Revealing Naïveté and Sophistication from Procrastination and Preproperation

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May 27, 2019

Abstract

This paper proposes a novel way of distinguishing whether a person is naïve or sophisticated about their own dynamic inconsistency using only their task completion behavior. It shows that adding an extra opportunity to complete the task that goes unused can lead a naïve (but not a sophisticated) person to complete it later, and can lead a sophisticated (but not a naïve) person to complete the task earlier. These results provide the framework for revealing preference and sophistication types from behavior in a general environment that includes that of O'Donoghue and Rabin (1999).

JEL Codes: D03, D84, D90.

Keywords: sophistication, naïveté, procrastination, preproperation, task completion, present bias, time inconsistency.

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Behavioral economic models of intertemporal choice following Strotz (1955) incorporate two assumptions. First, a person may be dynamically inconsistent — that is, her current preferences over future actions may differ from the future preferences she will use when she chooses. Second, a person may be imperfectly self-aware of her own dynamic inconsistency when she forms expectations of her own future behavior. Strotz proposed two ways a dynamically inconsistent person might form expectations: she can be naïve and expect her future selves to behave according to her current preferences, or she can be sophisticated and hold correct expectations about her future behavior.

Yet neither a person's preferences nor their self-awareness are directly observed. Moreover, common domains of study, such as consumption-savings decisions, are insufficiently rich to jointly infer both intertemporal preferences and self-awareness (Blow, Browning and Crawford, 2017). This makes it difficult to understand which assumption is more descriptively appropriate for any given application. In Strotz's model, however, a measure of self-awareness is necessary to accurately forecast the behavioral responses and welfare of an individual; for example, under different savings policies (Sprenger 2015, p. 283). This motivates the need to find ways to measure self-awareness on alternative domains of choice.[1]

This paper studies an individual's choice of when to complete a task that must be done exactly once, as in O'Donoghue and Rabin (1999). The paper's contribution is to show that sophistication and naïveté have sufficient force to yield distinct predictions in the domain of task completion. These predictions are directly testable and economically intuitive, and this is the case even in choice problems in which a person's higher-order beliefs about her future behavior are relevant and her preferences are not directly observed.

The main results of this paper establish that two separate classes of choice reversals are hallmarks of sophistication and naïveté. A person's behavior demonstrates a "reversal" if there is an instance in which adding an additional period to a set of completion opportunities changes when the person would complete the task, even

[1]Experiments that test for demand for commitment — a hallmark that indicates sophistication about a self-control problem — typically find that only a minority of participants demand commitment (e.g. Ashraf, Karlan and Yin (2006)); Section 6.4 reviews this literature. These findings further invite the question: are most people naïve, or time consistent, or is something else going on?

though the additional opportunity would not be used. Reversals are distinguished between "doing-it-earlier" reversals in which adding an opportunity results in the person's acting earlier, and "doing-it-later" reversals in which adding the opportunity results in their acting later in a sense my definition makes precise. Theorems 2 and 3 establish that doing-it-later reversals are a hallmark of naïveté, while doing-it-earlier reversals are a hallmark of sophistication, in the sense that a time-inconsistent naïve person will exhibit a doing-it-later reversal while a sophisticate never will, and vice versa.

The following example shows how a doing-it-later reversal reveals naïveté. A student must do an assignment exactly once in a given week, and we observe when she does it in each of two weeks. These weeks are directly comparable, in that the student's preferences over which day to do it are the same in both weeks. In the first week, she can only do it on either Tuesday or Thursday (because the lab is closed on Wednesday). In the second week, she also has the option of doing it on Wednesday. Suppose we observe that she does it on Tuesday in the first week and Thursday in the second week. Her first week's behavior reveals that on Tuesday, she prefers to do it now (i.e. on Tuesday) over Thursday. With this observation in hand, her decision to wait on Tuesday in the second week reveals information about her preferences: that on Tuesday, she must prefer to wait until Wednesday It also provides information about her beliefs: that she expects that she would act on Wednesday if she waits on Tuesday. Since she actually does it on Thursday in the second week, this expectation about her own behavior is incorrect, and she is revealed to be naïve about her Wednesday behavior.

The next example shows how a doing-it-earlier reversal reveals sophistication. Suppose instead that the lab is open on Tuesday and Wednesday in the first week, those days plus Thursday in the second week, and that her preferences over which day to do it are the same in each week. Suppose we observe that she does it on Wednesday in the first week and Tuesday in the second week. Her first week's behavior reveals that on Tuesday she prefers to wait to do it Wednesday. With this observation in hand, her decision to do it on Tuesday in the second week reveals information about her beliefs and preferences at that time: she expects that she would not do it on Wednesday and she would rather do it now than on Thursday.

She is thus not naïve about her Wednesday behavior.

My main results show that the intuition from the above two examples can be extended more broadly. For a naïf who is not aware of her time inconsistency, adding an additional opportunity to act makes waiting appear weakly more attractive. Due to her time inconsistency, this can lead her to delay doing it even if the added opportunity is not used.

In contrast, a sophisticate correctly anticipates her future behavior; thus an additional opportunity will affect her earlier behavior only if she would act if she were to reach it. This limits the extent to which adding an additional opportunity can lead a sophisticate to delay. Moreover, if she would act on this added opportunity against an earlier self's preferences, that earlier self may preemptively do it. This can lead her to complete the task even earlier than without the added opportunity.

My results for distinguishing naïfs from sophisticates relate to O'Donoghue and Rabin's (1999) results that a naïve person will always do it later than a sophisticated person with the same preferences, and both will violate the Independence of Irrelevant Alternatives. My results also relate to O'Donoghue and Rabin's (2001) results that adding an additional completion opportunity can lead a partially naïve (but not a sophisticated) person to exhibit extreme procrastination. Unlike O'Donoghue and Rabin (1999; 2001), my formal choice environment assumes that the stream of payoffs associated with doing the task at a given point in time can be arbitrary and unobserved by the analyst. The treatment and choice properties here apply regardless of whether the combination of the choice environment and the nature of dynamically inconsistent preferences induces future selves to want to delay (i.e. procrastinate) or do it early (i.e. preproperate) against earlier selves' wishes, or some combination of both. Furthermore, my results (unlike those of O'Donoghue and Rabin) do not make parametric assumptions about preferences.

The main results, Theorems 2 and 3, establish doing-it-later and doing-it-earlier reversals as hallmarks of naïveté and sophistication, respectively. Theorem 4 provides complete characterizations of naïveté and sophisticated behavior on the domain studied here, and Theorem 5 provides such a characterization for a model of partial naïveté. Corollary 1 and Proposition 1 provide conditions under which the relative sophistication of two partially naïve choice functions can be compared. Under the restrictions imposed by the task completion domain here, Propositions 2-3 show that naïveté and sophistication each imply that behavior has an alternative representation in terms of a well-studied two-stage model of boundedly rational choice, though each assumption implies a distinct model of the first stage (respectively, the models of Manzini and Mariotti 2007 and Masatlioglu, Nakajima and Ozbay 2012). Section 6 discusses possible extensions and reviews related literature.

1 Modeling naïveté and sophistication

1.1 Environment

Consider a person who faces a set of opportunities — periods at which a task can be done — and must act to complete a task in exactly one of the available periods. If the person has not already completed the task, the person can either act (i.e. do the task) or wait. The person cannot commit the behavior of her future selves except by acting.

Let $T \geq 3$ be a finite integer. Let $\bar{A} = \{1, \ldots, T\}$ denote the set of all possible periods; a period will typically be denoted by $t \in \bar{A}$. Let $\mathscr{A}$ denote the collection of all non-empty finite subsets of the set $\bar{A}$ of all possible opportunities. An opportunity set will typically be denoted by $A \in \mathscr{A}$.[2] Assume that a choice function $c : \mathscr{A} \to \bar{A}$ is observed, where $c(A)$ denotes the time a person acts when A is their opportunity set.[3] While c is formally equivalent to a choice function on the domain $\mathscr{A}$ in the usual sense, its interpretation is different from that of a usual static choice function, since it represents a dynamic choice problem and this is embedded in the temporal structure in $\bar{A}$. Interpret the behavior $t = c(A)$ as a result of choosing to wait at all periods in A prior to t, and then choosing act at time t over waiting until later.

Given a set $A \in \mathscr{A}$ and $t \in \bar{A}$, let $A_{>t} = \{t' \in A : t' > t\}$ and define $A_{\geq t}, A_{<t}$, and $A_{\leq t}$ analogously.

[2] Each $A \in \mathscr{A}$ is finite and thus induces a finite-horizon problem.

[3] c is (technically) a choice function, rather than a correspondence, which rules out the possibility of genuine indifference. If indifference is always broken in the same deterministic way, this assumption is innocuous, since T is finite.

Consider below two examples that can fit into this framework.

Example 1. A statistics assignment must be done in a computer lab in a single sitting. The lab is only open on a set number of weekdays, announced in advance. In this case, A could be given by any non-empty subset of $\{1,2,3,4,5\}$, and t denotes the opportunity to do the homework assignment on the t^{th} weekday.

Example 2. Let each $t \in \bar{A}$ be associated with a vector in $x^t \in \mathbb{R}^T$ that specifies a stream of costs/rewards associated with acting at time t. Setting $T = 2$ yields the setup of O'Donoghue and Rabin (1999), where $x^t = \left(x_1^t, x_2^t\right)$ means that acting at t yields an immediate utility benefit or cost of x_1^t at time t and a delayed cost or benefit of x_2^t realized at some time $\bar{T} > T$, which does not depend on t.

Notice that the setup here makes no distinction between choice environments with immediate costs and delayed rewards that are likely to induce procrastination, versus those with immediate benefits and delayed costs likely to invoke preproperation. Indeed, this setup can include periods with a mixture of types of payoff structures, as well as periods that do not neatly fit either structure.

1.2 Preferences

Consider the following model of preferences that allows for changing tastes (following Strotz 1955). Each person has a set of time-dependent utility functions. For each $t \in \bar{A}$, let $U_t : \bar{A}_{\geq t} \to \mathbb{R}$ denote her time-t utility function, that is, the utility function she uses at time t when she evaluates the desirability of each completion opportunity. Let $\mathscr{U} = \langle U_1, \ldots, U_T \rangle$ denote the ordered collection of utility functions for each t, and assume each is one-to-one.[4]

The structure of the completion opportunity space and preferences in this general model are minimally restricted. The magnitude and timing of flow utility the person expects to experience now and/or in the future if she acts in a particular period need not be observed by the analyst. This allows for arbitrary time-variant preferences and attitudes toward the timing of acting.

[4] Since $\bar{A}$ is finite, this assumption rules out indifferences and is without loss of generality so long as we assume that all indifferences are broken in the same way.

1.3 Beliefs and behavior under time inconsistency

The behavior of a time-inconsistent person will depend both on her preferences (represented by $\mathscr{U}$) and on her expectations about her future behavior in each period.

Model a person's beliefs about her future behavior through a set of perceived future utility functions, where each such function captures an earlier period's beliefs about the utility function that will apply in a later period. For each $t_1, t_2 \in \bar{A}$ with $t_1 < t_2$, the function $\hat{U}_{t_2|t_1} : \bar{A}_{\geq t_2} \to \mathbb{R}$ denotes the utility function that time-t_1 self believes her time-t_2 self will apply, referred to as a perceived future utility function. Given any $t \in \bar{A}$, let $\hat{\mathscr{U}}_{\cdot|t} = \left\langle \hat{U}_{t+1|t}, \ldots, \hat{U}_{T|t} \right\rangle$ denote the ordered collection of the t-self's perceived future utility functions and let $\hat{\mathscr{U}} = \left\langle \hat{\mathscr{U}}_{\cdot|1}, \ldots, \hat{\mathscr{U}}_{\cdot|T-1} \right\rangle$ denote her ordered collection of all perceived future utility functions. This formulation rules out higher-order beliefs about future utility functions (perceived perceived future utility functions and so on): a person at time t forecasts that her future beliefs and behavior will both be given by $\hat{\mathscr{U}}_{\cdot|t}$.

Given the preferences and beliefs captured by $\mathscr{U}$ and $\hat{\mathscr{U}}$, the perception perfect equilibrium concept of O'Donoghue and Rabin specifies how a person would behave when facing any opportunity set. Let the function s denote a strategy, where $s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t}) =$ wait or $=$ act specifies a wait-or-act decision in period t in opportunity set A given a current utility function U_t and and perceived future utility functions $\hat{\mathscr{U}}_{\cdot|t}$. A strategy is perception perfect if in each period t, given that her beliefs about her future utility functions and beliefs about future beliefs thereof are given by $\hat{\mathscr{U}}_{\cdot|t}$, as defined below, the person best responds according to her utility function U_t. That is, at time t, she believes that her behavior at future time t' will be determined by predicted strategy $s(t', A, \hat{U}_{t'|t}, \hat{\mathscr{U}}_{\cdot|t})$, which is perception perfect for the utility functions she predicts she will apply; letting τ_t denote the period after t in which she predicts she first would act if she waits at t, she acts at t if $U_t(t) > U_t(\tau_t)$ and waits otherwise.

Definition. The <u>perception-perfect strategy</u> corresponding to opportunity set $A \in \mathscr{A}$, set of utility functions $\mathscr{U}$, and set of perceived future utility functions $\hat{\mathscr{U}}$ is a strategy s that, for each $t \in A$, satisfies:

$$s(t,A,U_t,\hat{\mathscr{U}}_{\cdot|t}) = \begin{cases} \text{act} & \text{if } U_t(t) > U_t(\hat{\tau}_t) \text{ or } A_{>t} = \emptyset \\ \text{wait} & \text{otherwise} \end{cases}$$

for $\hat{\tau}_t = \min\left\{\tau > t : s(\tau,A,\hat{U}_{\tau|t},\hat{\mathscr{U}}_{\cdot|t}) = \text{act}\right\}$.

