Biased graphs, group labelled graphs, and well-quasi-ordering

Daryl Funk
joint with
Matt DeVos
Irene Pivotto (UWA)

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Outline

1. Biased graphs and group-labelled graphs
2. A topological characterisation of group labelled graphs
3. Excluded minors for classes of group labellable biased graphs
4. The blurry line between civilisation and the wilds
Theorem (DeVos, F., Pivotto) Given a biased graph \((G, B)\), construct a 2-cell complex \(K\). Then \((G, B)\) is group labellable \(\iff\) its set of balanced cycles \(B\) are precisely those contractible in \(K\).
Theorem (DeVos, F., Pivotto)

Given a biased graph \((G, \mathcal{B})\), construct a 2-cell complex \(K\). Then \((G, \mathcal{B})\) is group labellable \iff its set of balanced cycles \(\mathcal{B}\) are precisely those contractible in \(K\).
A *minor* of a graph: any graph obtained by a sequence of deletions or contractions of edges (and removing isolated vertices)
Minor closed classes and excluded minor characterisations

Planar graphs
  • a *minor closed* class
Minor closed classes and excluded minor characterisations

Planar graphs

- a *minor closed* class

Theorem (Kuratowski)

*A graph is not planar if and only if it has* $K_{3,3}$ *or* $K_5$ *as a minor.*
Minor closed classes and excluded minor characterisations

Equivalently

- $K_5$ and $K_{3,3}$ are the only minimally non-planar graphs

Theorem (Kuratowski)

The excluded minors for the class of planar graphs are $K_{3,3}$ and $K_5$. 

\[ K_5 \]

\[ K_{3,3} \]
Robertson & Seymour’s Graph Minors Project

Theorem (Generalised Kuratowski; Robertson & Seymour)
For each surface, there are only a finite number of excluded minors for the class of graphs that embed in the surface.
Robertson & Seymour’s Graph Minors Project

Theorem (Generalised Kuratowski; Robertson & Seymour)
For each surface, there are only a finite number of excluded minors for the class of graphs that embed in the surface.

Theorem (Robertson & Seymour)
Every minor closed class of graphs has only finitely many excluded minors.
Theorem (DeVos, F., Pivotto)

For every \( t \geq 3 \), there are minor closed classes of biased graphs having infinitely many excluded minors on \( t \) vertices.

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Biased graphs

Theorem (DeVos, F., Pivotto)

For every $t \geq 3$, there are minor closed classes of biased graphs having infinitely many excluded minors on $t$ vertices.
Biased graphs

A biased graph is a pair \((G, B)\)

- a graph \(G\)
- together with a collection of distinguished cycles \(B\)
  - called balanced
  - obeying the theta property:

contains two balanced cycles \(\implies\) all three cycles balanced
Biased graphs

A biased graph is a pair \((G, B)\)

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- together with a collection of distinguished cycles \(B\)
  - called balanced
  - obeying the theta property:

\[
\text{contains two balanced cycles} \implies \text{all three cycles balanced}
\]

- cycles not in \(B\) are unbalanced
- the bias of a cycle is its property of being balanced or unbalanced
Graphs on surfaces

Given a graph embedded on a surface

- put $\mathcal{B} = \{\text{contractible cycles}\}$
Signed graphs

Given a graph

- label each edge + or −
- let $B = \{ \text{cycles with even number of } − \text{ edges} \}$
Signed graphs

Given a graph

- label each edge $+$ or $-$
- let $\mathcal{B} = \{\text{cycles with even number of } - \text{ edges}\}$
Signed graphs

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Signed graphs

Given a graph

- label each edge $+$ or $-$
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Graphs are biased graphs

- put $B = \{ \text{all cycles} \}$
Group labelled graphs

Given $G = (V, E)$

- orient $E(G)$
- choose a group $\Gamma$
- label each edge with an element of $\Gamma$
- set $\mathcal{B} = \{C : \text{product of labels along simple closed walk } = 1\}$

\[
\gamma(C) = \alpha \beta \delta^{-1} \beta \epsilon \gamma^{-1} \delta
\]
Group labelled graphs

Given $G = (V, E)$

- orient $E(G)$
- choose a group $\Gamma$
- label each edge with an element of $\Gamma$
- set $\mathcal{B} = \{C : \text{product of labels along simple closed walk} = 1\}$

\[ \gamma(C) = \alpha \beta \delta^{-1} \beta^{-1} \delta \]

