 20)) \\
\text{for } g(p) = \sqrt{p}, &< \sum_{i=1}^{4} [g(.25i) - g(.25 (i - 1))] \, w_i \\
&= c(\tilde{w})
\end{aligned}$$

thus the DM would reject $\hat{p}$. The analysis in this case closely resembles the case without background risk.

2.4 Reduction of Compound Lotteries and Safra and Segal (2008)

Below, I provide an interpretation of Safra and Segal's [22, 23] analysis of the problem, and offer a summary of their results. Suppose for now that the DM evaluates an offered gamble $\hat{p}$ in the presence of background risk $\tilde{w}$ according to (3), and satisfies ROCL. Then she turns down gamble $\hat{p}$ whenever $c((\hat{p}\oplus\tilde{w})^R) \leq c(\tilde{w})$. Notice that $(\hat{p}\oplus\tilde{w})^R$ is the lottery Safra and Segal [22, 23][3] assume the DM evaluates when deciding whether to accept $\hat{p}$ given background risk $\tilde{w}$. Under this interpretation of the problem of risk-taking with background risk, Safra and Segal [22, 23] implicitly assume that a DM satisfies ROCL. Proposition 0 summarizes the main economic implication of Safra and Segal's [22, 23] results that apply to both continuously Gateaux differentiable and various non-differentiable non-expected preferences.

Proposition 0. *Suppose that a DM evaluates an offered gamble $\hat{p}$ given background risk $\tilde{w}$ according to $(\hat{p}\oplus\tilde{w})^R$ and c is the certainty equivalent function. Then for $\hat{p} = (-l, .5; +g, .5)$ with $g > l$, it cannot be the case that $c((\hat{p}\oplus\tilde{w})^R) \leq c(\tilde{w})$ for all $\tilde{w}$ with support in $[a, b]$ unless c also rejects "extremely favorable" gambles.*

Proof. See Theorem 3 in Safra and Segal [22], which makes the term "extremely favorable" precise. □

Many preference functionals studied in the non-expected utility literature are either (i) 'smooth' and thus subject to an extension of Rabin's critique covered in Theorem 2 of Safra and Segal [22] which applies even when the DM faces no background risk, or (ii) covered by Proposition 0. This led Safra and Segal to conclude that a descriptively reasonable model of risk aversion must drop the final wealth assumption. The rest of this paper shows that very different conclusions follow when ROCL is replaced with RNEU.

[3] This is in the definition of "Stochastic B3" in Safra and Segal [22], and its weaker version in Safra and Segal [23].

Example 3. Take V from Example 1, and revisit the decision in Example 2, but suppose now that the DM reduces compound lotteries, evaluating them according to a U^{ROCL} that satisfies (1). Then she would evaluate $\hat{p}$ given background risk $\tilde{w}$ (from Example 2) according to

$$\begin{aligned} U^{ROCL}(\hat{p} \oplus \tilde{w}) &= V((\hat{p} \oplus \tilde{w})^R) \\ &= \sum_{i=1}^{4} \Big([g(.25i - .125) - g(.25i - .25)]\,(w_i - 10) \\ &\qquad + [g(.25i) - g(.25i - .125)]\,(w_i + 20)\Big) \\ \text{for } g(p) = \sqrt{p}, \quad &> \sum_{i=1}^{4} [g(.25i) - g(.25\,(i-1))]\,w_i \\ &= c(\tilde{w}) \end{aligned}$$

thus the DM would accept $\hat{p}$. The analysis in this case differs from Examples 1 and 2. This is because the background risk mostly determines the rank of outcomes of the merged lottery $(\hat{p} \oplus \tilde{w})^R$. This attenuates the impact of non-linear probability weighting on the DM's risk aversion over the offered gamble $\hat{p}$.

3 Non-reduction, as-if narrow bracketing, and small-stakes risk aversion

RNEU does not assume that a gamble is bracketed narrowly. However, Theorem 1 demonstrates that when c is in the CARA class, an RNEU DM behaves as if she brackets narrowly: that is, her choices among offered gambles are independent of the background risk she faces.