Next, introduce representations of behavior. A representation for a choice function is called a Strotzian representation if it models choice generated by a perception-perfect strategy for some sets of utility functions $\mathscr{U}$ and perceived future utility functions $\hat{\mathscr{U}}$. A Strotzian representation is (fully) naïve if every time-t self's perceived time-t' utility function is given by her current utility function U_t. In contrast, a Strotzian representation is sophisticated if every time-t self's perceived time-t' utility function is given by her actual time-t' utility function $U_{t'}$. Partial naïveté is an intermediate case between these two extremes, in which sophistication is only allowed to fail in predictions about behavior in which there is a conflict between current and future preferences. That is, a Strotzian representation is termed partially naïve if every time-t self's perceived time-t' utility function's ranking between-t' and a later period is either consistent with her actual time-t' preferences or with her time-t preferences between these two periods. Naïve and sophisticated representations are thus each special cases of a partially naïve representation. These terms are defined formally below.

Definition. The choice function c has a Strotzian representation if there exist a $\mathscr{U}$ and a $\hat{\mathscr{U}}$ such that for each $A \in \mathscr{A}$, $c(A) = \min_t\{t : s(t,A,U_t,\hat{\mathscr{U}}_{\cdot|t}) \neq \text{wait}\}$. A Strotzian representation is naïve if $\hat{U}_{t_2|t_1}(t_3) = U_{t_1}(t_3)$ for all $t_1,t_2,t_3 \in \bar{A}$ with $t_1 < t_2 < t_3$. A Strotzian representation is sophisticated if $\hat{U}_{t_2|t_1}(t_3) = U_{t_2}(t_3)$ for all $t_1,t_2,t_3 \in \bar{A}$ with $t_1 < t_2 < t_3$. A Strotzian representation is partially naïve if, for all $t_1,t_2,t_3 \in \bar{A}$ with $t_1 < t_2,t_3$, $U_{t_1}(t_2) > U_{t_1}(t_3)$ and $U_{\min\{t_2,t_3\}}(t_2) > U_{\min\{t_2,t_3\}}(t_3)$ implies $\hat{U}_{\min\{t_2,t_3\}|t_1}(t_2) > \hat{U}_{\min\{t_2,t_3\}|t_1}(t_3)$.

1.4 Time consistency

Behavior is time consistent if all of her utility functions are consistent with the same ranking over periods; thus no two periods' selves would disagree on when to act. In this setting, we only observe when the decision-maker acts. Therefore, observationally, time consistency is equivalent to the requirement that the following two conditions hold for every opportunity set: (i) at all periods before she acts she would rather wait until the time she actually acts and (ii) at the time she acts she prefers acting to waiting until any available future period.

Definition. A choice function c is observationally time consistent if for every $A \in \mathscr{A}$, $t = c(A)$ implies that $t = c(\{t,t'\})$ for all $t' \in A$. Otherwise, c is observationally time inconsistent.

In the Strotzian model, completion times in two-opportunity choice sets reveal preferences at the earlier of the two periods. Thus, observational time consistency requires that each act or wait decision is made by a utility function that, as far as can be detected from observable choice, is consistent with the preferences of subsequent selves.

Example 3. Let $\bar{A} = \{1,2,3\}$ and suppose $3 = c(\{1,2,3\})$, $2 = c(\{1,2\})$, $3 = c(\{1,3\}) = c(\{2,3\})$. This c is observationally time consistent. Yet c has a Strotzian representation with $U_1(2) > U_1(3) > U_1(1)$, $U_2(3) > U_2(2)$, which would typically be viewed as a time-inconsistent representation. However, observational time inconsistency is a property of c, rather than $\mathscr{U}$. Notice that c also has a representation with $U_1(3) > U_1(2) > U_1(1)$, so choice alone cannot conclusively determine whether this c ought to be represented with a $\mathscr{U}$ that is time consistent in the usual sense; this motivates the notion of observational time consistency used in this paper.

The definition of observational time (in)consistency is based on comparing choice in arbitrary choice sets to choice in two-opportunity choice sets; the latter allow for clear inferences about preferences. However, the definition of observation time inconsistency does not indicate whether or how one can draw clear inferences about a person's beliefs about her own inconsistency. This motivates an alternative way to test for time inconsistency that will enable such inferences. To that end, introduce the notion of a reversal, based on comparing when a person does the task in two choice sets where one of them has an extra available opportunity.

Definition. c exhibits a reversal if there exists an $A \in \mathscr{A}$ and a t_2 such that $t_1 = c(A)$, $t_3 = c(A \cup \{t_2\})$, and $t_1, t_2 \neq t_3$. c exhibits no reversals if for all $A \in \mathscr{A}$ and $t_2 \in \bar{A}$, $c(A \cup \{t_2\}) \in \{c(A), t_2\}$.

The no-reversals property defined above is a variation of the Independence of Irrelevant Alternatives axiom and is equivalent to observational time consistency, as formalized below.[5]

Theorem 1. *Let c be a choice function. Then the following are equivalent: (i) c is observationally time consistent, (ii) c exhibits no reversals, and (iii) c has a Strotzian representation that is both sophisticated and naïve.*

Observational time inconsistency can thus be revealed from reversals (an insight that builds on O'Donoghue and Rabin 1999, pp. 114-115). The next section shows that specific classes of reversals allow an analyst to jointly infer time inconsistency and sophistication or naïveté about that inconsistency.

2 Choice reversals under naïveté and sophistication

Consider two types of reversals a choice function might exhibit.

Definition. Consider a reversal with $t_1 = c(A)$, $t_3 = c(A \cup \{t_2\})$, and $t_1, t_2 \neq t_3$. The reversal is a doing-it-later reversal if $t_1 < t_3$ and either $t_2 < t_3$ or $t_1 = c(\{t_1, t_3\})$ (or both). The reversal is a doing-it-earlier reversal if $t_3 < t_1$.

A doing-it-earlier reversal occurs when adding the period t_2 to A results in the person's acting earlier than without the added opportunity. The definition of a doing-it-later reversal requires that a reversal satisfies both of two conditions. First, adding period t_2 to A results in the person's acting even later than without the added opportunity. Second,[6] it is either the case that (i) their pairwise choice

[5]The no-reversals property is weaker than but equivalent to Sen's (1971, p. 313) Property α. Sen shows that his Property α is a necessary and sufficient condition for a (static) choice function on a finite domain to be rationalizable by an antisymmetric, complete, and transitive binary relation.

[6]Example 6 illustrates why this second part of the definition of a doing-it-later reversal is needed for such reversals to reveal naïveté.

directly reveals that at $t_1 < t_3$, she prefers acting at t_1 over t_3, or (ii) she acts later in $A \cup \{t_2\}$ than the added period, t_2 (or both). The results below show that naïve time inconsistency implies the existence of doing-it-later reversals and the absence of doing-it-earlier reversals. In contrast, sophistication allows doing-it-earlier but not doing-it-later reversals.

Intuitively, a naïve person can be induced to delay when she fails to anticipate that her future self will do something different from what she would currently prefer. Adding new options to a time-inconsistent person's opportunity set might tempt her at some point, but a naïve person will not anticipate any inconsistency with her current tastes when deciding whether to act now. Putting these pieces of intuition together, it seems possible that adding an opportunity might lead a naïve person to delay when her earlier selves do not anticipate that her later selves will be tempted to delay by the added opportunity, even if the added opportunity is not itself taken. By this same logic, a naïve person will never exhibit a doing-it-earlier reversal because, since she thinks her future behavior will be consistent with her current preferences, adding new opportunities to her opportunity set will make waiting appear weakly more attractive at earlier periods.

Example 4 gives an example of a doing-it-later reversal.

Example 4. Revisit Example 1. Suppose we observe that $1 = c(\{1,5\})$ and $5 = c(\{1,3,5\})$. These choices exhibit a doing-it-later reversal, since adding the irrelevant option of doing it on Wednesday (3, which is later than 1, Monday) leads the person to delay until Friday (which is later than Monday and Wednesday).

Example 5 gives an example of a doing-it-earlier reversal.

Example 5. Revisit Example 1. Suppose we observe that $3 = c(\{1,3\})$ and $1 = c(\{1,3,5\})$. Choices exhibit a doing-it-earlier reversal, since adding the unused option to do it on Friday ($t = 5$) leads the person to complete the assignment earlier than Wednesday.

Examples 4 and 5 show that there exist choices that exhibit doing-it-later and doing-it-earlier reversals.

Theorem 2 shows that not only is it possible that an added unused option leads to delay for a naïf, but that any reversal exhibited by a naïf implies delay: they will never exhibit a doing-it-earlier reversal, and if they are time inconsistent they will exhibit at least one doing-it-later reversal.[7] An implication of this result is that the choices in Example 5 are inconsistent with naïve decision-making, while those in Example 4 violate time consistency but are consistent with naïve decision-making.

Theorem 2. *If c has a naïve representation, then c does not exhibit any doing-it-earlier reversal. If c is also time inconsistent, then c exhibits a doing-it-later reversal.*

Proof. Suppose c has a naïve representation with set of utility functions $\mathscr{U}$ and set of perceived utility functions $\hat{\mathscr{U}}$. Further suppose c exhibits the reversal $t_1 = c(A)$, $t_3 = c(A \cup \{t_2\}) \neq t_1, t_2$ and let $A' = A \cup \{t_2\}$. Let s denote a perception perfect strategy.

Applying the restrictions that $\hat{U}_{t'|t} = U_t$ on $\bar{A}_{\geq t'}$, and that $A \subseteq A'$, we obtain that for any $t < t_2$,

$$\begin{aligned} U_t(\tau_t) &= \max_{\tilde{t} \in A_{>t}} U_t(\tilde{t}) \\ &\leq \max_{\tilde{t} \in A'_{>t}} U_t(\tilde{t}) \\ &= U_t(\tau'_t) \end{aligned}$$

where $\tau_t = \min\{\tau > t : s(\tau, A, \hat{U}_{\tau|t}, \hat{\mathscr{U}}_{\cdot|t}) = \text{act}\}$ and $\tau'_t = \min\{\tau > t : s(\tau, A', \hat{U}_{\tau|t}, \hat{\mathscr{U}}_{\cdot|t}) = \text{act}\}$. Thus $s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{wait}$ implies $s(t, A', U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{wait}$ for such t. Thus $t_3 \geq \min\{t_1, t_2\}$.

First suppose $t_1 > t_2$. Then, $s(t, A', U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{wait}$ for all $t < t_2$. Since $c(A') \neq t_2$, we have $s(t_2, A', U_{t_2}, \hat{\mathscr{U}}_{\cdot|t_2}) = \text{wait}$ as well. Additionally, for all $t > t_2$ we have $A'_{>t} = A_{>t}$, from which it follows from the representation that $s(t, A', U_t, \hat{\mathscr{U}}_{\cdot|t}) = s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t})$. It thus follows that $t_3 = t_1$, which contradicts our initial assumption. Thus $t_1 > t_2$ cannot hold, and since $t_1 \neq t_2$, it follows that $t_2 > t_1$.

Now suppose $t_2 > t_1$. Then since $t_3 \neq t_1$ and $t_3 \geq \min\{t_1, t_2\} = t_1$ we can conclude that $t_3 > t_1$. This establishes that a choice function with a naïve representation

[7] Theorem 2 here is related to Proposition 5 of O'Donoghue and Rabin (2001), which shows that for a person who faces the same set of options to complete a task each period, adding a new option to that set can lead to extreme procrastination if she is partially naïve, but not if she is sophisticated, under quasi-hyperbolic discounting. Neither result nests the other.

cannot exhibit a doing-it-earlier reversal.

Furthermore, since $t_1 = c(A)$ and $t_3 \in A_{>t_1}$, we must have $U_{t_1}(t_1) > U_{t_1}(\hat{\tau}_{t_1}) = \max_{t' \in A_{>t_1}} U_{t_1}(t') \geq U_{t_1}(t_3)$, which implies $t_1 = c(\{t_1, t_3\})$. Thus this reversal is a doing-it-later reversal.

If c is time inconsistent, then by Theorem 1 it exhibits at least one reversal and by the preceding argument this must be a doing-it-later reversal. □

Turning to sophisticates, intuition suggests that sophisticates act earlier because they anticipate their future self-control problems. Thus, they might act earlier to avoid the temptation to which they anticipate their future selves will succumb, exercising the only type of commitment to which they have access in this choice environment. When adding a new opportunity leads a person to do it at an earlier but previously available time, a person exhibits a doing-it-earlier behavior that cannot be accommodated by time-consistent preferences. Such behavior is, however, allowed for under time-inconsistent sophistication.

Notice that a sophisticated choice function can still exhibit a reversal with $t_3 > t_1$; Example 6 illustrates such a case.

Example 6. Revisit Example 1, and suppose that $1 = c(\{1,2,3\})$ but $2 = c(\{1,2,3,4\})$. If $2 = c(\{1,2\})$, this is not a doing-it-later reversal, since $2 < 4$. These choices would be generated by the sophisticated choice function with preferences $U_1(2) > U_1(1) > U_1(3) > U_1(4)$, $U_2(3) > U_2(2) > U_2(4)$, and $U_3(4) > U_3(3)$. Here, adding the opportunity to do the assignment on Thursday leads to delay compared to when the homework had to be completed by Wednesday. This occurs because the sophisticate waits on Monday because she recognizes that her Tuesday self will do it in order to avoid delaying until Thursday.

Theorem 3 states that any sophisticated choice function exhibits no doing-it-later reversals, but exhibits a doing-it-earlier reversal if it is time inconsistent.

Theorem 3. *If c has a sophisticated representation, then c does not exhibit any doing-it-later reversal. If c is also time inconsistent, then c exhibits a doing-it-earlier reversal.*

Proof. Suppose c has a sophisticated representation with set of utility functions $\mathscr{U}$ and set of perceived utility functions $\hat{\mathscr{U}}$. Further suppose c exhibits the reversal $t_1 = c(A), t_3 = c(A \cup \{t_2\}) \neq t_1, t_2$ and let $A' = A \cup \{t_2\}$. Let s denote the perception perfect strategy.