- let $\gamma : E(G) \to \Gamma$, extend $\gamma$ to closed walks
- set $\mathcal{B}_\gamma = \{C : \gamma(C) = 1\}$

- $\gamma$ is probably not well-defined on cycles, but $\mathcal{B}_\gamma$ is
Group labelled graphs are biased graphs

- always obey the theta property
  two balanced cycles $\implies$ all three cycles balanced
Group labelled graphs are biased graphs

- always obey the theta property
  - two balanced cycles $\implies$ all three cycles balanced
Not all biased graphs are group labellable

• \((G, B)\) is \textit{group labellable} if there is a group \(\Gamma\) and a labelling \(\gamma : E(G) \rightarrow \Gamma\) with \(B\gamma = B\)
Relabelling

- orient all edges out from a vertex $v$ replacing label $\alpha$ with $\alpha^{-1}$ if necessary
- left multiply label on each edge incident to $v$ by a group element $g$
Relabelling

- orient all edges out from a vertex $v$ replacing label $\alpha$ with $\alpha^{-1}$ if necessary
- left multiply label on each edge incident to $v$ by a group element $g$
- $B$ is unchanged
Not all biased graphs are group labellable
Not all biased graphs are group labelable

- Suppose we have a group labelling
  - w.m.a. edges oriented clockwise

\[ \mathcal{B} = \]

\[
\begin{array}{c}
  1 \\
  1 \\
  1 \\
  1 \\
\end{array}
\]
Not all biased graphs are group labelable

• Suppose we have a group labellabling
  • w.m.a. edges oriented clockwise

\[ B = \begin{pmatrix} 1 & 1 \\ \alpha & \alpha^{-1} \end{pmatrix} \]
Not all biased graphs are group labellable

- Suppose we have a group labellabling
  - w.m.a. edges oriented clockwise

\[
B = \begin{pmatrix}
1 & 1 \\
1 & \alpha^{-1} \\
\beta & 1 \\
1 & \beta^{-1}
\end{pmatrix}
\]
Which biased graphs are group labellable?

Theorem (DeVos, F, Pivotto)

Let \((G, B)\) be a biased graph. Let \(K\) be the 2-cell complex obtained by adding a disc with boundary \(C\) for each \(C \in B\). TFAE:

1. \((G, B)\) is \(\pi_1(K)\)-labellable.
2. \((G, B)\) is group labellable.
3. No unbalanced cycle can be moved to a balanced cycle via a sequence of balanced reroutings on closed walks.
4. A cycle \(C\) is contractible in \(K\) if and only if \(C \in B\).
Balanced rerouting

- $W$ a closed walk
Balanced rerouting

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Balanced rerouting

- $W'$ a closed walk obtained by a *balanced rerouting* of $W$
Balanced rerouting

- $W'$ a closed walk obtained by a balanced rerouting of $W$

- If $W, W'$ are cycles, the theta property $\Rightarrow W$ and $W'$ have same bias
Balanced rerouting in a group labelled graph

- $\gamma(W) = \beta \alpha = \gamma(W')$

- A sequence of balanced reroutings on closed walks will never move an unbalanced cycle to a balanced cycle.
Not all biased graphs are group labellable

Example: balanced rerouting

\[ \mathcal{B} = \]

- Diagram showing a balanced rerouting example.
Not all biased graphs are group labellable

Example: balanced rerouting
Not all biased graphs are group labelable

Example: balanced rerouting

\[ B = \]

\[ \begin{array}{c}
\begin{array}{cccc}
6 & 4 & 5 & 1 \\
3 & 2 & 6 & 7 \\
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{cccc}
6 & 4 & 5 & 1 \\
3 & 2 & 6 & 7 \\
\end{array}
\end{array} =
\begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
\end{array}
\end{array} \]

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Not all biased graphs are group labellable

Example: balanced rerouting
Which biased graphs are group labellable?

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\[1 \implies 2 \implies 3 \quad \checkmark\]
Which biased graphs are group labelable?

