Risk Taking with Background Risk under Recursive Rank-Dependent Utility

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Abstract

Evidence from both the lab and field suggest that the presence of background risk increases a decision-maker's risk aversion. Evidence from the lab suggests that behaviour deviates from expected utility, however, existing analyses of non-expected utility theories suggest many such theories cannot accommodate both findings. Additionally motivated by evidence that people fail to reduce compound lotteries in lab experiments, this paper assumes that a decision-maker treats a risk-taking decision in the presence of background risk as a compound lottery, and evaluates this compound lottery recursively using rank-dependent utility, and studies how background risk affects risk-taking behaviour in the recursive rank-dependent utility model. I find that adding background risk increases risk aversion in this model under the same conditions whenever the utility-for-wealth function is risk vulnerable.

Keywords: non-expected utility; recursive preferences; risk aversion; background risk.

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1 Introduction

Risk is a fact of life. People who face risky decisions almost invariably have uninsurable pre-existing risks. Evidence from lab experiments (Beaud and Willinger, 2015), survey data (Guiso and Paiella, 2008), and observational data on portfolio choices (Guiso et al., 1996) suggests that pre-existing risks tend to make people more risk averse. This evidence has been interpreted in terms of a property of the utility-for-wealth function known as "risk vulnerability," which is a necessary and sufficient condition for an expected utility (EU) maximizer to exhibit more risk aversion when she faces a background risk (Gollier and Pratt, 1996).

However, evidence also suggests that many people violate the Independence Axiom, as suggested by the Allais paradox (e.g. Camerer and Ho 1994). This has motivated non-EU theories that relax the Independence Axiom, including rank-dependent utility (RDU, Quiggin 1982; Yaari 1987), that accommodate such behavior by weighting probabilities non-linearly. In RDU and related models, risk aversion due to probability weighting is attenuated by the presence of uninsurable background risk whenever a decision-maker (DM) integrates any additional risk and her background risk into a single lottery using the laws of probability (Quiggin, 2003; Safra and Segal, 2008).

If we view an additional risk plus background risk as a two-stage lottery with the additional risk resolving first, then Quiggin's and Safra and Segal's results rely on the assumption that a DM reduces this compound lottery. However, evidence from lab experiments suggest that most people do not reduce compound lotteries (Camerer and Ho, 1994; Halevy, 2007), and are better captured by recursive non-EU preferences, as developed in Segal (1990) and applied in Dillenberger (2010). Previous work (Freeman, 2015; Artstein-Avidan and Dillenberger, 2015) shows that recursive non-EU preferences, including recursive RDU (RRDU), are immune to the calibration critique of RDU and related models in (Safra and Segal, 2008) and can accommodate descriptively reasonable small- and large- stakes risk aversion even in the presence of background risk. However, Freeman (2015) leaves as open question: when does the presence of background risk raise a DM's risk aversion over additional risks? This paper applies the modeling approach of Freeman (2015) to show that under RRDU, a background risk with a weakly negative expected value increases a DM's risk aversion over an additional risk whenever her utility-for-wealth function is risk vulnerable, regardless of her probability weighting function.

2 Model

2.1 Preliminaries

The setup follows Freeman (2015). Let $W = [a, b] \subset \mathbb{R}_+$ denote the set of feasible final wealth levels; let $\Delta(W)$ denote the set of all finite-support probability distributions over W, and refer to $\Delta(W)$ as the set of *one-stage lotteries* over W. A one-stage lottery over W can be written as $q = [w_1, q_1; ...; w_m, q_m] \in \Delta(W)$, where q_i denotes the probability of receiving prize w_i; for such lotteries, adopt the convention that $w_1 \leq ... \leq w_m$. Given $q \in \Delta(W)$, let F_q denote the cumulative distribution function (CDF) of q. For a given $q \in \Delta(W)$ and $y \in \mathbb{R}$, let $q + y = [w_1 + y, q_1; ...; w_m + y, q_m]$; the resulting $q + y$ is only in $\Delta(W)$ if $w_1 + y \geq 0$; I omit this caveat in statements below for exposition.