Since $A_{>t_2} = A'_{>t_2}$, $s(t, A', U_t, \hat{\mathscr{U}}_{\cdot|t}) = s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t})$ for all $t > t_2$. Suppose $s(t_2, A', U_{t_2}, \hat{\mathscr{U}}_{\cdot|t_2}) =$ wait. For $t_a = \max A_{<t_2}$, by sophistication, for each $t' > t_a$, $\hat{U}_{t'|t_2} = \hat{U}_{t'|t_a} = U_{t'}$ on $A_{\geq t'}$, so $s(t_2, A', \hat{U}_{t_2|t_a}, \hat{\mathscr{U}}_{\cdot|t_a}) = s(t_2, A', U_{t_2}, \hat{\mathscr{U}}_{\cdot|t_2}) =$ wait, and thus $\tau'_{t_a} = \min\{\tau > t_a : s(\tau, A', \hat{U}_{\tau|t_a}, \hat{\mathscr{U}}_{\cdot|t_a}) = \text{act}\} = \min\{\tau > t_a : s(\tau, A, \hat{U}_{\tau|t_a}, \hat{\mathscr{U}}_{\cdot|t_a}) = \text{act}\} = \tau_{t_a}$. By the same argument, if $t_b \in A_{<t_a}$ and $s(t, A', U_t, \hat{\mathscr{U}}_{\cdot|t}) = s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t})$ for all $t \in A$ with $t_b < t < t_a$, then $\tau'_{t_b} = \tau_{t_b}$ and thus $s(t_b, A', U_{t_b}, \hat{\mathscr{U}}_{\cdot|t_b}) = s(t_b, A, U_{t_b}, \hat{\mathscr{U}}_{\cdot|t_b})$. But then we have $s(t, A', U_t, \hat{\mathscr{U}}_{\cdot|t}) = s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t})$ for all $t \in A$. But this implies $t_3 = t_1$, a contradiction. Conclude that $s(t_2, A', U_{t_2}, \hat{\mathscr{U}}_{\cdot|t_2}) =$ act, and thus $t_3 < t_2$. Therefore c cannot exhibit a doing-it-later reversal with $t_3 > t_2$.

Now suppose that $t_1 < t_3$ and $t_1 = c(\{t_1, t_3\})$. By the representation, $t_1 = c(\{t_1, t_3\})$ implies $U_{t_1}(t_1) > U_{t_1}(t_3)$, and $t_3 = c(A \cup \{t_2\})$ implies that $t_3 = \min\left\{t : s(t, A', U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{act}\right\}$. But then by the definition of a perception perfect strategy, act $= s(t_3, A', U_{t_3}, \hat{\mathscr{U}}_{\cdot|t_3})$ and wait $= s(t, A', U_t, \hat{\mathscr{U}}_{\cdot|t})$ for all $t \in A_{<t_3} \cap A_{>t_1}$. Since $U_{t_1}(t_1) > U_{t_1}(t_3)$, we must also have $s(t_1, A', U_{t_1}, \hat{\mathscr{U}}_{\cdot|t_1}) =$ act, which would imply $t_3 = c(A') \leq t_1$, a contradiction. Thus if c has a sophisticated representation, c cannot exhibit a doing-it-later reversal.

If c is time inconsistent, then by Theorem 1 it exhibits at least one such reversal. If $t_1 > t_3$, then this is a doing-it-earlier reversal. Next, suppose $t_1 < t_3$. By the argument two paragraphs above, $s(t_2, A', U_{t_2}, \hat{\mathscr{U}}_{\cdot|t_2}) =$ act. Since $t_3 = c(A')$, it follows that $t_3 < t_2$, $s(t_3, A', U_{t_3}, \hat{\mathscr{U}}_{\cdot|t_3}) =$ act, and $s(t, A', U_t, \hat{\mathscr{U}}_{\cdot|t}) =$ wait for all $t \in A'_{<t_3}$. Working backward, if $s(t_3, A, U_{t_3}, \hat{\mathscr{U}}_{\cdot|t_3}) =$ act, then sophistication and $A_{<t_3} = A'_{<t_3}$ imply that $s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t}) = s(t, A', U_t, \hat{\mathscr{U}}_{\cdot|t}) =$ wait for all $t < t_3$ (including t_1), which contradicts that $t_1 < t_3$ and $t_1 = c(A)$; thus, $s(t_3, A, U_{t_3}, \hat{\mathscr{U}}_{\cdot|t_3}) =$ wait. By the sophisticated representation, it follows that $c(A_{\geq t_3}) > t_3$ and $c(A'_{\geq t_3}) = t_3$ is a doing-it-earlier reversal. $\square$

To summarize, Theorems 2 and 3 show that doing-it-later reversals are a hallmark of naïveté in the sense that a sophisticated person will not exhibit such reversals, but a time-inconsistent naïve person will, whereas doing-it-earlier reversals are a hallmark of sophistication in the sense that a naïve person will never exhibit them, but a time-inconsistent sophisticated person will.

3 Characterizations of naïve and sophisticated choice

This section provides testable conditions that completely characterize the behavioral content of naïve and sophisticated choice on this domain.

Example 7 shows that without assuming additional structure, the absence of doing-it-later reversals does not guarantee that choice has a sophisticated representation.

Example 7. Consider a choice function c with a Strotzian representation with $U_1(2) > U_1(1) > U_1(3) > U_1(4), U_2(3) > U_2(2) > U_2(4), U_3(4) > U_3(3), \hat{U}_{3|1}(3) > \hat{U}_{3|1}(4)$, and $\hat{U}_{2|1} = U_2$, $\hat{U}_{3|2} = U_3$. In this representation, the time-1 self is naïve about her time 3 behavior, but all selves are otherwise sophisticated. We can see that the c corresponding to this $\mathscr{U}, \hat{\mathscr{U}}$ pair exhibits no doing-it-later reversals, and exhibits doing-it-earlier reversals since, applying the perception perfect strategy, $2 = c(\{1,2,4\})$ but $1 = c(\{1,2,3,4\})$, and also $3 = c(\{2,3\})$ but $2 = c(\{2,3,4\})$. However, a fully sophisticated choice function $\tilde{c}$ with the same $\mathscr{U}$ would have $2 = \tilde{c}(\{1,2,3,4\})$. Since U_2 and U_3 are pinned down by choices from two-option sets, it follows that c exhibits no doing-it-later reversals yet does not have a sophisticated representation.

Next, consider three conditions that will be used to characterize naïve and sophisticated choice.

First, consider the Irrelevant Alternatives Delay condition, which rules out doing-it-earlier reversals while also requiring that the added option (t_2) in any reversal be at a later date than that of the initial choice (t_1).

Irrelevant Alternatives Delay. If $t_1 = c(A)$, and $t_1, t_2 \neq t_3 = c(A \cup \{t_2\})$, then $t_2, t_3 > t_1$.

The following property, Exclusion Consistency (Rubinstein and Salant, 2008),[8] restricts that if adding a period (t_2) generates a reversal, then the previously chosen option (t_1) is not chosen in any set in which the added period (t_2) is available.

Exclusion Consistency. If $t_1 = c(A)$, and $t_1, t_2 \neq c(A \cup \{t_2\})$, and $t_2 \in A'$, then $t_1 \neq c(A')$.

The Recursivity condition strengthens the central postulate of sophisticated consistent planning (Strotz, 1955; Pollak, 1968) as a choice-based condition. It requires that the added earlier option is taken if and only if it would be taken in the opportunity set that contains only two periods: the added earlier opportunity and the opportunity that would have been taken were the added option not available.[9]

Recursivity. For any A and t, $c(A_{>t} \cup \{t\}) = c(\{t, c(A_{>t})\})$.

Recursivity characterizes sophistication in this domain, while Irrelevant Alternatives Delay and Exclusion Consistency jointly characterize naïveté.

Theorem 4. *(i) c has a naïve representation if and only if c satisfies Irrelevant Alternatives Delay and Exclusion Consistency.*

(ii) c has a sophisticated representation if and only if c satisfies Recursivity.

These characterization results provide a complete set of testable implications of naïve and sophisticated representations. A full proof is available in the Appendix. Notice that beliefs about future behavior are trivial in opportunity sets with only two periods, and thus behavior in such sets cleanly reveals preferences that applied at the earlier period. A key step of the proof uses choices from two opportunity sets to characterize preferences, defining that $t = c(\{t, t'\})$ implies $U_{\min\{t,t'\}}(t) >$

[8]Rubinstein and Salant (2008) use the Exclusion Consistency property to characterize a model of a person who chooses by sequentially applying two different binary relations. Their characterization uses an abstract choice space that lacks the temporal structure used here.

[9]This property plays a similar role to Gul and Pesendorfer's (2005) "No Compromise" axiom and Noor's (2011) "Sophistication" axiom.

$U_{\min\{t,t'\}}(t')$, and then shows that the conditions in either (i) or (ii) are sufficient to guarantee that $\mathscr{U}$ can be so constructed.

The following example shows that Irrelevant Alternatives Delay is not sufficient to guarantee that a choice function has a naïve representation.

Example 8. Let $T = 3$, $1 = c(\{1,2\}) = c(\{1,3\})$ and $2 = c(\{2,3\}) = c(\{1,2,3\})$. By construction, c satisfies Irrelevant Alternatives Delay. But in a naïve representation $1 = c(\{1,2\}) = c(\{1,3\})$ can occur only if $U_1(1) > U_1(2), U_1(3)$, which would imply $1 = c(\{1,2,3\})$ regardless of expectations about $t = 2$ behavior.

4 Partial naïveté

Some people may be neither fully naïve nor fully sophisticated. This section studies the behavior of such partially naïve people. First, the predictive content of the class of Strotzian and partially naïve representations will be characterized in terms of two behavioral conditions, as explained below.

If t_{n-1} and t_n are the last two available opportunities to do the task in choice set A and t_1 is an earlier opportunity in A, then a decision-maker's time t_1 decision should be invariant to replacing the last two periods, t_{n-1} and t_n, with the period in which she expects (at t_1) that she would first act if she were to reach t_{n-1}; this reasoning applies regardless of the decision-maker's sophistication. By this logic, if the time t_1 choice depends on which period is removed, then for every other choice set A' where t_{n-1} and t_n are the last two available opportunities and t_1 is in A', the decision-maker would hold the same t_1 beliefs about when she would act were she to reach t_{n-1}. The Penultimate Replaceability condition imposes this restriction on behavior. To facilitate statement of the condition and those that follow, for each $t < T$ define the function $c_t : \{A \in \mathscr{A} : \ c(A) \leq t\} \to \{\text{wait}, \text{act}\}$ by $c_t(A) = \text{wait}$ if $c(A) > t$ and $c_t(A) = \text{act}$ if $c(A) = t$.

Penultimate Replaceability. Let $t_1 < t_2 < t_3$. Then either $c_{t_1}(A \cup \{t_2, t_3\}) = c_{t_1}(A \cup \{t_2\})$ for all $A \subseteq \{t_1, \ldots, t_2 - 1\}$, or $c_{t_1}(A \cup \{t_2, t_3\}) = c_{t_1}(A \cup \{t_3\})$ for all $A \subseteq \{t_1, \ldots, t_2 - 1\}$.

When Penultimate Replaceability holds, the option that can be replaced for t_{n-1} and t_n will, in the representation, be the time at which the person believes (at time t_1) she would end up doing it if she reaches the penultimate opportunity t_{n-1}. However, the condition places no restriction on when beliefs can be incorrect. But in a partially naïve representation, a person at time t_1 would only make an incorrect prediction about her t_{n-1} behavior by incorrectly applying t_1 preferences that differ from t_{n-1} preferences. When such behavior identifies incorrect t_1 expectations about what she would do if she reached t_{n-1}, if c has a partially naïve representation, this also identifies the earlier self's preferences. The Wishfulness condition requires that for each earlier period t, time-t preferences can be constructed to align with any revealed mispredictions about behavior in the last two periods without generating any preference cycles. To state the condition, for each $t \in \bar{A}$, define R_t on $\bar{A}_{\geq t}$ as follows: if $t = c(\{t, t_1\})$, then tR_tt_1, if $t_1 = c(\{t, t_1\})$, then t_1R_tt, and if there exists an $A \subseteq \{t, \ldots, \min\{t_1, t_2\} - 1\}$ such that $c_t(A \cup \{t_1, t_2\}) = c_t(A \cup \{t_1\}) \neq c_t(A \cup \{t_2\})$ but $t_2 = c(\{t_1, t_2\})$, then $t_1R_tt_2$.

Wishfulness. For each $t \in \bar{A}$, R_t is acyclic.

Theorem 5 shows that choice has a Strotzian representation if and only if c satisfies Penultimate Replaceability, and this is a partially naïve representation if and only if c also satisfies Wishfulness.

Theorem 5. *(i) c has a Strotzian representation if and only if c satisfies Penultimate Replaceability.*

(ii) c has a partially naïve representation if and only if c satisfies Penultimate Replaceability and Wishfulness.

Different partially naïve people may fall at different places on the continuum between "naïve" and "sophisticated". This raises the question of how to compare the degree of sophistication of different Strotzian representations using behavior. Such an exercise can be conducted where behavior can possibly reveal beliefs; to do so, restrict comparisons to representations that are minimally naïve in the sense that when beliefs cannot be determined from behavior, they are modeled as sophisticated.

Definition. A partially naïve representation $\mathscr{U}, \hat{\mathscr{U}}$ for choice function c is minimally naïve if $c_{t_1}(A \cup \{t_2,t_3\}) = c_{t_1}(A \cup \{t_2\}) = c_{t_1}(A \cup \{t_3\})$ for all $A \subseteq \{t_1, \ldots, \min\{t_2,t_3\} - 1\}$, and $t_2 = c(\{t_2,t_3\})$ implies $\hat{U}_{\min\{t_2,t_3\}|t_1}(t_2) > \hat{U}_{\min\{t_2,t_3\}|t_1}(t_3)$.