Theorem 1. *Suppose a recursive non-EU decision-maker treats an offered gamble in the presence of background risk as a compound lottery as in (3), and has lottery preferences that satisfy Constant Absolute Risk Aversion. Then c has a unique extension to $\Delta(\mathbb{R})$, $\hat{c}$, such that, $U(\hat{p} \oplus \tilde{w}) = \hat{c}(\hat{p}) + c(\tilde{w})$ represents preferences whenenver $\hat{p} \oplus \tilde{w} \in \Delta(\Delta(W))$.*

Proof. For any $q \in \Delta(\mathbb{R})$ with support bounded from below, define $\underline{w} = -\inf\{\text{support } q\}$, and extend c to lotteries over $\mathbb{R}$, $\Delta(\mathbb{R})$, by defining $\hat{c}(q) = c(q+\underline{w})-\underline{w}$. Since c satisfies CARA, this extension to $\hat{c}$ is unique. Under RNEU,

$$\begin{array}{lrcl} & U(\hat{p} \oplus \tilde{w}) & = & c([c(\tilde{w}+y_1), p_1; ...; c(\tilde{w}+y_n), p_n]) \\ \text{by CARA,} & & = & c([c(\tilde{w})+y_1, p_1; ...; c(\tilde{w})+y_n, p_n]) \\ \text{by CARA and using } \hat{c}, & & = & \hat{c}(\hat{p})+c(\tilde{w})\,. \end{array}$$

□

Theorem 1 shows that when c satisfies Constant Absolute Risk Aversion, background risk does not affect how a DM evaluates an offered gamble under RNEU. Thus CARA versions of non-expected utility theories – including RDU and Gul's [13] disappointment aversion with linear utility-for-wealth – that avoid ridiculous calibration results in the absence of background risk are also immune to such calibration results under RNEU.

These linear utility-for-wealth versions of RDU and of Gul's [13] disappointment aversion also belong to the class of constant risk averse preferences. Corollary 1 shows that for the constant risk averse class of non-expected utility preferences, tight calibration implications can be drawn from turning down a gamble $\hat{p}$, but these implications seem reasonable.

Corollary 1. *Suppose c satisfies Constant Risk Aversion. For $\hat{p}^t = (ty_1, p_1; ...; ty_n, p_n)$, $U(\hat{p}^t \oplus \tilde{w}) = t\hat{c}(\hat{p}) + c(\tilde{w})$ represents preferences whenenver $\hat{p}^t \oplus \tilde{w} \in \Delta(\Delta(W))$, where $\hat{c}$ is the extension of c from Theorem 1.*

An implication of Corollary 1 is that whenever we know that lottery preferences satisfy Constant Risk Aversion, then the DM turns down $\hat{p}$ at $\tilde{w}$ at which $U(\hat{p} \oplus \tilde{w})$ is well defined if and only if she turns down $\hat{p}^t$ for all $\tilde{w}$ and for all $t > 0$. So suppose a DM would turn down $\hat{p} = (-10, .5; +11, .5)$ given any distribution of background risk. Corollary 1 says that if c is constant risk averse, then the RNEU will also turn down $\hat{p} = (-10t, .5; +11t, .5)$ for all $t > 0$ at all $\tilde{w}$, but no stronger conclusion is possible.

