On 2-separated excluded minors for the class of frame matroids

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joint with

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Theorem (DeVos, F.)

Let $M$ be an excluded minor for the class of frame matroids, and suppose $M$ has a 2-separation. Then either

1. $M$ is a member of a finite list, or
2. $M$ is the 2-sum of $U_{2,4}$ and a 3-connected frame matroid.
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• excluded minor?
• frame matroid?
• 2-separation?
• 2-sum?

• $U_{2,4}$?
• 3-connected?
• what’s the list?
• so what?
Abstracting circuits

Let $E = \{e_1, e_2, \ldots, e_m\}$ be a set.
Let $\mathcal{C}$ be a collection of subsets of $E$, called circuits, such that

- $\emptyset$ is not a circuit
- no circuit is properly contained in another
- the circuit elimination axiom holds
Abstracting circuits: Matroids

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Let $C$ be a collection of subsets of $E$, called circuits, such that

- $\emptyset$ is not a circuit
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Such an ordered pair $(E, C)$ is called a matroid.
Abstracting circuits: Matroids

\[ M = (E, \mathcal{C}) \]

- \( \emptyset \) is not a circuit
- no circuit is properly contained in another
- the \textit{circuit elimination axiom} holds

\textbf{Question:} When is \( \mathcal{C} \) the set of circuits of a graph?
Example: a non-graphic matroid

\( U_{r,n} \)

- ground set \( \{1, \ldots, n\} \)
- circuits \( C \) are all subsets of size \( r + 1 \)

- \( U_{2,4} \) is not graphic
Example: a non-graphic matroid

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Put $E = \{a, b, c, d\}$

<table>
<thead>
<tr>
<th>Graph</th>
<th>Circuits?</th>
</tr>
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<tbody>
<tr>
<td>${a, b, c}$</td>
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<td>$\times$</td>
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Minors

- A *minor* is obtained by applying any sequence of *deleting* or *contracting* of elements
Minors

- deletion

\[ G \rightarrow G \setminus e \]
Minors

- contraction

\[ G \xrightarrow{f} \text{G} \]

\[ \text{G} / f \]
Minor-closed families

- a family (of graphs, of matroids) is minor-closed if every minor of every member is in the family
Minor-closed families

- a family (of graphs, of matroids) is *minor-closed* if every minor of every member is in the family
- An *excluded minor theorem* characterises a minor-closed family by exhibiting a list of minimal (w.r.t. minors) graphs/matroids not in the family
  - matroid $\in$ Family $\iff$ no minor in list
An excluded minor theorem

Theorem (Wagner/Kuratowski)

A graph can be embedded in the plane if and only if it has no $K_5$ or $K_{3,3}$ as a minor.
When is $\mathcal{C}$ the set of circuits of a graph?

Theorem (Tutte)

A matroid is graphic if and only if it has no $U_{2,4}$, $F_7$, $F_7^*$, $M^*(K_5)$, or $M^*(K_{3,3})$ minor.
When is $\mathcal{C}$ the set of circuits of a biased graph?

A theorem we would like

A matroid $(E, \mathcal{C})$ is frame if and only if it has no minor in the list \{N, $N'$, $N''$, \ldots\}. 

Frame matroids

A *frame* matroid is a matroid which can be extended to posses a basis $B_0$ (a *frame*) such that every element is spanned by two elements of $B_0$. 
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• elements of $B_0 = \text{vertices}$
• element spanned by $u, v \in B_0 = uv \text{ edge}$
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- elements of $B_0 = \text{vertices}$
- element spanned by $u, v \in B_0 = uv \text{ edge}$
- a minor closed class
Frame matroids - What do circuits look like?

- Forests are independent
- A leaf edge is not spanned by the elements remaining after removing its leaf; iterate
- \( k+1 \) edges on \( k \) vertices are dependent
- They are spanned by a set of rank \( k \)
- So a circuit \( C \) on \( k \) vertices has no leaf edge and \( k \) or \( k+1 \) edges w.m.a. connected.

\[ \implies \]
- \( C \) is a cycle (if \( |C| = k \)), or
- \( C \) is a pair of cycles joined by a path, or
- \( C \) is a theta (if \( |C| = k+1 \)).