Next, consider a working definition of what it means for one partially naïve choice function to be more sophisticated than another. Consider two partially naïve representations, each with the same collection of utility functions $\mathscr{U}$ but with different forecasts of future utility functions at each period, captured by $\hat{\mathscr{U}}$ and $\hat{\mathscr{U}}'$. Intuitively, $\hat{\mathscr{U}}$ is more sophisticated than $\hat{\mathscr{U}}'$ if, whenever the perceived utility function in $\hat{\mathscr{U}}'$ makes a correct forecast about future behavior, so does the corresponding perceived utility function in $\hat{\mathscr{U}}$.[10]

Definition. Given two choice functions, c and c', with partially naïve representations with the same $\mathscr{U}$ and $\hat{\mathscr{U}}$ and $\hat{\mathscr{U}}'$, respectively, c is more sophisticated than c' if for all $t_1,t_2,t_3 \in \bar{A}$ with $t_1 < t_2,t_3$, $U_{\min\{t_2,t_3\}}(t_2) > U_{\min\{t_2,t_3\}}(t_3)$ and $\hat{U}'_{\min\{t_2,t_3\}|t_1}(t_2) > \hat{U}'_{\min\{t_2,t_3\}|t_1}(t_3)$ implies $\hat{U}_{\min\{t_2,t_3\}|t_1}(t_2) > \hat{U}_{\min\{t_2,t_3\}|t_1}(t_3)$.

The "more sophisticated than" definition applies to two representations, rather than their choices directly. The following comparative leverages intuition from Penultimate Replaceability to compare two choice functions based on penultimate accuracy: when an earlier self's expectations about when she would act in the last two periods can be revealed from behavior, these expectations may, but need not be, accurate relative to when she would actually act if the last two periods were reached. This gives a notion of penultimate accuracy that can be used to compare choice functions.

Definition. c is more penultimately accurate than c' if for all $t_1,t_2,t_3 \in \bar{A}$ with $t_1 < t_2 < t_3$ and $A \in \mathscr{A}$ such that $A \subseteq \{t_1, \ldots, t_2 - 1\}$, $t_j = c'(\{t_2,t_3\})$ and $c'_{t_1}(A \cup \{t_2,t_3\}) = c'_{t_1}(A \cup \{t_j\}) \neq c'_t(A \cup \{t_{-j}\})$ implies $c_{t_1}(A \cup \{t_2,t_3\}) = c_{t_1}(A \cup \{c(\{t_2,t_3\})\})$.

[10] The comparative notion of sophistication below is an ordinal version of Ahn et al.'s (2016) "more u-aligned" notion for comparing the relative sophistication of two representations, except that the comparative below restricts such comparisons to individuals with the same preferences over opportunities at all periods (i.e. same $\mathscr{U}$).

The following corollary to Theorem 5 clarifies the tight link between the more sophisticated than relationship between representations and the more penulatimately accurate than relationship between choice functions when restricted to minimally naïve representations.

Corollary 1. *Let c and c' be minimally naïve representations, both with the set of utility functions $\mathscr{U}$ and with the sets of perceived future utility functions $\hat{\mathscr{U}}$ and $\hat{\mathscr{U}}'$, respectively. c is more sophisticated than c' if and only if c is more penultimately accurate than c'.*

The penultimately accurate relation is not obviously related to the reversals studied earlier. The next example show why a simple comparison of reversals does not necessarily rank the relative sophistication of two choice functions. To that end, Example 9 shows that a partially naïve person can exhibit a doing-it-earlier reversal not exhibited by a fully sophisticated person with the same preferences. This is because believing that behavior will better align with current preferences can work in both directions. In the partially naïve model here, if a decision-maker is currently more optimistic about her behavior in the more distant future, then she also expects her less-distant-future-selves to share that optimism. This could lead her to expect future selves to delay against her current wishes, which could lead her to do it earlier.

Example 9. O'Donoghue and Rabin (1999) show that a naïf will always do it weakly later than a sophisticate with the same preferences facing the same choice set. This result does not transfer to a comparison between sophistication and partial naïveté. Consider $\bar{A} = \{1,2,3,4\}$ and $\mathscr{U}$ such that $U_1(2) > U_1(1) > U_1(3) > U_1(4)$, $U_2(3) > U_2(2) > U_2(4)$, and $U_3(4) > U_3(3)$. A sophisticate with these preferences has $2 = c(\{1,2,4\}) = c(\{1,2,3,4\})$. However, a partially naïve person who, at $t = 1$, incorrectly projects that $\hat{U}_{3|1}(3) > \hat{U}_{3|1}(4)$ but correctly predicts that $\hat{U}_{2|1}(4) < \hat{U}_{2|1}(2) < \hat{U}_{2|1}(3)$, will expect herself to delay at $t = 2$ and do it at $t = 3$ and would thus do it at $t = 1$. This leads to a doing-it-earlier reversal, since it implies $2 = c(\{1,2,4\})$ but $1 = c(\{1,2,3,4\})$. Thus for this $\mathscr{U}$, adding $t = 3$ to the opportunity set $\{1,2,4\}$ generates a doing-it-earlier reversal for this partially naïve person, but not for a fully sophisticated one.[11] This demonstrates that the intuition that a more sophisticated person is more prone to doing-it-earlier reversals must be qualified.

It can also be shown that this partially naïve person exhibits no doing-it-later reversals on this domain. But the lack of a doing-it-later reversal that reveals their naïveté can be viewed as an artifact of the restrictive domain of this example: if doing it at $t = 1$ could be made less attractive so that $U_1(3) > U_1(1) > U_1(4)$, then a doing-it-later reversal would be observed.

When a person faces at most three completion opportunities, she makes at most one forecast of a future behavior, which allows an analyst to draw clear inferences about her predicted future behavior from such choices.

As shown in Example 8, the interaction between elements of sophistication and of naïveté can confound a clear comparison of degrees of sophistication by a simple comparison of their propensity for doing-it-earlier/later reversals when a person needs to form beliefs about future beliefs. When looking at opportunity sets with only three opportunities, such interactions do not arise, which allows for simple comparisons. This motivates a definition of "three-opportunity revealable" which delineates the class of beliefs that can be directly revealed by only looking at choices involving three or fewer opportunities.

Definition. A belief $\hat{U}_{\min\{t_2,t_3\}|t_1}(t_2) \gtreqless \hat{U}_{\min\{t_2,t_3\}|t_1}(t_3)$ is <u>three-opportunity revealable</u> if $t_1 < \min\{t_2,t_3\}$ and $t_1 = c(\{t_1,t_2\}) \neq c(\{t_1,t_3\})$.

Next, a comparison of the reversals involving only three options.

Definition. Let c and c' have partially naïve representations. Say that c <u>exhibits more three-opportunity doing-it-later reversals than</u> c' if for every triple $t_1,t_2,t_3 \in \bar{A}$ with $t_1 < t_2,t_3$, $t_1 = c'(\{t_1,t_2\}) = c(\{t_1,t_2\})$ and $t_2 = c'(\{t_1,t_2,t_3\})$ implies $t_2 = c(\{t_1,t_2,t_3\})$. Say that c <u>exhibits more three-opportunity doing-it-earlier reversals than</u> c' if for every triple $t_1,t_2,t_3 \in \bar{A}$ with $t_1 < t_2,t_3$, $t_2 = c'(\{t_1,t_2\}) = c(\{t_1,t_2\})$

[11] This example also demonstrates that a more sophisticated person need not complete the task earlier than a less sophisticated person with the same preferences.

and $t_1 = c'(\{t_1,t_2,t_3\})$ implies $t_1 = c(\{t_1,t_2,t_3\})$.[12]

Proposition 1 clarifies the relationship between these notions of exhibiting more three-opportunity doing-it-later/earlier reversals and the relative sophistication of two representations compared only for revealable beliefs.

Proposition 1. *Let c and c' have partially naïve representations with the same set of utility functions $\mathscr{U}$ and sets of perceived future utility functions $\hat{\mathscr{U}}$ and $\hat{\mathscr{U}}'$, respectively. Then the following are equivalent:*

(i) For any t_1,t_2,t_3 such that $\hat{U}_{\min\{t_2,t_3\}|t_1}(t_2) \gtrless \hat{U}_{\min\{t_2,t_3\}|t_1}(t_3)$ is three-opportunity revealable and $U_{\min\{t_2,t_3\}}(t_2) > U_{\min\{t_2,t_3\}}(t_3)$, then $\hat{U}_{\min\{t_2,t_3\}|t_1}(t_2) > \hat{U}_{\min\{t_2,t_3\}|t_1}(t_3)$ implies $\hat{U}'_{\min\{t_2,t_3\}|t_1}(t_2) > \hat{U}'_{\min\{t_2,t_3\}|t_1}(t_3)$;

(ii) c exhibits more three-opportunity doing-it-later reversals than c';

(iii) c' exhibits more three-opportunity doing-it-earlier reversals than c.

An implication of Proposition 1 is that if we were willing to enlarge the space of opportunities or make sufficiently strong structural assumptions on the desirability of different opportunities and to guarantee that all beliefs are three-opportunity revealable, then comparing only three-opportunity reversals provides a complete basis for comparing the relative sophistication of two choice functions; a working paper version of this paper proved such a result using a richer domain.

5 Relationship to models of boundedly rational choice

In the setting considered in this paper, naïve and sophisticated representations have alternative representations in terms of boundedly rational choice procedures that have been well studied in the choice theory literature. This section derives these

[12] Since picking an earlier action commits to a decision, exhibiting more three-opportunity doing-it-earlier reversals is a special case of Gul and Pesendorfer's (2001) "greater preference for commitment" comparative restricted to sets with up to three options. In a similar vein, exhibiting more three-opportunity doing-it-later reversals is a special case of Ahn et al.'s (2016) notion of being "more naïve" on the domain here.

connections. As a preliminary to the analysis below, given any binary relation R define $m(\cdot,R):\mathscr{A}\to\mathscr{A}$ by $m(A,R)=\{t\in A:\nexists t'\in A \text{ for which } t'Rt\}$.

First, introduce the rational shortlist method (RSM) of Manzini and Mariotti (2007).

Definition. A choice function c has an RSM representation if there exist asymmetric binary relations P_1 and P_2 such that $c(A)=m(m(A,P_1),P_2)$ for all $A\in\mathscr{A}$.[13]

Naïve choice behavior is equivalent to that of an RSM representation where P_2 is the "less than" order $<$ on $\mathbb{N}$ restricted to $\bar{A}$ and P_1 is a subset of $>$.

Proposition 2. *c has a naïve representation if and only if it also has an RSM representation with $P_1\subseteq>$ and $P_2=<$.*

Rubinstein and Salant (2008) prove that a choice function c has an RSM representation if and only if it satisfies Exclusion Consistency. Thus Proposition 2 can be viewed as a corollary of the conjunction of Rubinstein and Salant (2008) with Proposition 4. The proof here explicitly constructs P_1 and P_2 from $\mathscr{U}$ by providing an alternative characterization of a naïve representation.

Proof. Suppose c has a naïve representation $\mathscr{U},\hat{\mathscr{U}}$. Then naïveté requires $\hat{U}_{t_2|t_1}(t_3)=U_{t_1}(t_3)$ for all t_1,t_2,t_3 with $t_1<t_2\leq t_3$. Plugging this into the definition of a predicted strategy and working backward from the last period with available opportunities in A, we obtain that at each t_1,t_2 with $t_1<t_2$, act $=s(t_2,A,\hat{U}_{t_2|t_1},\hat{\mathscr{U}}_{\cdot|t_1})$ if and only if $t_2=\arg\max_{t\in A_{\geq t_2}}U_{t_1}(t)$. Similarly, plugging the above formula for forecasts of behavior into the definition of s, we obtain an expression for $s(t,A,U_t,\hat{\mathscr{U}}_{\cdot|t})$:

$$s(t,A,U_t,\hat{\mathscr{U}}_{\cdot|t})=\begin{cases}\text{act} & \text{if } U_t(t)>\max\limits_{t'\in A_{>t}}U_t(t')\\ \text{wait} & \text{otherwise}\end{cases}\tag{1}$$

which yields the representation that $c(A)$ is the earliest time at which that period's self prefers doing the task immediately over each available future opporunity. Let

[13]Excuse the notational sloppiness in this definition that conflates each singleton set $m(m(A,P_1),P_2)$ with its element.

P_1 be a binary relation that ranks an opportunity above another if it prevents this doability condition from being satisfied. That is, define P_1 by $t'P_1t$ if and only if $t' > t$ and $U_t(t') > U_t(t)$; by construction, $P_1 \subseteq >$. Thus the set $m(A, P_1) = \{t \in A : s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{act}\}$ captures periods in which, if reached, the decision-maker would act. Then by construction, $m(m(A, P_1), <)$ returns the earliest time in A at which $s(t, A, U_t, \hat{\mathscr{U}} \cdot |t) = \text{act}$.

Conversely, suppose that c has an RSM representation P_1, P_2 where $P_1 \subseteq >$ and $P_2 =<$. Then define $\mathscr{U}$ so that $U_{\min\{t,t'\}}(t) > U_{\min\{t,t'\}}(t')$ if not $t'P_1t$. For any t and $t' > t$, define $\hat{U}_{t'|t} = U_t$ on $A_{\geq t'}$, and let s denote a perception-perfect strategy. By construction of $\mathscr{U}$ and $\hat{\mathscr{U}}$, given any set A, $m(A, P_1) = \{t \in A : U_t(t) > U_t(t')\ \forall t' \in A_{>t}\} = \{t \in A : s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{act}\}$, thus $m(m(A, P_1), P_2) = \min\{t \in A : s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{act}\}$. This establishes that this RSM representation has an alternative naïve representation. $\square$

Just as any naïve choice function has an RSM representation, so too any sophisticated representation has a representation in terms of a model from the bounded rationality literature: the choice with limited attention (CLA) model of Masatlioglu, Nakajima and Ozbay (2012).