Many classes of preferences considered in the non-expected utility literature contain a constant risk averse special case that can generate small-stakes risk aversion – including the class of RDU preferences, betweenness preferences, and Gateaux differentiable preferences. Corollary 1 shows that these classes of preferences are immune to a calibration critique, since each class of preferences contains constant risk averse special cases in which observing a DM turn down $\hat{p}^t = (ty_1, p_1; ...; ty_n, p_n)$ at small t only implies that this DM also turns down $\hat{p}^t$ at large t. This conclusion hardly seems unreasonable for a DM who rejects Rabin's gambles.[4]

Typical applications of non-EU preferences do not assume Constant Risk Aversion, but rather allow for diminishing marginal utility of wealth. However, one might expect that over small stakes, non-expected utility preferences tend to behave like constant risk averse preferences since u is almost linear locally whenever u is differentiable. To formalize this intuition, I introduce the notion of dual differentiability for a certainty equivalent function c based on the idea of taking a Gateaux derivative with respect to comonotonic mixtures in the sense of Yaari [26]. In Appendix B, I provide a set of sufficient conditions for the dual differentiability of c. I show that RDU is dually differentiable under mild conditions, and also provide sufficient conditions for dual differentiability for betweenness preferences.

Given $q \in \Delta(W)$, let $F_q^{-1} : [0,1] \to W$ denote the inverse of the CDF q, that is, $F_q^{-1}(p) = \min\{w : \ F_q(w) \geq p\}$; let I denote the set of all inverse CDFs.[5] For $q, r \in \Delta(W)$, define the *dual mixture operation* $(1-\lambda)q \boxplus \lambda r$ as yielding the lottery s whose inverse CDF is given by $F_s^{-1} = (1-\lambda)F_q^{-1} + \lambda F_r^{-1}$. Say that a non-expected utility function V is *dually differentiable* if for each $q \in \Delta(W)$, there exists a linear functional $dV(q) : I \to \mathbb{R}$ such that $V((1-\lambda)q \boxplus \lambda r) - V(q) = \lambda(dV(q))(F_q^{-1} - F_r^{-1}) + o(\lambda)$.

Theorem 2 shows that for dually differentiable preferences, behavior over small-stakes gambles given a fixed distribution of background wealth risk is well approximated by a constant risk averse certainty equivalent function under RNEU.

Theorem 2. *If c is dually differentiable then for each $\tilde{w} \in \Delta(W)$ there exists a*

[4] Safra and Segal [21, Proposition 1] show that the only c that is constant risk averse but not first-order risk averse is the expected value function. So this discussion only implies immunity to a calibration critique for classes of preferences that contain a first-order risk averse subclass, and does not apply, for example, to the class of Fréchet differentiable preferences studied by Machina [17].

[5] The mappings $q \mapsto q^{-1}$ and $q^{-1} \mapsto q$ are one-to-one, which allows for the analysis that follows.

constant risk averse certainty equivalent function on $\Delta(\mathbb{R})$, $\hat{c}$, such that for any $a \in \mathbb{R}$, $U(\hat{p}^t \oplus \tilde{w}) = t\hat{c}(\hat{p}+a) - ta + c(\tilde{w}) + o(t)$. Moreover, $\hat{c}$ is first-order risk averse if and only if c is.

Proof. See Appendix A. □

Theorem 2 shows that an RNEU DM's preferences over offered gambles can be well approximated by a constant risk averse preference, at least over small stakes and under a dual differentiability assumption. Moreover, this approximating constant risk averse preference $\hat{c}$ will inherit first-order risk aversion from the DM's underlying lottery preferences. Non-negligible risk aversion over small stakes is the defining feature of first-order risk aversion. Theorem 2 tells us that this defining feature is still behaviorally evident for a RNEU DM who faces background risk, when c is dually differentiable. Theorem 3 shows that this result extends to the case in which c is not necessarily dually differentiable, but satisfies mild regularity conditions.

Theorem 3. *Suppose c is first-order risk averse, $\lim_{y\to 0^+} \frac{1}{y}\left[c(\tilde{w}+y) - c(\tilde{w})\right] \leq \lim_{y\to 0^-} \frac{1}{y}\left[c(\tilde{w}+y) - c(\tilde{w})\right]$, and $\lim_{t\to 0^+} \frac{1}{t}\left[c(q+ty) - c(q)\right]$ exists for any $q \in \Delta(W)$, $y \in \mathbb{R}$. Then, $U(\hat{p}^t \oplus \tilde{w})$ demonstrates first-order risk aversion over the offered gamble $\hat{p}^t$.*

Proof. See Appendix A. □

Under first-order risk aversion, small-stakes risk aversion is compatible with reasonable large-stakes risk attitudes. Theorem 3 shows that in a RNEU specification, first-order risk aversion is maintained even in the presence of background risk, under mild assumptions. For example, Theorem 2 does not apply to Gul's [13] disappointment aversion, but Theorem 3 does.