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![Diagram](image-url)

- tight handcuffs
- loose handcuffs
- odd theta
Biased graphs

A *biased graph* is a pair \((G, \mathcal{B})\)

- a graph \(G\)
- together with a collection of distinguished cycles \(\mathcal{B}\)
  - called *balanced*
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- together with a collection of distinguished cycles \(\mathcal{B}\)
  - called *balanced*
  - obeying the *theta property*:

\[
\text{contains two balanced cycles} \quad \Rightarrow \quad \text{all three cycles balanced}
\]
Biased graphs

A *biased graph* is a pair \((G, B)\)

- A graph \(G\)
- Together with a collection of distinguished cycles \(B\)
  - Called *balanced*
  - Obeying the theta property:
    
    \[
    \text{contains two balanced cycles} \implies \text{all three cycles balanced}
    \]

- Zaslavsky: this yields a frame matroid \(M(G, B)\)
- We say \((G, B)\) represents \(M(G, B)\)
Example: Graphs on surfaces

Given a graph embedded on a surface

- put $\mathcal{B} = \{ \text{contractible cycles} \}$
Example: Signed graphs

Given a graph

- label each edge with $+1$ or $-1$
- put $\mathcal{B} = \{ \text{cycles with product of edge labels} = +1 \}$
Example: Signed graphs

Given a graph

- label each edge with $+1$ or $-1$
- put $\mathcal{B} = \{ \text{cycles with product of edge labels} = +1 \}$

- giving every edge label $-1$ we get $\mathcal{B} = \{ \text{even cycles} \}$
Graphs are biased graphs

- put $\mathcal{B} = \{ \text{all cycles} \}$
$U_{2,4}$ is frame

e_1 \ e_2 \ e_3 \ e_4

tight handcuffs
loose handcuffs
odd theta
$U_{2,4}$ is frame

- there are three biased graphs whose circuits are the circuits of $U_{2,4}$

- all cycles unbalanced
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2-sum $\iff$ 2-separation - Graphs

$G$

$e$

$G$

$H$

$e'$

$H$

$G$

$ee'$

$H$

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2-sum $\iff$ 2-separation - Graphs

$G \oplus_2 H$

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2-sum $\iff$ 2-separation - Graphs

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2-sum $\iff$ 2-separation - Graphs

$G \oplus_2 H$
2-sums of matroids

- Matroids $M, N$
- choose $e \in E(M), e' \in E(N)$
- $M \oplus_2 N$ on elements $e, e'$ is the matroid with

\[
G \oplus_2 H
\]
2-sums of matroids

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2-sum $\iff$ 2-separation - Matroids

Suppose $M$ has 2-separation $(A, B)$

Then $M = Ab \oplus_2 Ba$

- where the 2-sum is taken on elements $a$ and $b$
2-sums of biased graphs

balanced

handcuffs

odd theta

balanced $G$ $e$

unbalanced $H$ $e'$

$G \oplus_2 H$

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2-sums of biased graphs

balanced
handcuffs
odd theta

balanced
\(G\)

unbalanced
\(H\)

\(e\)

\(e'\)

\(G \oplus_2 H\)

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SFU
2-sums of biased graphs

balanced

handcuffs

odd theta

balanced

G ⊕ H

balanced unbalanced

G ⊕₂ H

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2-sums of biased graphs

balanced

handcuffs

odd theta

$G \oplus_2 H$
Not all 2-sums of frame matroids are frame

- all cycles unbalanced

- is a perfectly good matroid

- cannot be represented by a biased graph

\[ U_{2,4} \oplus_2 U_{4,6} \]
2-separated excluded minors for the class of frame matroids

Let $M = (E, C)$ be an excluded minor for the class of frame matroids, i.e.

- there is no biased graph $(G, B)$ with $M = M(G, B)$
2-separated excluded minors for the class of frame matroids

Let $M = (E, C)$ be an excluded minor for the class of frame matroids, i.e.