Definition. A function $\Gamma : \mathscr{A} \to \mathscr{A}$ is an attention filter if, for each $A \in \mathscr{A}$, $\emptyset \neq \Gamma(A) \subseteq A$ and for all $t \in A \backslash \Gamma(A)$ we have $\Gamma(A \backslash \{t\}) = \Gamma(A)$. A choice function c has a CLA representation if there exists an asymmetric, complete, and transitive binary relation P and an attention filter Γ such that $c(A) = m(\Gamma(A), P)$ for all $A \in \mathscr{A}$.

Proposition 3. *If c has a sophisticated representation, then it has a CLA representation with $P =<$.*

The proof explicitly constructs Γ and P from $\mathscr{U}$.

Proof. Let c have a sophisticated representation corresponding to $\mathscr{U}$. Define Γ by $\Gamma(A) = \{t \in A : s(t, A_{\geq t}, U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{act}\}$ for each A. That is, $\Gamma(A)$ gives the set of all periods in A at which, if reached, the person would act. Let $P =<$. Then $m(\Gamma(A), P)$ picks out the earliest period in A at which the strategy specifies to act.

It remains to verify that Γ is an attention filter. So suppose $t \in A \backslash \Gamma(A)$. Then by our choice of Γ, $s(t, A_{\geq t}, U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{wait}$. But then $s(t', A_{\geq t'} \backslash \{t\}, U_{t'}, \hat{\mathscr{U}}_{\cdot|t'}) =$

$s(t', A_{\geq t'}, U_{t'}, \hat{\mathscr{U}}_{\cdot|t'})$ for all $t' \neq t$: the $t' < t$ case follows by the definition of a sophisticated representation because $s(t, A_{\geq t}, U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{wait}$, and the $t' > t$ case follows trivially, since $A_{\geq t'} = (A \backslash \{t\})_{\geq t'}$. □

The converse to Proposition 3 does not hold: the restrictions on Γ imposed by a CLA representation need not allow one to constuct a $\mathscr{U}$ that generates the choices in a sophisticated representation. The following example shows that even a choice function with a CLA representation that exhibits no doing-it-later reversals can fail to have a sophisticated representation.

Example 10. Consider $\bar{A} = \{1,2,3,4\}$, let $\Gamma(\bar{A}) = \{1,2,3,4\}$, $\Gamma(A) = A \backslash \{1\}$ for all $A \subsetneq \bar{A}$ with $A \neq \{1\}$, and let $P =<$. The choice function c with this CLA representation has $1 = c(\bar{A})$ and also has $2 = c(\{2,t\})$ for all t. By the latter, any sophisticated representation for c must have $U_1(2) > U_1(1)$ and $U_2(2) > U_2(3), U_2(4)$. These restrictions on U_1 and U_2 are sufficient to infer that 2 must be chosen from $\bar{A}$ in any sophisticated representation for c, which would contradict $1 = c(\bar{A})$. Thus c has no sophisticated representation.

6 Extensions and relation to existing work

6.1 Structured domains

The theoretical literature on time-inconsistent preferences studies an individual's preferences over dated rewards (Ok and Masatlioglu, 2007; Dziewulski, 2015; Ericson and Noor, 2015; Chakraborty, 2016) or consumption streams (Montiel Olea and Strzalecki, 2014; Galperti and Strulovici, 2015; Echenique, Imai and Saito, 2015; Noor and Takeoka, 2017), or lotteries over consumption streams (Hayashi, 2003).[14] These rich and structured domains allow these authors to study specific models of the weighing of earlier versus later rewards or consumption. However, these papers do not speak to how a decision-maker resolves conflicting preferences

[14]Another example of a structured domain is that of optimal stopping with deterministic or stochastic payoff flows, as studied by Quah and Strulovici (2013). In Section V of their paper, they show that under partially naïve quasi-hyperbolic discounting, a more sophisticated agent will stop weakly earlier in their environment.

when she cannot commit her future behavior. My setting is exactly suited to this task.

Also unlike the aforementioned papers on time preferences, my setting does not assume that the analyst observes a cardinal measure of the payoffs realized at each time period following an action. My model also does not impose structural assumptions on preferences. In contrast, most papers on time-inconsistent choice assume that preferences are time invariant (see Halevy 2015) and study particular functional forms for discounting current versus future payoffs (e.g. quasi-hyperbolic discounting). That being said, if the stream of payoffs associated with acting in each period were observable, variations on doing-it-later/earlier reversals can be generated to create new testable implications of naïveté and sophistication. This is illustrated with two short examples.

Suppose that a person makes choices from sets of dated rewards in $\mathbb{R} \times T$, where (x,t) indicates an opportunity at time t that yields immediate reward x. Suppose that the person always prefers larger rewards to smaller ones, that is, each $U_t((x,t'))$ is increasing in x and $U_t((0,t')) = 0$. Let $0 < x \leq y < z' < y' < z$, and assume $(z,3) = c(\{(x,1),(y,2),(z,3)\})$ but $(x,1) = c(\{(x,1),(y',2),(z,3)\})$. Such behavior is not a doing-it-earlier reversal, but has the same intuition: in $\{(x,1),(y,2),(z,3)\}$, y is sufficiently low that the person would wait at $t = 2$ and anticipate this at $t = 1$, but y' is sufficiently high that when facing $\{(x,1),(y',2),(z,3)\}$ the person anticipates at $t = 1$ that he would do it at $t = 2$, and avoids that by doing it immediately. As in a doing-it-earlier reversal, a naïve person would never exhibit such a reversal, since she would only expect at $t = 1$ that she would wait at $t = 2$ if her $t = 1$ self prefers to wait until $t = 3$.

Now instead suppose that $(x,1) = \tilde{c}(\{(x,1),(y,2),(z',3)\})$ but $(y,2) = \tilde{c}(\{(x,1),(y,2),(z,3)\})$. This clearly indicates naïveté, since if the person did $(y,2)$ over $(z,3)$, she would also do $(y,2)$ over $(z',3)$. Thus, when facing $\{(x,1),(y,2),(z',3)\}$, doing it immediately and either naïveté or sophistication implies that the person prefers $(x,1)$ over $(y,2)$ at $t = 1$. Then waiting at $t = 1$ when facing $\{(x,1),(y,2),(z,3)\}$ implies naïveté.

Many prior experiments study pairwise choices among dated rewards to estimate intertemporal utility functions. The examples above show that observing behavior in a modification of that domain can be similarly used to estimate a person's degree of sophistication.

6.2 Limited datasets

The doing-it-later and earlier reversals each only apply in a restricted set of comparisons, and the characterizations in Theorem 4 assume that the analyst observes how a person would behave in all possible opportunity sets $A \in \mathscr{A}$. Yet the analysis here broadly points to the testable content of both naïveté and sophistication that could be harnessed with appropriate tests that could apply even when only a small number of choices are observed that do not permit direct tests for reversals or for the properties in Section 3. When only a subset of choice problems is observed, the no-reversals property (which is equivalent to the Weak Axiom of Revealed Preference on this domain) is insufficient to guarantee that choices will maximize a complete and transitive binary relation, motivating the Strong Axiom of Revealed Preference (Houthakker, 1950). Appendix A provides analogous solutions for testing naïveté and sophistication. It proposes tests that work by checking whether it is possible to construct a set $\{R_t\}_{t=1}^{T}$ of complete, transitive, and asymmetric binary relations corresponding to $\mathscr{U}$ that generate a naïve/sophisticated representation consistent with observed choices. These tests are in the spirit of limited dataset tests of models of boundedly rational choice outlined by De Clippel and Rozen (2014).

6.3 Relation with work on "choice over menus" in decision theory

Work on decision theory has used a person's explicit preferences over choice sets to characterize the implications of sophistication for people with time-inconsistent preferences (Gul and Pesendorfer, 2005) or intrinsic preferences for smaller choice sets due to self-control costs (Noor, 2011). These papers have not sought analogous chacterizations for naïve choice. The comparison can be made more explicit by nesting the choice environment of this paper within that of Gul and Pesendorfer (2005); such an extension also allows Theorem 3 to be extended to sophisticated models with intrinsic self-control costs (like the model of Gul and Pesendorfer

(2001; 2004)).[15] In recent work in a two-period choice setting, Ahn et al. (2016) provide a way of comparing the relative naïveté of decision-makers using their preferences over choice sets of lotteries and their subsequent choice. Their main result relates their comparative notion to properties of the $t = 1$ and $t = 2$ utility functions of expected utility maximizers, and they use the rich domain of lotteries to ensure that the extent of an individual's sophistication can be elicited. Since a person's higher-order beliefs about behavior are potentially relevant whenever a person faces a decision in more than two periods, none of their results directly apply to my setting when $T > 3$.

6.4 Relation with empirical work on measuring sophistication

Many existing experiments look for whether subjects demand commitment — a hallmark that indicates sophistication about a self-control problem. Experiments that are often cited as finding evidence of demand for commitment typically find that only 25-40% of participants — a minority — demand commitment (Ashraf, Karlan and Yin, 2006; Kaur, Kremer and Mullainathan, 2010; Duflo, Kremer and Robinson, 2011; John, 2016). Augenblick, Niederle and Sprenger (2015) is the notable exception that further demonstrates the robustness of this finding: while 59% of their subjects demand commitment at a price of $0, only 9% of subjects are willing to pay a positive price for it.

A related line of work takes as a hallmark of (partial) naïveté a person's willingness to commit to a contract that leads to an ex post outcome that is strictly suboptimal according to ex ante preferences. Indeed, a doing-it-later reversal can be viewed as an example of such behavior. In another example, Giné, Karlan and Zinman (2010) find that of the (small minority of) smokers who took up a commitment savings account tied to passing a subsequent nicotine test, only 19% of those tested passed the nicotine test and the remainder lost all of their savings in that account. Heidhues and Kőszegi (2010) show that (partial or full) naïveté could lead a borrower to choose a contract that offers a cheap front-loaded repayment schedule with a high late-payment penalty and then end up paying the penalty, achieving a

[15] Results along these lines appeared in an earlier version of this paper.

worse outcome for the contract-signing-self than she would have achieved from initially choosing an alternative back-loaded repayment. Della-Vigna and Malmendier (2004; 2006) find that many people purchase gym memberships at a price that is cost-inefficient compared to a pay-per-use price given ex post use, and similarly interpret their findings as evidence of partial naïveté.

Two other prior attempts to estimate a degree of sophistication estimate that people are, on average, completely naïve. But these papers' estimates require exceedingly strong assumptions. Fang and Wang (2015) assume that demographic heterogeneity related to one's expectation of contracting cancer is unrelated to degree of sophistication. Augenblick and Rabin (2015) assume that subjects' experimentally elicited individual beliefs about their future behavior correspond to the beliefs subjects use in making real decisions. In my view, this growing evidence that suggests that people are truly naïve motivates the need for choice-based hallmarks of naïveté (like doing-it-later reversals) and measures based on them.

Other previous experiments study the domain of task completion (Ariely and Wertenbroch, 2002; Burger, Charness and Lynham, 2011; Bisin and Hyndman, 2014) but do not attempt to use their data to distinguish naïveté from sophistication.

7 Discussion

This paper has established separate hallmarks of naïveté and sophistication in the domain of task completion. The examples and results here suggest how empirical work on task completion can be used to measure naïveté versus sophistication based on behavior, and how related but model-specific results can be obtained for models of partial naïveté. The results of such tests would provide a much-needed means of evaluating the appropriateness of alternative assumptions about sophistication and naïveté in applications of models of time-inconsistent preferences.

One implication of the analysis here is that firms that have data on task completion can learn the degree of sophistication or naïveté of their clients or employees, especially if they can experiment. For example, a financial institution that observes when a client pays her bills can use this information to target her with financial products that exploit her degree of naïveté without having to offer her a menu of contracts that screen for this — targeting that Heidhues and Kőszegi (2017) show can lower welfare. Similarly, a manager who observes when an employee completes work assignments can use this information to infer the degree of sophistication of the employee, and can use this to better tailor his work responsibilities and deadlines.

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Appendix: Proofs

Proof of Theorem 1

(i) $\implies$ (ii) Suppose c is observationally time consistent. Suppose $t_1 = c(A)$ and $t_3 = c(A \cup \{t_2\})$. If $t_3 \neq t_2$, then time consistency and $t_1 = c(A)$ implies that $t_1 = c(\{t_1, t_3\})$, but time consistency and $t_3 = c(A \cup \{t_2\})$ implies that $t_3 = c(\{t_1, t_3\})$, and thus, $t_3 = t_1$ must hold. This proves that c cannot exhibit a reversal.

(ii) $\implies$ (i)

Suppose that c exhibits no reversals. If $|A| = 2$, $t = c(A)$ and $t' \in A$, then $t = c(\{t, t'\})$ since $\{t, t'\} = A$. Thus observational time consistency holds for all $A \in \mathscr{A}$ with $|A| = 2$. Suppose that, for some $n > 2$, observational time consistency holds whenever $|A| < n$. Now consider some A with $|A| = n - 1$ and $t' \notin A$ and $t = c(A)$; let $A' = A \cup \{t'\}$. Let $t^c = c(A')$. By the no reversals property, $t^c = t$ or $t^c = t'$. For any $t'' \in (A') \backslash \{t^c\}$, the no reversals property implies that $c(A' \backslash \{t''\}) = c(A')$. By observational time consistency on $A' \backslash \{t''\}$, $c(A') = c(\{t^c, t'''\})$ for all $t''' \in A' \backslash \{t''\}$. But since the choice of t'' was arbitrary, $c(A \cup \{t'\}) = c(\{c(A \cup \{t'\}), t''\})$ as well. Thus observational time consistency is satisfied for $A \cup \{t'\}$.