4 Calibration

What constitutes descriptively reasonable risk aversion is a quantitative question. This section studies small- and large- stakes risk attitudes in a calibrated model based on recursive RDU.

I adopt the standard power utility-for-wealth function $u(w) = w^{1-\gamma}/(1-\gamma)$, and the probability weighting function $g(p) = p^{\nu}$ axiomatized in Grant and Kajii [11] and used in Safra and Segal [22]. This weighting function is only one parameter richer than EU, is consistent with small-stakes risk aversion and Allais-type choices when $0 < \nu < 1$, and captures expected utility as a special case when $\nu = 1$. I use $\gamma = .71$, suggested by Chetty [5] based on previous studies of labor supply responses to wage changes.[6]

I calibrate $\nu \in [.5, .64]$ to match Holt and Laury's [15] experimental finding that 63% of their student subjects choose (\$40, .6; \$32, .4) over (\$77, .6; \$2, .4), and 60% of subjects choose (\$77, .7; \$2, .3) over (\$40, .7; \$32, .3). We might alternatively calibrate ν to field data. Cohen and Einav [8] estimate risk aversion in a CARA EU specification based on Israelis' choices of automobile insurance deductible,[7] and find that "an individual with the average risk aversion parameter in our sample is indifferent about participating in a 50–50 lottery in which he gains \$100 or loses \$56." Calibrating ν to match this finding would yield $\nu = .64$.

While the risk in $\tilde{w}$ only has a second-order effect on decisions among offered gambles in recursive RDU, the risk in $\tilde{w}$ reduces risk aversion under RDU with reduction. To allow for comparison, take $\tilde{w}$ as a discrete uniform distribution with support on dollar-values in [\$100000, \$500000] to capture background wealth risk.[8]

Table 1 summarizes how different calibrated models discussed above would predict that a DM would make choices in $(-L, .5; G, .5)$ gambles. In each row of the table, L is fixed at the level in the left-hand column, while the entry in the table lists the G at which a DM would be indifferent to either taking or turning down the listed gamble.

Table 1 (Columns 1 and 2) indicates that for $\nu = .5, .64$ recursive RDU can produce descriptively reasonable risk aversion over both small and large stakes. RDU with reduction produces barely any risk aversion over small stakes (Column 3). Even for stakes into the thousands of dollars, EU induces preferences over gambles that are extremely close to expected value maximization (Column 4). Even with a higher value for γ, EU would induce preferences over gambles that are extremely close to

[6] While Chetty assumes expected utility in his calculations, the approach he takes fully carries through to RDU in the case where utility is separable in consumption and leisure; I use Chetty's estimates from this case.

[7] Available deductibles range from about \$250 to about \$1000.

[8] I derive quantitative results using a discrete approximation of the uniform distribution. I assume that lifetime wealth has an expected present value of \$300,000, since this figure is emphasized in Rabin [19], but the quantitative results are not particularly sensitive to this assumption.

Table 1: Calibration results - small and large stakes risk aversion

Loss	$\nu = .5$	$\nu = .64$	$\nu = .5$, reduction	$\nu = 1$ (EU)
10	24.14	17.91	10.10	10.00
100	241.60	179.21	103.36	100.03
500	1211.84	898.12	538.77	500.83
1000	2433	1801	1112	1003
5000	12555	9219	6403	5084
10000	26133	18992	14364	10343
50000	185392	123239	137678	60105

Gain required for a DM to take (-Loss, .5; Gain, .5)

expected value maximization over stakes of hundreds of dollars. These quantitative results are not sensitive to the choice of a distribution for background wealth risk.