- there is no biased graph $(G, B)$ with $M = M(G, B)$
- for every $e \in E$, there is a biased graph representing $M \setminus e$ and a biased graph representing $M/e$
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Suppose $M$ has 2-separation $(A, B)$
2-separated excluded minors for the class of frame matroids

Then \( M = Ab \oplus_2 Ba \)
2-separated excluded minors for the class of frame matroids

Then $M = Ab \oplus_2 Ba$

- each of $Ab$ and $Ba$ is frame
- so each must be non-graphic
When is $C$ the set of circuits of a graph?

Theorem (Tutte)

A matroid $(E, C)$ is graphic if and only if it has no $U_{2,4}$, $F_7$, $F_7^*$, $M^*(K_5)$, or $M^*(K_{3,3})$ minor.

$dashed edges are signed −1$
2-separated excluded minors for the class of frame matroids

\[ M = Ab \oplus_2 Ba \]

\[ M = \begin{array}{c}
  M \\
  = \\
  ?? \\
  M =
\end{array} \begin{array}{c}
  M^* (K_{3,3}) \\
  U_{2,4} \\
  b \oplus_2 a
\end{array} \begin{array}{c}
  M^* (K_5) \\
  U_{2,4} \\
  a
\end{array} \begin{array}{c}
  M^* (K_5) \\
  U_{2,4}
\end{array} \]
On the list: $M = Ab \oplus_2 Ba$
On the list: \( M = Ab \oplus_2 Ba \)
Not on the list

\[ \emptyset \oplus 2 \cup 2, 4 = \emptyset \]

- \( U_{2,4} \oplus_2 U_{2,4} \) is frame
Let $M = (E, C)$ be an excluded minor for the class of frame matroids, and suppose $M$ has a 2-separation $(A, B)$, and $M$ is not one of the previous slides. Then

- one of $A \cup B$ contains $U_{2,4} \oplus_2 U_{4,6}$
- while the other contains $U_{2,4}$ as a proper minor.

$U_{2,4} \oplus_2 U_{4,6}$ is not frame
Proposition (DeVos, F.)

Let $M = (E, C)$ be an excluded minor for the class of frame matroids, and suppose $M$ has a 2-separation $(A, B)$, and $M$ is not on one of the previous slides. Then

- one of $Ab, Ba \cong U_{2,4}$
- while the other contains $U_{2,4}$ as a proper minor.
$\mathcal{L}$-excluded minors

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$\mathcal{L}$-excluded minors
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$U_{2,4} \setminus e = U_{2,3}$
\( L \)-excluded minors
The rest of the list

• we find nine more excluded minors:
$|\mathcal{L}| \geq 3$
$|\mathcal{L}| \geq 3$
The rest of the list: $|\mathcal{L}| \geq 3$

- the 2-sum of $U_{2,3}$ with 3 $U_{2,4}$'s, one on each element
The rest of the list: $|\mathcal{L}| \geq 3$

- the 2-sum of $U_{2,3}$ with 3 $U_{2,4}$'s, one on each element of $U_{2,3}$
The rest of the list: $|\mathcal{L}| = 2$

- Each $l_i$ gets 2-summed with a $U_{2,4}$
The rest of the list: \(|\mathcal{L}| = 2\)

- \(\mathcal{W}^3\)
  - \(B = \{l_2 cd\}\)

- \(Q_6\)
  - \(B = \emptyset\)

- \(M(W_4)\)
  - \(B = \{l_2 ac, l_2 bd\}\)

- each \(l_i\) gets 2-summed with a \(U_{2,4}\)

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$|\mathcal{L}| = 1$

- Any other excluded minor not in our list having a 2-separation looks like this:

$$U_{2,4} \oplus_2 3\text{-connected}$$
\(|L| = 1\)

- Any other excluded minor \( \notin \) our list having a 2-separation looks like this:

![Diagram](image)

**Theorem (DeVos, F.)**

*Let \( M \) be an excluded minor for the class of frame matroids, and suppose \( M \) has a 2-separation \((A, B)\). Then either*

1. \( M \in \) our list, or
2. \( M \) is the 2-sum of \( U_{2,4} \) and a 3-connected frame matroid.*
...or this?

![Diagram with text](image-url)
\[ |\mathcal{L}| = 1 \]

- we know 20 excluded minors of this form (so far)
- these all have rank 3 or 4

\[ U_{2,4} \oplus 2 \]

3-connected