Define a *compound lottery* as a finite lottery over lotteries over final wealth levels. A compound lottery can be written as $Q = [q^1, p_1; ...; q^n, p_n]$ where $q^i \in \Delta(W)$ and p_i is the probability of receiving lottery q^i. Let $\Delta(\Delta(W))$ denote the set of compound lotteries.

2.2 Preferences over compound lotteries

I assume that a DM has RDU (Quiggin, 1982; Yaari, 1987) preferences over single-stage lotteries. $V : \Delta(W) \to \mathbb{R}$ is a *rank-dependent utility function* if there exist strictly increasing functions $u : W \to \mathbb{R}$ and $g : [0, 1] \to [0, 1]$ such that for any $q \in \Delta(W)$, $V(q) = \sum_{i=1}^{n} [g(\sum_{j=1}^{i} q_j) - g(\sum_{j=1}^{i-1} q_j)]u(w_i)$. The function g is called a *probability weighting function*, and is required to satisfy $g(0) = 0$ and $g(1) = 1$; the function u is called a *utility-for-wealth function*. Notice that EU corresponds to the special case of RDU in which the probability weighting function is linear. Given an RDU function V, let c denote its corresponding certainty equivalent function defined by

$c = u^{-1} \circ V$. In the analysis that follows, assume that V is risk averse – that is, averse to mean-preserving spreads. This is equivalent to assuming that g and u are both weakly concave (Chew et al., 1987).

An RDU function V can be extended to compound lotteries in two possible ways. A normatively appealing way is to use the laws of probability to reduce any two-stage lottery $Q = [q^1, p_1; ...; q^n, p_n]$ to a single-stage lottery and apply V to this reduced lottery, consistent with the Reduction of Compound Lotteries axiom. This leads to the following definition: a DM *reduces compound lotteries* if she evaluates the desirability of any two-stage lottery Q according to $V(Q^R)$, where $Q^R = [w_1, \sum_{i=1}^{n} p_i q_1^i; ...; w_K, \sum_{i=1}^{n} p_i q_K^i]$.

A DM might alternatively apply her single-stage lottery preferences to a compound lottery recursively. That is, the DM could evaluate $Q = [q^1, p_1; ...; q^n, p_n]$ by first applying c to each q^i to obtain the single-stage lottery $[c(q^1), p_1; ...; c(q^n), p_n]$, to which she could apply c a second time. This assumes that the DM applies the same single-stage lottery preferences at each stage of the compound lottery, as in Segal's (1990) Time Neutrality axiom. Define that a function $U : \Delta(\Delta(W)) \to \mathbb{R}$ is a *Recursive RDU (RRDU) function* over two-stage lotteries if there is an RDU function V with corresponding certainty equivalent function c such that for any $Q \in \Delta(\Delta(W))$, $U(Q) = c([c(q^1), p_1; ...; c(q^n), p_n])$.

2.3 Risk-taking with background risk

Define a *gamble* as a finite lottery over gain and loss prizes (as opposed to wealth levels). Consider a DM with wealth level w who faces background wealth risk described by the gamble $\hat{q} = [y'_1, q_1; ...; y'_m, q_m]$, which is not the subject of choice. This DM is offered the gamble over prizes $\hat{p} = (y_1, p_1; ...; y_n, p_n)$. Each $y_i \in \mathbb{R}$ is a monetary prize added to or taken away from the DM's final wealth after lottery $\hat{p}$ resolves and each $y'_i \in \mathbb{R}$ is similarly a monetary gain or loss similarly added to or taken away from the DM's final wealth after $\hat{q}$ resolves.

Let $\hat{p} \oplus \hat{q} + w$ denote the compound lottery formed by the simple gamble over prizes $\hat{p}$, which resolves at the first stage, and independent background risk $\hat{q}$, which resolves at the second stage, given initial wealth w. The compound lottery $\hat{p} \oplus \hat{q} + w$

is given by

$$\hat{p} \oplus \hat{q} + w = [\hat{q} + y_1 + w, p_1; ...; \hat{q} + y_n + w, p_n] \tag{1}$$

where $\hat{q} + y_i + w = [y'_1 + y_i + w, q_1; ...; y'_m + y_i + w_m, q_m]$ denotes the lottery over final wealth states that the DM faces if prize y_i is won in the gamble $\hat{p}$.