5 Discussion

This paper has shown that RNEU can produce non-negligible small-stakes risk aversion without implying ridiculous large-stakes risk aversion, and can generate 'as-if' narrow bracketing over small-stakes gambles. A calibration exercise demonstrated that recursive RDU can be calibrated to provide descriptively reasonable levels of risk aversion over small and large stakes. Many non-expected utility theories have clear axiomatic foundations, have been well studied, and have proven tractable in applications. The results here show that many of these theories can accommodate descriptively reasonable risk aversion over both small and large stakes. The RNEU approach to applying non-expected utility preserves this tractability for a DM who faces multiple risks. Furthermore, the departures RNEU makes from expected utility theory are each well supported by experimental evidence on the Allais paradox and non-reduction of compound lotteries. This suggests that RNEU preferences provide a descriptively reasonable alternative to prospect theory while retaining utility over final wealth levels.

Appendix A: Proofs

Proof of Theorem 2.

Suppose c is weakly dually differentiable at $\tilde{w}$. Then, taking the dual derivative local to the pairs $\tilde{w}, \tilde{w}+y$ and $\tilde{w}, \tilde{w}-y$ implies that there exists $d_{\tilde{w}}$ such that $c(\tilde{w}+ty) = c(\tilde{w}) + d_{\tilde{w}}ty + o(t)$.

Then:

$$\begin{aligned} U(\hat{p}^t \oplus \tilde{w}) &= c([c(\tilde{w}+ty_1), p_1; ...; c(\tilde{w}+ty_n), p_n]) \\ &= c([c(\tilde{w}) + d_{\tilde{w}}ty_1 + o(t), p_1; ...; c(\tilde{w}) + d_{\tilde{w}}ty_n + o(t), p_n]) \\ &= c(c(\tilde{w}) + (d_{\tilde{w}}ty_1 + o(t), p_1; ...; d_{\tilde{w}}(\tilde{w})ty_n + o(t), p_n)) \end{aligned}$$

For any gamble $\hat{p}$, define $F_{\hat{q}}^{-1}$ in the obvious fashion (given the definition for lotteries). Since c is dually differentiable at $c(\tilde{w})$, take $dc(c(\tilde{w}))$ and define $\hat{c}$ by $\hat{c}(\hat{p}) = d_{\tilde{w}}(dc([c(\tilde{w}), 1]))(F_{\hat{p}}^{-1})$ for each gamble $\hat{p}$, and $\hat{c}$ is constant risk averse by construction. Then, $c(c(\tilde{w}) + \hat{p}^t)) = c(\tilde{w}) + t\hat{c}(\hat{p}+a) - ta + o(t)$ for any $a \in \mathbb{R}$.

By definition, c satisfies first-order risk aversion if and only if for any $\hat{p}^t = (ty_1, p_1; ...; ty_n, p_n)$ with an expected value of zero,

$$\begin{aligned} 0 &> \lim_{t \to 0^+} \frac{1}{t} \left[c(c(\tilde{w}) + \hat{p}^t)) - c(\tilde{w}) \right] \\ &= \lim_{t \to 0^+} \frac{1}{t} \left[t\hat{c}(\hat{p}) + o(t) \right] \\ &= \hat{c}(\hat{p}) \end{aligned}$$

Since $\hat{c}(a + \hat{p}^t) = t\hat{c}(\hat{p}) + a$ for any $a \in \mathbb{R}$, the preceding argument implies that $\hat{c}$ is first-order risk averse if and only if c is.

By the form of $\hat{c}$, we can take the sup and inf of the $o(t)$ terms in $\hat{c}(ty_1 + o(t), p_1; ...; ty_n + o(t), p_n)$, and bound from above and below by $\hat{c}(ty_1, p_1; ...; ty_n, p_n) \pm o(t)$. Thus $U(\hat{p}^t \oplus \tilde{w}) = c(\tilde{w}) + \hat{c}(\hat{p}^t) + o(t) = c(\tilde{w}) + t\hat{c}(\hat{p}) + o(t)$. Since $\hat{c}$ is constant risk averse, $t\hat{c}(\hat{p}) = t\hat{c}(\hat{p}+a) - at$ for any $a \in \mathbb{R}$.