(i) $\iff$ (iii)

First, show that c is observationally time consistent if and only if there exists a complete, transitive, and antisymmetric binary relation R such that $c(A)$ equals the R-maximal element in A for every $A \in \mathscr{A}$.

Define R by tRt' if and only if there exists a $A \in \mathscr{A}$ such that $t = c(A)$ and $t' \in A$. Since $c(\{t, t'\}) = t$ or $= t'$, R is complete. Since $t = c(A)$ and $t' \in A$ implies $t = c(\{t, t'\})$ while $t' = c(A')$ and $t \in A'$ implies $t' = c(\{t, t'\})$, R is antisymmetric. To show that R is transitive, suppose $t_1 R t_2$ and $t_2 R t_3$. Then by the definition of R and its antisymmetry, we must have (a) $t_1 = c(\{t_1, t_2\})$ and (b) $t_2 = c(\{t_2, t_3\})$.

By (a) and observational time consistency, $t_2 \neq c(\{t_1,t_2,t_3\})$. Similarly by (b), $t_3 \neq c(\{t_1,t_2,t_3\})$. Thus $t_1 = c(\{t_1,t_2,t_3\})$, so $t_1 R t_3$, proving that R is symmetric.

Conversely, suppose that there exists a complete and transitive binary relation R such that $c(A)$ equals the R-maximal element in A for every $A \in \mathscr{A}$. Now take an arbitrary $A \in \mathscr{A}$ and suppose $t = c(A)$. Since t must be R-maximal in c, $tRt'$for all $t' \in A$, which implies $t = c(\{t,t'\})$ for all $t' \in A$. Thus c must be observationally time consistent.

Second, show that such a representation is equivalent to a Strotzian representation that is both sophisticated and naïve.

First, define a function $V : \bar{A} \to \mathbb{N}$ by $V(t) = |\{t' \in \bar{A} : tRt'\}|$. By transitivity and antisymmetry of R, tRt' implies $V(t) > V(t')$; conversely, by completeness and transitivity of R, $V(t) > V(t')$ implies tRt'. By antisymmetry of R, V is an injection. Thus $c(A) = \arg\max_{t\in A} V(t)$ for all A. For each t, define U_t as the restriction of V to the domain $A_{\geq t}$. For each t_1 and $t_2 > t_1$ define $\hat{U}_{t_2|t_1} = U_{t_2}$, and note that by construction, $\hat{U}_{t_2|t_1} = U_{t_1}$ on $\bar{A}_{\geq t_1}$. By choice of $\mathscr{U}$, working backwards implies that at each t_1 and each $t_2 > t_1$ a person's perceived future decision $s(t_2, A, \hat{U}_{t_2|t_1}, \hat{\mathscr{U}}_{\cdot|t_1}) =$ act if and only if $t_2 = \arg\max_{t'\in A_{\geq t_2}} V(t')$, and similarly $s(t_1, A, U_{t_1}, \hat{\mathscr{U}}_{\cdot|t_1}) =$ act if and only if $t_1 = \arg\max_{t'\in A_{\geq t_1}} V(t')$. Thus $c(A) = \min\{t \in A : s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{act}\}$; by construction, $\mathscr{U}, \hat{\mathscr{U}}$ is a Strotzian representation that is both naïve and sophisticated.

□

Proof of Theorem 4

Part (i): Necessity.

Suppose c has a naïve representation $\mathscr{U}, \hat{\mathscr{U}}$.

First, suppose that $t_1 = c(A)$ and $t_3 = c(A \cup \{t_2\})$ where $t_1 \neq t_3 \neq t_2$. Since for each t, $\max_{t'\in(A\cup\{t_2\})_{>t}} U_t(t') \geq \max_{t'\in A_{>t}} U_t(t')$ and by naïveté $\hat{U}_{t'|t}(t'') = U_t(t'')$ for any $t < t' \leq t''$, we have that $s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t}) =$ wait implies $s(t, A \cup \{t_2\}, U_t, \hat{\mathscr{U}}_{\cdot|t}) =$ wait for each $t \neq t_2$. Since $t_2 \neq t_3$, it follows from the naïve representation that $t_3 > t_1$.

But since $s(t_1, A, U_{t_1}, \hat{\mathscr{U}}_{\cdot|t_1}) =$ act and $s(t_1, A \cup \{t_2\}, U_{t_1}, \hat{\mathscr{U}}_{\cdot|t_1}) =$ wait, it follows by the representation that $\max_{t' \in A_{>t_1}} U_{t_1}(t') < U_{t_1}(t_1) < \max_{t' \in (A \cup \{t_2\})_{>t_1}} U_{t_1}(t')$, thus it must be the case that $t_2 > t_1$ and $U_{t_1}(t_2) > U_{t_1}(t_1)$. Since $t_2, t_3 > t_1$, this proves that c satisfies the Irrelevant Alternatives Delay property. In addition, since $U_{t_1}(t_2) > U_{t_1}(t_1)$, for any A' with $t_2 \in A'$, $U_{t_1}(t_1) < U_{t_1}(t_2) \leq \max_{t' \in A'_{>t_1}} U_{t_1}(t')$, thus wait $= s(t_1, A', U_{t_1}, \hat{\mathscr{U}}_{\cdot|t_1})$, thus $t_1 \neq c(A')$. This proves that c satisfies Exclusion Consistency.

Part (i): Sufficiency.

Consider the following, alternative expression of the definition of naïve perception perfect equilibrium:

(i) $A_{>t} = \emptyset$ implies $s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t}) =$ act,
(ii) $U_t(t) > \max_{t' \in A_{>t}} U_t(t')$ implies $s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t}) =$ act, and
(iii) $U_t(t) < \max_{t' \in A_{>t}} U_t(t')$ implies $s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t}) =$ wait.

This definition gives $t = c(A)$ if and only if it is the case that $U_t(t) > U_t(t')$ for all $t' \in A_{\geq t}$, whereas for each $t' \in A_{<t}$, there exists a $t'' \in A_{\geq t'}$ such that $U_{t'}(t'') > U_{t'}(t')$.

Construct utility functions from choices. Construct U_t by setting $U_t(t) > U_t(t')$ if $t' > t$ and $t = c(\{t, t'\})$, and setting $U_t(t') > U_t(t)$ if $t' = c(\{t, t'\})$.

Such a U_t is well-defined, since if $t_1 < t_2, t_3$ and c has the cycle with (i) $t_1 = c(\{t_1, t_2\})$, (ii) $t_2 = c(\{t_2, t_3\})$ and $t_3 = c(\{t_1, t_3\})$, we would conclude that $U_{t_1}(t_3) > U_{t_1}(t_1) > U_{t_1}(t_2)$ and $U_{\min\{t_2,t_3\}}(t_2) > U_{\min\{t_2,t_3\}}(t_3)$. Intuitively, in this construction of U_{t_1}, choice only dictates for each $t_2 > t_1$ whether $U_{t_1}(t_1) \gtrless U_{t_1}(t_2)$; thus choice cannot generate cycles that prevent such a construction. Thus U_t is well-defined.

Show that the naïve perception perfect equilibrium option in a set is chosen. Suppose $t_1 < \cdots < t_n$ and for some $j > i$ we have $U_{t_i}(t_i) < U_{t_i}(t_j)$. Then by the definition of U_{t_i}, $t_j = c(\{t_i, t_j\})$. Irrelevant Alternatives Delay requires that a newly added option that leads to a reversal must lead to delay. Since and $t_i < t_k$ for all $k > i$, by repeatedly applying Irrelevant Alternatives Delay we have:

$$
\begin{aligned}
t_i \neq \quad & c(\{t_i, t_{i+1}, t_j\}) \\
\vdots \quad & \\
t_i \neq \quad & c(\{t_i, \ldots, t_n\})
\end{aligned}
$$

Irrelevant Alternatives Delay requires that if a newly added option leads to a reversal, it must be a later option than that originally chosen. Since $k < k'$ implies $t_k < t_{k'}$, by repeatedly applying Irrelevant Alternatives Delay we have:

$$
\begin{aligned}
t_i \neq \quad & c(\{t_{i-1}, t_i, \ldots, t_n\}) \\
\vdots \quad & \\
t_i \neq \quad & c(\{t_1, \ldots, t_n\})
\end{aligned}
$$

Thus, if $t_j > t_i$, $U_{t_i}(t_j) > U_{t_i}(t_i)$, and $t_j \in A$, then $t_i \neq c(A)$. Call this result (A).

Next, suppose that $t_1 < \cdots < t_n$ and $U_{t_1}(t_1) > U_{t_1}(t_i)$ for all $i > 1$. Then by the definition of U_{t_1}, $t_1 = c(\{t_1, t_i\})$ for each i. Thus, by Exclusion Consistency, for each i and j, we have $t_1 = c(\{t_1, t_i, t_j\})$. Now suppose that for any $A \subseteq \{t_1, \ldots t_n\}$ with $|A| < k$, we have $t_1 = c(A)$. Now consider A with $|A| = k$. Since $t_1 = c(A \backslash \{t_i\})$ for each $i \neq 1$, and $t_1 = c(\{t_1, t_j\})$, by Exclusion Consistency it must be the case that $t_1 = c(A)$ (since picking $i \neq j$ would generate a contradiction if $t_j = c(A)$ for $j > 1$). By applying this argument until $k = n$, we obtain that if $t_1 < \cdots < t_n$ and $U_{t_1}(t_1) > U_{t_1}(t_i)$ for all $i > 1$, then $t_1 = c(\{t_1, \ldots, t_n\})$. Call this result (B).

Next, suppose $t_1 < \cdots < t_{i-1} < t_i < \cdots < t_n$. Suppose further that $U_{t_i}(t_i) > U_{t_i}(t_j)$ for all $j > i$, and for each $j < i$, there exists a $k > j$ such that $U_{t_j}(t_k) > U_{t_j}(t_j)$. Then, by the definition of U_{t_i}, $t_i = c(\{t_i, t_j\})$ for each $j > i$. Then by result (B), we have $t_i = c(\{t_i, \ldots, t_n\})$. So now suppose that for some $j \leq i$, we have that $t_i = c(\{t_j, \ldots, t_n\})$. Then by result (A), $t_{j-1} \neq c(\{t_{j-1}, \ldots, t_n\})$; but since $t_{j-1} < t_i$, by Irrelevant Alternatives Delay, it follows that $t_i = c(\{t_{j-1}, \ldots, t_n\})$. Repeating the argument until $j=1$ yields that $t_i = c(\{t_1,\dots,t_n\})$. Thus, if t_i is the naïve perception perfect equilibrium prediction for $\{t_1,\dots,t_n\}$ under $\mathscr{U}$, then $t_i = c(\{t_1,\dots,t_n\})$. Call this result (C).

Show that the chosen option is the naïve perception perfect equilibrium. Suppose $t_i = c(\{t_1,\dots,,t_n\})$. Given $\mathscr{U}$, the naïve perception perfect equilibrium concept makes a unique choice prediction; by result (C), this must be equal to t_i.

Part (ii): Necessity.

Suppose c has a sophisticated representation $\mathscr{U},\hat{\mathscr{U}}$ and suppose $t' = c(A_{>t})$ and $t \in A$.

Then by the sophisticated representation,

$$\begin{aligned} t' &= \min\left\{\tau : s(\tau, A_{>t}, U_\tau, \hat{\mathscr{U}}_{\cdot|\tau}) = \text{act}\right\} \\ &= \min\left\{\tau > t : s(\tau, A_{\geq t}, U_\tau, \hat{\mathscr{U}}_{\cdot|\tau}) = \text{act}\right\} \\ &= \min\left\{\tau > t : s(\tau, A_{\geq t}, U_\tau, \hat{\mathscr{U}}_{\cdot|t}) = \text{act}\right\} \end{aligned}$$

Thus,

$$s(t, A_{\geq t}, U_t, \hat{\mathscr{U}}_{\cdot|t}) = \begin{cases} \text{act} & \text{if } U_t(t) > U_t(t') \\ \text{wait} & \text{if } U_t(t) < U_t(t') \end{cases}.$$

Thus, if $U_t(t) > U_t(t')$, then $t = \min\left\{\tau : s(\tau, A_{\geq t}, U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{act}\right\} = c(A_{\geq t})$. If instead $U_t(t) < U_t(t')$, then $t' = \min\left\{\tau : s(\tau, A_{\geq t}, U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{act}\right\} = c(A_{\geq t})$. Since these cases are exhaustive, it follows that Recursivity holds.

Part (ii): Sufficiency.

Construct $\{U_t\}_{t=1}^T$ using choices from two-element sets as in the proof in part (i) – the same arguments directly apply. For each t and each $t' > t$, define $\hat{U}_{t'|t} = U_{t'}$ on $\bar{A}_{\geq t'}$. It follows by this construction that $c(\{t_1,t_2\}) = \min\{t \in \{t_1,t_2\} : s(t, \{t_1,t_2\}, U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{act}\}$ for all $t_1, t_2 \in \bar{A}$, establishing a sophisticated representation on two-element sets. Now proceed by induction.