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- \(1 \implies 2 \implies 3\) \(\checkmark\)
- to show \(3 \implies 4 \implies 1\):
  - \(4 \implies 1\)
  - not \(4 \implies\) not \(3\)
Proof

• assume $G$ connected (else apply to each component)
• arbitrarily orient $E(G)$
• choose a spanning tree $T$
• label edges in $T$ with 1
• take a generator $g_e$ for each $e \in E(G) \setminus E(T)$
• take a simple closed walk $W$ around each balanced cycle
• add the group product given by $W$ to the set of relations
Proof

- note: there are no edges in a disc
Proof

\[ \Gamma = \langle \{g_1, g_2, \ldots, g_{10}\} \mid g_1g_2g_4 = 1, g_8g_9 = 1, g_6 = 1 \rangle \]
\[ \cong \pi_1(K) \]

• closed walk \( W \) is contractible in \( K \) \( \iff \gamma(W) = 1 \)

(Van Kampen’s theorem)
Proof

\[ \Gamma = \langle \{g_1, g_2, \ldots, g_8\} \mid g_1g_2g_4 = 1, g_5g_7 = 1, g_6 = 1 \rangle \]

\[ \cong \pi_1(K) \]

- every \( C \in \mathcal{B} \) is contractible in \( K \)
- if also every contractible cycle \( \in \mathcal{B} \) then this labelling realises \( \mathcal{B} \)

* i.e. \( (G, \mathcal{B}) \) is \(\pi_1(K)\)-labellable
Proof

If there is $C \notin B$ contractible in $K$

- take closed walk $W$ around $C$ with $\gamma(W) = h_1h_2 \cdots h_k = 1$
- the group relations in $\Gamma$ which reduce $h_1h_2 \cdots h_k$ to 1 yield a sequence of closed walks moving $C$ to a balanced cycle via balanced reroutings
Which biased graphs are group labellable?

Theorem (DeVos, F, Pivotto)

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\[1 \iff 2 \iff 3 \iff 4 \]

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Example

\[ \mathcal{B} = \]

![Diagram](image)
Example
Example

\[ \mathcal{B} = \]

\[ \mathcal{B} = \]

\[ \mathcal{B} = \]
Example

\[ \mathcal{B} = \]

\[ \mathcal{B} = \]

\[ \mathcal{B} = \]
Example

$B =$

$B =$

$B =$
Example

$\mathcal{B} =$

$\mathcal{B} =$

$\mathcal{B} =$
Example

\[ B = \]

\[ B = \]

\[ B = \]
Example

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\[ B = \]

\[ B = \]
Quasi-orders

- a *quasi-order* \(\preceq\) is a reflexive, transitive relation
- an *antichain* is a set of pairwise incomparable elements
- a *well-quasi-order* has no infinite antichain
Quasi-orders

Example

• for positive integers, say \( a \preceq b \) if \( a \) divides \( b \)
• \( \{4, 15, 18, 34\} \) is an antichain
• are the positive integers well-quasi-ordered by \( \preceq \)?
Quasi-orders

Example

- for positive integers, say $a \preceq b$ if $a$ divides $b$
- $\{4, 15, 18, 34\}$ is an antichain
- are the positive integers well-quasi-ordered by $\preceq$?

- No. The primes form an infinite antichain.
“is a minor of” is a quasi-order on graphs

- $G \preccurlyeq H$ if $G$ is a minor of $H$
“is a minor of” is a quasi-order on graphs

- $G \preceq H$ if $G$ is a minor of $H$

Theorem (Robertson & Seymour)
Every minor closed class of graphs has only finitely many excluded minors.

Equivalently
Graphs are well-quasi-ordered by the minor relation.
- i.e., the set of all graphs contains no infinite antichain.
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Equivalently

*Graphs are well-quasi-ordered by the minor relation.*

- *i.e.,* the set of all graphs contains no infinite antichain.

- $X$ is an infinite antichain $\implies \{G : G \text{ has no minor in } X\}$ is a minor-closed class having infinitely many excluded minors
“is a minor of” is a quasi-order on graphs

- $G \preceq H$ if $G$ is a minor of $H$

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Equivalently

Graphs are well-quasi-ordered by the minor relation.

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- $\mathcal{H}$ is a minor-closed class having infinitely many excluded minors $\Rightarrow$ its set of excluded minors is an infinite antichain
Biased graph minors

- \((G, \mathcal{B}) \setminus e = (G \setminus e, \mathcal{B} \setminus \{\text{cycles containing } e\})\)
- for \(e\) not an unbalanced loop:
  \((G, \mathcal{B})/e = G/e\) with cycles \(C\) balanced if \(C \in \mathcal{B}\) or \(C \cup e \in \mathcal{B}\)

\[(K_5 \setminus e, \emptyset)\]
\[(K_5, \mathcal{B})\]
\[(K_5/e, \mathcal{B})\]
Biased graph minors

Might similar statements hold for biased graphs?