$\square$

Proof of Theorem 3.

By the assumption in the theorem, there exist $d^+_{\tilde{w}}, d^-_{\tilde{w}}$ such that $c(\tilde{w} + ty) = c(\tilde{w}) + d^{\text{sign}(y)}_{\tilde{w}} ty + o(t)$.

Modifying the proof of Theorem 2,

$$\begin{aligned}
U(\hat{p}^t \oplus \tilde{w}) &= c([c(\tilde{w} + ty_1), p_1; ...; c(\tilde{w} + ty_n), p_n]) \\
&= c([c(\tilde{w}) + d^{\text{sign}(y_1)}_{\tilde{w}} ty_1 + o(t), p_1; ...; c(\tilde{w}) + d^{\text{sign}(y_n)}_{\tilde{w}} ty_n + o(t), p_n]) \\
&= c(c(\tilde{w}) + (d^{\text{sign}(y_1)}_{\tilde{w}} ty_1 + o(t), p_1; ...; d^{\text{sign}(y_n)}_{\tilde{w}}(\tilde{w}) ty_n + o(t), p_n)) \\
&\leq c(c(\tilde{w}) + (d^+_{\tilde{w}} ty_1 + o(t), p_1; ...; d^+_{\tilde{w}}(\tilde{w}) ty_n + o(t), p_n)) \\
&= c(c(\tilde{w}) + (d^+_{\tilde{w}} ty_1, p_1; ...; d^+_{\tilde{w}}(\tilde{w}) ty_n, p_n)) + o(t)
\end{aligned}$$

If c is first-order risk averse, $\lim\limits_{t \to 0^+} \frac{1}{t} \left[c(c(\tilde{w}) + (d^+_{\tilde{w}} ty_1, p_1; ...; d^+_{\tilde{w}}(\tilde{w}) ty_n, p_n)) - c(\tilde{w}) \right] < 0$ if $\hat{p}$ has an expected value of zero. By the analysis above, $\lim\limits_{t \to 0^+} \frac{1}{t} \left[U(\hat{p}^t \oplus \tilde{w}) - c(\tilde{w}) \right] < 0$.

$\square$

Appendix B: Sufficient Conditions for Dual Differentiability

Proposition 1. *Suppose V takes a RDU functional form with a differentiable u. Then V is dually differentiable.*

Proof. Write:

$$
\begin{aligned}
\frac{1}{\lambda}\left[V((1-\lambda)q \boxplus \lambda r) - V(q)\right] &= \int_0^1 \frac{1}{\lambda}\left[u((1-\lambda)F_q^{-1}(p) + \lambda F_r^{-1}(p)) - u(F_q^{-1}(p))\right] dg(p) \\
\text{as } \lambda \to 0^+, \quad &\to \int_0^1 u'(F_q^{-1}(p))[F_r^{-1}(p) - F_q^{-1}(p)]dg(p)
\end{aligned}
$$

Thus the linear functional is given by $(dV(q))(F_r^{-1}) = \int_0^1 u'(F_q^{-1}(p))F_r^{-1}(p)dg(p)$. $\square$

Proposition 2. *If V has a betweenness representation $V(q) = \int_0^1 v(F_q^{-1}(p), V(q))dp$ in which v is continuously differentiable, then V is dually differentiable.*

Proof. Since v is continuously differentiable in both arguments, we can proceed as in the RDU case and apply a chain rule to get the linear functional:

$$(dV(q))(F_r^{-1}) = \left(1 - \int_0^1 v_2(F_q^{-1}(p), V(q))dp\right)^{-1} \left(\int_0^1 v_1(F_q^{-1}(p), V(q))F_r^{-1}(p)dp\right) \quad \square$$

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