Suppose that $c(A) = \min\{t \in A : s(t, A, \mathscr{U}, \hat{\mathscr{U}}_{\cdot|t}) = t\}$ and $t_0 < \min A$. Applying Recursivity and then the construction of U_{t_0},

$$\begin{aligned} c(A \cup \{t_0\}) &= c(\{t_0, c(A)\}) \\ &= \begin{cases} t_0 & \text{if } U_{t_0}(t_0) > U_{t_0}(c(A)) \\ c(A) & \text{otherwise} \end{cases} \end{aligned}$$

Thus, since $s(t_0|A \cup \{t_0\}, \mathscr{U}, \hat{\mathscr{U}}^{s|\mathscr{U}}) = \begin{cases} t_0 & \text{if } U_{t_0}(t_0) > U_{t_0}(\hat{\tau}_{t_0}) \\ \text{wait} & \text{otherwise} \end{cases}$ for $\hat{\tau}_{t_0} = \min\{t \in A : s(t, A \cup \{t_0\}, U_t, \hat{\mathscr{U}}_{\cdot|t}) = t\} = \min\{t \in A : s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t}) = t\} = c(A)$, and $s(t, A \cup \{t_0\}, U_t, \hat{\mathscr{U}}_{\cdot|t}) = s(t, A, U_t, \hat{\mathscr{U}}_{\cdot|t})$ for all $t > t_0$, it follows that $c(A \cup \{t_0\}) = \min\{t \in A \cup \{t_0\} : s(t, A \cup \{t_0\}, U_t, \hat{\mathscr{U}}_{\cdot|t}) = \text{act}\}$. The sophisticated representation thus extends to all of $\mathscr{A}$ by induction.

□

Proof of Theorem 5

Necessity.

(i) Suppose c has a Strotzian representation $\mathscr{U}, \hat{\mathscr{U}}$. Then, given any t_0, t_1, t_2 with $t_0 < t_1 < t_2$ we either have $\hat{U}_{t_1|t_0}(t_1) > \hat{U}_{t_1|t_0}(t_2)$ or $\hat{U}_{t_1|t_0}(t_2) > \hat{U}_{t_1|t_0}(t_1)$. In the former case, for all $A \subseteq \{t_0, \ldots, t_1 - 1\}$ we have $t_1 = s(t_1, A \cup \{t_1, t_2\}, \hat{U}_{t_1|t_0}, \hat{\mathscr{U}}_{\cdot|t_0})$, and thus $s(t, A \cup \{t_1, t_2\}, \hat{U}_{t|t_0}, \hat{\mathscr{U}}_{\cdot|t_0}) = s(t, A \cup \{t_1\}, \hat{U}_{t|t_0}, \hat{\mathscr{U}}_{\cdot|t_0})$ for all t with $t_0 < t < t_1$, thus, $c_{t_0}(A \cup \{t_1, t_2\}) = s(t_0, A \cup \{t_1, t_2\}, U_{t_0}, \hat{\mathscr{U}}_{\cdot|t_0}) = s(t_0, A \cup \{t_1\}, U_{t_0}, \hat{\mathscr{U}}_{\cdot|t_0})$. An analogous argument applies in the latter case, implying $c_{t_0}(A \cup \{t_1, t_2\}) = s(t_0, A \cup \{t_1, t_2\}, U_{t_0}, \hat{\mathscr{U}}_{\cdot|t_0}) = s(t_0, A \cup \{t_2\}, U_{t_0}, \hat{\mathscr{U}}_{\cdot|t_0})$ for all $A \subseteq \{t_0, \ldots, t_1 - 1\}$. Thus, c satisfies Penultimate Replaceability.

(ii) Next, suppose that c has partially naïve representation $\mathscr{U}, \hat{\mathscr{U}}$, and $c(\{t_1, t_2\}) = c(\{t_0, t_2\}) = t_2$ but $c(\{t_0, t_1\}) = t_0$. Thus, $U_{\min\{t_1, t_2\}}(t_2) > U_{\min\{t_1, t_2\}}(t_1)$ and $U_{t_0}(t_2) > U_{t_0}(t_0) > U_{t_0}(t_1)$. Then by partially naïve restriction, $\hat{U}_{\min\{t_1, t_2\}|t_0}(t_2) > \hat{U}_{\min\{t_1, t_2\}|t_0}(t_1)$. By the Strotzian representation, $t_2 = c(\{t_0, t_1, t_2\}) \neq t_0$. Thus, c must satisfy Wishfulness.

Sufficiency.

(i) Let c satisfy Penultimate Replaceability.

Construct $\mathscr{U}$ and $\hat{\mathscr{U}}$ to satisfy requirement 1 and 2 below.

Requirement 1. If $t_1 = c(\{t_1,t_2\})$, then require $U_{\min\{t_1,t_2\}}(t_1) > U_{\min\{t_1,t_2\}}(t_2)$.

Requirement 2. If $t_1,t_2 > t_0$, and $\exists A$ such that $t_0 \in A \subseteq \{t_0,\dots,\min\{t_1,t_2\}-1\}$ and $c_{t_0}(A \cup \{t_1,t_2\}) = c_{t_0}(A \cup \{t_1\}) \neq c_{t_0}(A \cup \{t_2\})$, then require that $\hat{U}_{\min\{t_1,t_2\}|t_0}(t_1) > \hat{U}_{\min\{t_1,t_2\}|t_0}(t_2)$. By Penultimate Replaceability, each $\hat{U}_{\min\{t_1,t_2\}|t_0}$ can be constructed to satisfy this requirement.

Given $\hat{\mathscr{U}}$, define $\hat{c}^{|t_0}$ mapping from subsets of $\mathscr{A}_{>t_0}$ to $\bar{A}_{>t_0}$, by $\hat{c}^{|t_0}(A_{>t_0}) = \min\{t \in A_{>t_0} : s(t,A,\hat{U}_{t|t_0},\hat{\mathscr{U}}_{\cdot|t_0}) = \text{act}\}$. The function $\hat{c}^{|t_0}$ is the time t_0 self's expected future choice function in a Strotzian representation.

It remains to show that for arbitrary t_0 and A that $c_{t_0}(\{t_0\} \cup A_{>t_0}) = c_{t_0}(\{t_0,\hat{c}^{|t_0}(A_{>t_0})\})$. We know that the representation holds whenever $|A_{>t_0}| = 1$ by the representation on two element choice sets. So suppose that, for some $m \geq 2$, $c_{t_0}(\{t_0,\dots,t_n\}) = c_{t_0}(\{t_0,\hat{c}^{|t_0}(\{t_1,\dots,t_n\})\})$ whenever $n < m$ (where $t_0 < t_1 < \cdots < t_n$). Now let $n = m$ and consider $\{t_0,\dots,t_n\}$. By Penultimate Replaceability, $c_{t_0}(\{t_0,\dots,t_n\}) = c_{t_0}(\{t_0,\dots,t_{n-2},t_n\})$ or $= c_{t_0}(\{t_0,\dots,t_{n-1}\})$; by the construction of $\hat{U}_{\min\{t_1,t_2\}|t_0}$ and $\hat{c}^{|t_0}$, we have $c_{t_0}(\{t_0,\dots,t_n\}) = c_{t_0}(\{t_0,\dots,t_{n-2},\hat{c}^{|t_0}(\{t_{n-1},t_n\})\})$. Since $\{t_1,\dots,t_{n-2},\hat{c}^{|t_0}(\{t_{n-1},t_n\})\}$ has only $m-1$ elements, it follows that

$$c_{t_0}(\{t_0,\dots,t_n\}) = c_{t_0}(\{t_0,\hat{c}^{|t_0}(\{t_1,\dots,t_{n-2},\hat{c}^{|t_0}(\{t_{n-1},t_n\})\})\}) \qquad (2)$$

Applying our recursive definition of $\hat{c}^{|t_0}$, working forward from t_1,

$$\begin{aligned}
\hat{c}^{|t_0}(\{t_1,\dots,t_{n-1},\hat{c}^{|t_0}(\{t_{n-1},t_n\})\}) &= \hat{c}^{|t_0}(\{t_1,\hat{c}^{|t_0}(\{t_2,\dots,t_{n-2},\hat{c}^{|t_0}(\{t_{n-1},t_n\})\})\}) \\
&= \hat{c}^{|t_0}(\{t_1,\hat{c}^{|t_0}(\{t_2,\hat{c}^{|t_0}(\{t_3,\dots,t_{n-2},\hat{c}^{|t_0}(\{t_{n-1},t_n\})\})\})\}) \\
\vdots \quad &\vdots \quad \vdots \\
&= \hat{c}^{|t_0}(\{t_1,\hat{c}^{|t_0}(\{t_2,\hat{c}^{|t_0}(\{t_3,\hat{c}^{|t_0}(\{t_3,\dots\})\})\})\}) \\
&= \hat{c}^{|t_0}(\{t_1,\dots,t_n\}) \qquad (3)
\end{aligned}$$

Then, working backwards from the final expression, repeatedly apply the definition of $\hat{c}^{|t_0}$ to simplify: $\hat{c}^{|t_0}(\{t_{n-2},\hat{c}^{|t_0}(\{t_{n-1},t_n\})\}) = \hat{c}^{|t_0}(\{t_{n-2},t_{n-1},t_n\})$, $\hat{c}^{|t_0}(\{t_{n-3},\hat{c}^{|t_0}(\{t_{n-2},t_{n-1},t_n\})\}) = \hat{c}^{|t_0}(\{t_{n-3},t_{n-2},t_{n-1},t_n\})$, and so on, until obtaining $\hat{c}^{|t_0}(\{t_1,\hat{c}^{|t_0}(\{t_2,\ldots,t_n\})\}) = \hat{c}^{|t_0}(\{t_1,\ldots,t_n\})$. Combining this with (2) and (3), obtain that $c_{t_0}(\{t_0,\ldots,t_n\}) = c_{t_0}(\{t_0,\hat{c}^{|t_0}(\{t_1,\ldots,t_n\})\})$. Repeating the preceding steps of argument establishes this result, and thus the Strotzian representation for arbitrary choice sets of size n; the representation holds for arbitrary sets by induction. This proves part (i).

Next, suppose that c also satisfies Wishfulness.

For each $t \in \bar{A}$, R_t is acyclic, each has an asymmetric transitive closure, denote it by $\bar{R}_t$. By Szpilrajn's Theorem (see Ok 2007, p. 17) each $\bar{R}_t$ has a transitive and asymmetric completion $\bar{\bar{R}}_t$, which hence has a one-to-one utility representation U_t. Based on this reasoning, consider the following strengthening of Requirement 1 to 1' and the additional Requirement 3.

Requirement 1'. For each $t < T$, U_t represents a transitive and asymmetric completion $\bar{\bar{R}}_t$ of R_t.

Requirement 3. If $t_1,t_2 > t_0$ and $\forall A$ such that $t_0 \in A \subseteq \{t_0,\ldots,\min\{t_1,t_2\}-1\}$, $c_{t_0}(A\cup\{t_1,t_2\}) = c_{t_0}(A\cup\{t_1\}) = c_{t_0}(A\cup\{t_2\})$, $t_1 = c(\{t_1,t_2\})$, then require $\hat{U}_{\min\{t_1,t_2\}|t_0}(t_1) > \hat{U}_{\min\{t_1,t_2\}|t_0}(t_2)$.

By Penultimate Replaceability and Wishfulness, $\mathscr{U}$ and $\hat{\mathscr{U}}$ can be constructed to satisfy Requirements 1', 2, and 3.

It remains to show the partial naïve restriction holds. Take any t_1. From the definition of R_{t_1} and construction of $U_{\min\{t_2,t_3\}}$, $U_{\min\{t_2,t_3\}}(t_2) > U_{\min\{t_2,t_3\}}(t_3)$ and $\hat{U}_{\min\{t_2,t_3\}|t_1}(t_3) > \hat{U}_{\min\{t_2,t_3\}|t_1}(t_2)$ implies that $t_3 R_{t_1} t_2$, which implies $U_{t_1}(t_3) > U_{t_1}(t_2)$ – thus implying that the representation is partially naïve. Further, this constructed representation is minimally naïve.

□

Proof of Corollary 1

Take c and c' with minimally naïve representations with the same $\mathscr{U}$ and different $\hat{\mathscr{U}}$ and $\hat{\mathscr{U}}'$.

Suppose $\mathscr{U},\hat{\mathscr{U}}$ is more sophisticated than $\mathscr{U},\hat{\mathscr{U}}'$. If $t_1,t_2 > t_0$ and $\forall A$ such that

$t_0 \in A \subseteq \{t_0,\ldots,\min\{t_1,t_2\}-1\}$, $c'_{t_0}(A\cup\{t_1,t_2\}) = c'_{t_0}(A\cup\{t_1\}) \neq c'_{t_0}(A\cup\{t_2\})$ and $t_1 = c'(\{t_1,t_2\})$, then $\hat{U}'_{\min\{t_1,t_2\}|t_0}(t_1) > \hat{U}'_{\min\{t_1,t_2\}|t_0}(t_2)$ and $U_{\min\{t_1,t_2\}}(t_1) > U_{\min\{t_1,t_2\}}(t_2)$ by the representation for c'. Since $\mathscr{U},\hat{\mathscr{U}}$ is more sophisticated than $\mathscr{U},\hat{\mathscr{U}}'$, it follows that $\hat{U}_{\min\{t_1,t_2\}|t_0}(t_1) > \hat{U}_{\min\{t_1,t_2\}|t_0}(t_2)$, and thus by the representation for c, that $c_{t_0}(A\cup\{t_1,t_2\}) = c_{t_0}(A\cup\{t_1\}) = c_{t_0}(A\cup\{c(\{t_1,t_2\})\})$. Thus c is more penultimately accurate than c'.

Next, suppose c is more penultimately accurate than c'.

Suppose $\exists A$ such that $t_0 \in A \subseteq \{t_0,\ldots,\min\{t_1,t_2\}-1\}$, $c'_{t_0}(A\cup\{t_1,t_2\}) = c'_{t_0}(A\cup\{t_1\}) \neq c'_{t_0}(A\cup\{t_2\})$, then $\hat{U}'_{\min\{t_1,t_2\}|t_0}(t_1) > \hat{U}'_{\min\{t_1,t_2\}|t_0}(t_2)$ must hold by the representation. Consider two subcases.