In the results that follow, I will say that *adding unfavorable background risk increases risk aversion* if whenever $\hat{p}$ and $\hat{q}$ are gambles and $\hat{q}$ has a weakly negative expected value, $c(\hat{p} + w) - w \leq c((\hat{p} \oplus (\hat{q} + w))^R) - c(\hat{q} + w)$ for each $w \in W$. Analogously define the expression that *adding unfavourable background risk decreases risk aversion.*

A utility-for-wealth function u is *risk vulnerable* if for any $\hat{q} \in \Delta(Y)$ with a weakly negative expected value and for any w,

$$-\frac{\sum_{i=1}^{n} q_i u''(w + y_i)}{\sum_{i=1}^{n} q_i u'(w + y_i)} \geq -\frac{u''(w)}{u'(w)}.$$

Proposition 1 in Gollier and Pratt (1996) shows that in EU, risk vulnerability of u is a necessary and sufficient condition for the addition of an unfavorable background risk to increase risk aversion over offered gambles.

Proposition 1. *Under EU, adding an unfavorable background risk increases risk aversion if and only if u is risk vulnerable.*

Gollier and Pratt (1996) provide a set of intuitive conditions on utility-for-wealth functions that imply risk vulnerability under EU. In particular, Pratt and Zeckhauser's (1987) "properness" and Kimball's (1993) "standardness" each imply risk vulnerability. These requirements are satisfied by any u in the hyperbolic absolute risk aversion class, including the constant relative risk aversion cases.

Quiggin (2003) shows that very different results hold under RDU, as summarized in Proposition 2.[1]

[1] Quiggin (2003) works on the space of Savage acts with a known probability of each state. The stated result is an implication of Quiggin's result translated onto the space of objective lotteries here.

Proposition 2. *If a DM has RDU preferences with a linear u and she reduces compound lotteries, then adding an unfavorable background risk reduces risk aversion.*

Beaud and Willinger (2015) find experimentally that background risk increases risk aversions among subjects, and take this as evidence against this prediction of RDU.

3 The effect of background risk on risk taking in RRDU

Theorem 1 shows that if the utility-for-wealth function u is risk vulnerable then any RRDU function with utility-for-wealth function u and a concave g that evaluates offered gambles according to (1) has the property that adding unfavorable background risk increases risk aversion.

Theorem 1. *Suppose U is an RRDU function with a concave probability weighting function g and a concave utility-for-wealth function u. If u is risk vulnerable, then adding an unfavorable background risk increases risk aversion.*

Proof. Suppose U captures a DM's RRDU preferences, with corresponding probability weighting function g and a risk vulnerable utility-for-wealth function u. Let $\hat{p}, \hat{q} \in \Delta(Y)$ with $\int y d\hat{q}(y) \leq 0$.

Suppose c turns down $\hat{p}$ at wealth level w. Then,

$$\int u(w+y) dg(F_{\hat{p}}(y)) < u(w)\,. \tag{2}$$

Since g is a probability weighting function, $g \circ F_{\hat{p}}$ and $g \circ F_{\hat{q}}$ are CDFs corresponding to gambles in $\Delta(Y)$. Equation (2) is equivalent to an EU maximizer with utility-for-wealth function u turning down the gamble with CDF $g \circ F_{\hat{p}}$.

Since $\int y d\hat{q}(y) \leq 0$ and g is concave, it follows that $\int y dg(F_{\hat{q}}(y)) \leq \int y dF_{\hat{q}}(y) \leq 0$. Thus $g \circ F_{\hat{q}}$ is the CDF of an actuarially unfair gamble.

Since u is risk vulnerable and an EU maximizer with utility-for-wealth function u turns down the gamble with CDF $g \circ F_{\hat{p}}$ at wealth w, by Proposition 1, an EU-maximizer with utility-for-wealth function u, wealth w, and background risk $g \circ F_{\hat{q}}$ also turns down $\hat{p}$, that is,

$$\int\int u(w+y+x)dg(F_{\hat{p}}(y))dg(F_{\hat{q}}(x)) < \int u(w+x)dg(F_{\hat{q}}(x))\,.$$

Equivalently, $$u \circ U(\hat{p} \oplus (w+\hat{q})) < u \circ c(w+\hat{q})$$

Which implies that $$U(\hat{p} \oplus (w+\hat{q})) < c(w+\hat{q})\,.$$

Thus such a DM will also turn down $\hat{p}$ when she faces actuarially unfavorable background risk $\hat{q}$. Since our choice of $\hat{p}$ and $\hat{q}$ were arbitrary, conclude that adding an unfavorable background risk increases risk aversion for an RRDU DM with a risk vulnerable u. □

A remaining question is how these results could be generalized to provide a condition for g and u for which, under RRDU, an increase in background risk, from $\hat{q}$ to the riskier $\hat{r}$, raises risk aversion. I answer that question for the cases in which $\hat{r}$ is first- or second- order stochastically dominated by $\hat{q}$.

Define that *an FSD-deterioration in background risk increases risk aversion* if whenever $\hat{p}$, $\hat{q}$, and $\hat{r}$ are gambles for which $\hat{q}$ has a weakly negative expected value and first-order stochastically dominates $\hat{r}$, we have $c((\hat{p} \oplus (\hat{q}+w))^R) - c(\hat{q}+w) \leq c((\hat{p} \oplus (\hat{r}+w))^R) - c(\hat{r}+w)$ for each $w \in W$. Similarly, define that a *SSD-deterioration in background risk increases risk aversion* if this condition holds when $\hat{q}$ has a weakly negative expected value and second-order stochastically dominates $\hat{r}$.

Eeckhoudt et al. (1996) provide two conditions that, under EU, they show are necessary and sufficient for FSD- and SSD- deteriorations in background risk to increase risk aversion. Theorem 2 below shows that these conditions on u are also sufficient in RRDU.

Theorem 2. *Suppose U is an RRDU function with a concave probability weighting function g and a concave utility-for-wealth function u. If there exists a $\lambda \in \mathbb{R}$ such that $-\frac{u'''(w')}{u''(w')} \geq \lambda \geq -\frac{u''(w)}{u'(w)}$ $\forall w, w' \in [a,b]$, then an FSD-deterioration in background risk increases risk aversion. If, in addition, there exists a $\kappa \in \mathbb{R}$ such that $-\frac{u''''(w')}{u'''(w')} \geq \kappa \geq -\frac{u''(w)}{u'(w)}$ $\forall w, w' \in [a,b]$, then an SSD-deterioration in background risk increases* *risk aversion.*

Proof. If $\hat{q}$ first- (second-) order stochastically dominates $\hat{r}$, then $g \circ F_{\hat{q}}$ first- (second-) order stochastically dominates $g \circ F_{\hat{r}}$ (Chew et al., 1987).

The results then follow by applying this fact, the proof of Theorem 1, and Eeckhoudt et al.'s Proposition 2 (and 3). □

4 Discussion

Previous work has shown that in dual theory, when background risk and any risk being evaluated are integrated into a single lottery, background risk reduces risk aversion (Quiggin, 2003) and can lead to qualitatively very different choices of insurance contracts (Doherty and Eeckhoudt, 1995). These predictions seem counter to evidence and intuition.

The RRDU model here parsimoniously accommodates violations of the Independence Axiom and Reduction of Compound Lotteries as well as small-stakes risk aversion. Theorem 1 showed that, when u is risk vulnerable, RRDU is also consistent with evidence that background risk increases risk aversion. Thus model here provides a tractable and descriptively-motivated way to apply RDU in the presence of background risk that avoids the descriptively problematic predictions that result when RDU is combined with Reduction of Compound Lotteries.

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