First, if $t_1 = c'(\{t_1,t_2\})$, then $U_{\min\{t_1,t_2\}}(t_1) > U_{\min\{t_1,t_2\}}(t_2)$ by the representation for c'. Thus $t_1 = c(\{t_1,t_2\})$ by the representation for c, and thus since c is more penultimately accurate than c', $c_{t_0}(A\cup\{t_1,t_2\}) = c_{t_0}(A\cup\{t_1\})$ for all $A \subseteq \{t_0,\ldots,\min\{t_1,t_2\}-1\}$. Therefore, a minimally naïve representation for c must have $\hat{U}_{\min\{t_1,t_2\}|t_0}(t_1) > \hat{U}_{\min\{t_1,t_2\}|t_0}(t_2)$. In this subcase, $\hat{U}_{\min\{t_1,t_2\}|t_0}$ must be equally sophisticated as $\hat{U}'_{\min\{t_1,t_2\}|t_0}$ about that comparison.

Second, if $t_2 = c'(\{t_1,t_2\})$, then $U_{t_0}(t_1) > U_{t_0}(t_2)$ since $\mathscr{U},\hat{\mathscr{U}}'$ is a partially naïve representation for c'. Since $\hat{U}'_{\min\{t_1,t_2\}|t_0}$ is naïve about this comparison in this case, the definition of more sophisticated than puts no restriction on $\hat{U}_{\min\{t_1,t_2\}|t_0}$ about this comparison so $\hat{U}_{\min\{t_1,t_2\}|t_0}$ is (trivially) more sophisticated than $\hat{U}'_{\min\{t_1,t_2\}|t_0}$ about this comparison.

Since the choice of t_0,t_1,t_2 was arbitrary, it follows that $\mathscr{U},\hat{\mathscr{U}}$ is more sophisticated than $\mathscr{U},\hat{\mathscr{U}}'$.

□

Proof of Proposition 1

(i) $\Longrightarrow$ (ii). Suppose condition (i) holds, and c' exhibits doing-it-later reversal $c'(\{t_1,t_2,t_3\}) = t_2 > t_1 = c(\{t_1,t_2\})$. Then, $U_{t_1}(t_1) > U_{t_1}(t_2)$, thus by the Strotzian representation, waiting at t_1 when facing $\{t_1,t_2,t_3\}$ implies that $U_{t_1}(t_3) > U_{t_1}(t_1)$ and $\hat{U}'_{\min\{t_2,t_3\}|t_1}(t_3) > \hat{U}'_{\min\{t_2,t_3\}|t_1}(t_2)$; the fact that the decision-maker subsequently acts at t_2 implies $U_{\min\{t_2,t_3\}}(t_2) > U_{\min\{t_2,t_3\}}(t_3)$. Then by (i)

$\hat{U}_{\min\{t_2,t_3\}|t_1}(t_3) > \hat{U}_{\min\{t_2,t_3\}|t_1}(t_2)$, it follows from the Strotzian representation for c that $c(\{t_1,t_2,t_3\}) = t_2 > t_1 = c(\{t_1,t_2\})$. Thus, (i) implies (ii).

(ii) $\implies$ (iii). Now suppose that (ii) holds. Suppose that $t_1 < t_2, t_3$ generates the doing-it-earlier reversal $c(\{t_1,t_2,t_3\}) = t_1 < t_2 = c(\{t_1,t_2\})$ for c (if no such t_1,t_2,t_3 exist, then c exhibits no three opportunity doing-it-earlier reversals, so the desired conclusion holds trivially). Since $t_2 = c(\{t_1,t_2\})$, the Strotzian representation implies that $U_{t_1}(t_2) > U_{t_1}(t_1)$. Since $t_1 = c(\{t_1,t_2,t_3\})$, it must also be the case that $U_{t_1}(t_1) > U_{t_1}(t_3)$ and $\hat{U}_{\min\{t_2,t_3\}|t_1}(t_3) > \hat{U}_{\min\{t_2,t_3\}|t_1}(t_2)$. Since c has a partially naïve representation, it follows that $U_{\min\{t_2,t_3\}}(t_3) > U_{\min\{t_2,t_3\}}(t_2)$. If $\hat{U}'_{\min\{t_2,t_3\}|t_1}(t_3) < \hat{U}'_{\min\{t_2,t_3\}|t_1}(t_2)$, then by the representation for c', $t_3 = c'(\{t_1,t_2,t_3\})$; but by the representation for c' it is also the case that $t_3 > t_1 = c'(\{t_1,t_3\})$, thus this is a doing-it-later reversal for c' not exhibited by c, which contradicts that c exhibits more three opportunity doing-it-later reversals than c'. Thus, $\hat{U}'_{\min\{t_2,t_3\}|t_1}(t_3) > \hat{U}'_{\min\{t_2,t_3\}|t_1}(t_2)$ and $t_1 = c'(\{t_1,t_2,t_3\})$ and $t_2 = c'(\{t_1,t_2\})$, proving (iii).

(iii) $\implies$ (i). Now suppose that (iii) holds, $\hat{U}_{\min\{t_2,t_3\}|t_1}(t_2) \gtrless \hat{U}_{\min\{t_2,t_3\}|t_1}(t_3)$ is three opportunity revealable, $U_{\min\{t_2,t_3\}}(t_2) > U_{\min\{t_2,t_3\}}(t_3)$, and $\hat{U}_{\min\{t_2,t_3\}|t_1}(t_2) > \hat{U}_{\min\{t_2,t_3\}|t_1}(t_3)$. Then since c has a partially naïve representation and $\hat{U}_{\min\{t_2,t_3\}|t_1}(t_2) > \hat{U}_{\min\{t_2,t_3\}|t_1}(t_3)$ is three opportunity revealable, it must be the case that either (a) $U_{t_1}(t_3) > U_{t_1}(t_1) > U_{t_1}(t_2)$ or (b) $U_{t_1}(t_2) > U_{t_1}(t_1) > U_{t_1}(t_3)$. In case (b), since c' has a partially naïve representation and $U_{\min\{t_2,t_3\}}(t_2) > U_{\min\{t_2,t_3\}}(t_3)$, it follows that $\hat{U}'_{\min\{t_2,t_3\}|t_1}(t_2) > \hat{U}'_{\min\{t_2,t_3\}|t_1}(t_3)$. In case (a), the representation for c implies that $c(\{t_1,t_2,t_3\}) = t_1 < t_3 = c(\{t_1,t_3\})$, a doing-it-earlier reversal. But since c' exhibits more doing-it-earlier reversals than c, $c'(\{t_1,t_2,t_3\}) = t_1 < t_3 = c'(\{t_1,t_3\})$ as well. This implies that $\hat{U}'_{\min\{t_2,t_3\}|t_1}(t_2) > \hat{U}'_{\min\{t_2,t_3\}|t_1}(t_3)$ – the desired result.

□

A Supplementary Appendix: Limited Dataset Tests

In applications, the analyst will likely only observe a limited number of choices. The analysis below provides a testable characterizations of the class of Strotzian, sophisticated, and naïve representations that also apply to such limited datasets.[16]

First, introduce notation for partial datasets. Let $\mathscr{A}_{obs}$ denote a subset of $\mathscr{A}$, and let $c_{obs} : \mathscr{A}_{obs} \to \bar{A}$ denote a choice function on $\mathscr{A}_{obs}$. The function c_{obs} denotes the observed choice function, with its domain being the set of observed choices $\mathscr{A}_{obs}$. Next, consider testable conditions under which a given c_{obs} can be extended to a full-domain choice function $c : \mathscr{A} \to \bar{A}$ with a Strotzian representation.

Strotzian No-Cycle Condition. There exist $\{R_t\}_{t\in\bar{A}}, \{\hat{R}_{t'|t}\}_{t\in A, t'\in A_{>t}}$ with each relation antisymmetric on its domain that satsify the following:

if $t = c_{obs}(A)$, then for each $t' \in A$,

(i) there exists a chain $t_1, \ldots, t_n$ with $t_n = \max A$, and $t' < t_1 < \cdots < t_n$ such that $t_1 R_{t'} t'$ and for each $k < n$, $t_k \hat{R}_{t_k|t'} t_{k+1}$ and $t_{k+1} \hat{R}_{t''|t'} t''$ for all $t'' \in A_{<t_{k+1}} \cap A_{>t_k}$,

(ii) there exists a chain $t_1, \ldots, t_n$ with $t_n = \max A$, and $t < t_1 < \cdots < t_n$ such that $t R_t t_1$ and for each $k < n$, $t_k \hat{R}_{t_k|t} t_{k+1}$ and $t_{k+1} \hat{R}_{t''|t'} t''$ for all $t'' \in A_{<t_{k+1}} \cap A_{>t_k}$.

Proposition 4. *c_{obs} satisfies the Strotzian No-Cycle Condition if and only if there exists a choice function $c : \mathscr{A} \to \bar{A}$ with a Strotzian representation and $c_{obs} \subseteq c$.*

Proof. Necessity and sufficiency almost immediately follows from the axiom.

Suppose c has a partially naïve representation corresponding to $\mathscr{U}$ and $\hat{\mathscr{U}}$. Pick each R_t as the order implied by U_t and $\hat{R}_{t'|t}$ as the order implied by $\hat{U}_{t'|t}$. These orders are antisymmetric by construction. Now suppose $t = c_{obs}(A)$. Given any t', pick $t_1, \ldots, t_n$ to satisfy $t' < t_1 < \cdots < t_n = \max_{\tilde{t}\in\bar{A}} \tilde{t}$ and $t_k = s(t_k, A, \hat{U}_{t_k|t'}, \hat{\mathscr{U}}_{\cdot|t'})$ for each k. By the definition of a perception perfect strategy and the choice of $\{R_{\tilde{t}}\}_{\tilde{t}}, \{\hat{R}_{\tilde{t}'|\tilde{t}}\}_{\tilde{t},\tilde{t}'}$, these sequences verify that parts (i) and (ii) hold in the Strotzian No-Cycle Condition.

[16] De Clippel and Rozen (2014) note that given a set of axioms that characterize a given theory given a complete choice function, it may be possible for a partial dataset to pass direct tests of each axiom while not being consistent with the given theory. They show that this can be particularly important when testing models that violate standard choice axioms, motivating the exercise here.

Conversely, suppose c_{obs} satisfies the Strotzian No-Cycle Condition. Notice that for any t and $t' > t$, choice only pins down whether $U_t(t) \gtrless U_t(t')$; since each R_t is well defined, we can construct U_t to represent R_t for each t, and by a similar argument we can construct $\hat{U}_{t'|t}$ to represent $\hat{R}_{t'|t}$ for each $t, t' > t$. Then by construction and the Strotzian No-Cycle Condition, and the $c(A)$ is the perception perfect equilibrium choice from A given $\mathscr{U}, \hat{\mathscr{U}}$ for each $A \in \mathscr{A}_{obs}$. $\square$

Using the Strotzian No-Cycle Condition to test the Strotzian model requires checking $\sum_{t=1}^{|\mathscr{T}|}(t^2 - t)$ different binary relations. The large number of different binary relations reflects flexibility in the partially naïve model. Note, however, that the sophisticated and naïve models are restrictions of the Strotzian model in which each $\hat{U}_{t'|t}$ is tied to an element in $\mathscr{U}$ in a particular way — in the sophisticated model, $\hat{U}_{t'|t} = U_{t'}$ for each t and $t' > t$, while in the naïve model, $\hat{U}_{t'|t}$ is the restriction of U_t to $\bar{A}_{\geq t'}$. These restrictions correspond to analogous restrictions on the Strotzian No-Cycle Condition. These place additional restrictions on $\mathscr{U}$ that allow choice to identify whether $U_{t_1}(t_2) \gtrless U_{t_1}(t_3)$, even if $U_{t_1}(t_1)$ is higher than (or lower than) both of them. As a result, we must now ensure that each R_t has no cycles to ensure that each U_t can be constructed.

Naïve No-Cycle Condition. There exists $\{R_t\}_{t \in \mathscr{T}}$ with each complete, transitive, and antisymmetric on its domain that satsify the following:

if $t = c_{obs}(A)$, then for each $t' \in A$,

(i) if $t' \neq t$, there exists a chain $t_1, \dots, t_n$ with $t_n = \max \bar{A}$, and $t' < t_1 < \cdots < t_n$ such that $t_1 R_{t'} t'$ and for each $k < n$, $t_k R_{t'} t_{k+1}$ and $t_{k+1} R_{t'} t''$ for all $t'' \in A_{<t_{k+1}} \cap A_{>t_k}$.

(ii) there exists a chain $t_1, \dots, t_n$ with $t_n = \max \bar{A}$, and $t < t_1 < \cdots < t_n$ such that $t R_t t_1$ and for each $k < n$, $t_k R_t t_{k+1}$ and $t_{k+1} R_{t'} t''$ for all $t'' \in A_{<t_{k+1}} \cap A_{>t_k}$.

Sophisticated No Cycle Condition. There exists $\{R_t\}_{t \in \mathscr{T}}$ with each complete, transitive, and antisymmetric on its domain that satsify the following:

if $(a, t) = c_{obs}(A)$, then for each $t' \in A$,

(i) if $t' \neq t$, there exists a chain $t_1, \dots, t_n$ with $t_n = \max \bar{A}$, and $t' < t_1 < \cdots < t_n$ such that $t_1 R_{t'} t'$ and $t_k R_{t_k} t_{k+1}$ for each $k < n$,

(ii) there exists a chain $t_1, \ldots, t_n$ with $t_n = \max \bar{A}$, and $t < t_1 < \cdots < t_n$ such that $tR_t t_1$ and $t_k R_{t_k} t_{k+1}$ for each $k < n$.

Notice that checking either of the Sophisticated and Naïve No-Cycle Conditions only requires checking for the existence of $\bar{A}$ transitive completions of binary relations.

Corollary 2. *(i) c_{obs} satisfies the Naïve No-Cycle Condition if and only if there exists a choice function $c : \mathscr{A} \to \bar{A}$ with a naïve representation and $c_{obs} \subseteq c$.*

(ii) c_{obs} satisfies the Sophisticated No-Cycle Condition if and only if there exists a choice function $c : \mathscr{A} \to \bar{A}$ with a sophisticated representation and $c_{obs} \subseteq c$.