- *Every minor closed class of biased graphs has only finitely many excluded minors?*
- *Are biased graphs well-quasi-ordered by the minor relation?*
- *Are there infinite antichains of biased graphs?*
Biased graph minors

For a group $\Gamma$, let $\mathcal{G}_\Gamma$ be the class of $\Gamma$-labellable biased graphs

- $\mathcal{G}_\Gamma$ is minor closed
- What are the excluded minors for $\mathcal{G}_\Gamma$?
- Is the set of excluded minors for $\mathcal{G}_\Gamma$ finite?

Theorem (DeVos, F., Pivotto)

For every infinite group $\Gamma$ and every $t \geq 3$ there are infinitely many excluded minors for $\mathcal{G}_\Gamma$ on $t$ vertices.

Theorem (DeVos, F., Pivotto)

Let $\Gamma$ be an infinite group, and fix $t \geq 3$. There is an infinite antichain of $\Gamma$-labelled graphs on $t$ vertices.
Biased graph minors

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Let $\Gamma$ be an infinite group, and fix $t \geq 3$. There is an infinite antichain of $\Gamma$-labelled graphs on $t$ vertices.
Constructing minor minimal not group labelable biased graphs

Let $G$ be a simple graph embedded in the plane

- a subdivision of a 3-connected graph
- $t$-vertex coloured
- every colour appears on every face exactly once
- every cycle of size $\leq t$ is a face boundary

Let $(\tilde{G}, B)$ be biased graph obtained by

- identifying each colour class to a single vertex
- setting $B = \text{cycles corresponding to finite face boundaries of } G$
Constructing minor minimal not group labelable biased graphs

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- identifying each colour class to a single vertex
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Constructing minor minimal not group labellable biased graphs

\[(\tilde{G}, B)\] is not group labellable:

- Let \(K = \text{embedding of } G\) minus infinite face
- \(\tilde{K} \sim K\) with colour classes identified
- There is a contractible unbalanced cycle
Constructing minor minimal not group labellable biased graphs

\((\tilde{G}, \mathcal{B})\) is not group labellable:
- Let \(K = (\text{embedding of } G) \text{ minus infinite face}\)
- \(\tilde{K} \cong K\) with colour classes identified
- there is a contractible unbalanced cycle
Constructing minor minimal not group labellable biased graphs

- let $\Gamma$ be an infinite group, construct a $\Gamma$-labelling
  - of $(\tilde{G}, \mathcal{B}) \setminus e$ from embedding of $G \setminus e$ on cylinder
Constructing minor minimal not group labelable biased graphs

- let $\Gamma$ be an infinite group, construct a $\Gamma$-labelling
  - of $(\tilde{G}, B) \setminus e$ from embedding of $G \setminus e$ on cylinder
  - of $(\tilde{G}, B)/e$ for any $e$ from $G/e$ using fact $\tilde{G}$ has $\leq 2$ balanced cycles

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Where is the line between civilisation and the wilds?

- Biased graphs
- Labellable by infinite group
- Labellable by finite group
- Labellable by finite abelian group
- Labellable by trivial group (graphs)

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Where is the line between civilisation and the wilds?

Γ infinite:
- infinite sets of excluded minors, all on a fixed number of vertices, for perfectly sensible minor closed classes of biased graphs (namely, Γ-labelled graphs)
Where is the line between civilisation and the wilds?

Γ infinite:

- infinite sets of excluded minors, all on a fixed number of vertices, for perfectly sensible minor closed classes of biased graphs (namely, Γ-labelled graphs)
- infinite antichains of Γ-labelled graphs on a fixed number of vertices
Where is the line between civilisation and the wilds?

Γ infinite:

- infinite sets of excluded minors, all on a fixed number of vertices, for perfectly sensible minor closed classes of biased graphs (namely, Γ-labelled graphs)
- infinite antichains of Γ-labelled graphs on a fixed number of vertices

Geelen, Gerards, & Whittle know that if Γ is a finite abelian group, then the class of Γ-labelled graphs is well-quasi-ordered
Where is the line between civilisation and the wilds?

- Biased graphs

- Infinite antichains

  - Labellable by infinite group

- Labellable by finite group

- W.Q.O.

  - Labellable by finite abelian group

  - Labellable by trivial group (graphs)
Where is the line between civilisation and the wilds?

- We know that if $\Gamma$ is a finite group (not necessarily abelian), then the class of $\Gamma$-labelled graphs of bounded branch-width is well-quasi-ordered
  - $\text{branch-width} \leq k$: graphs that can be built by assembling small graphs in a tree-like way
Where is the line between civilisation and the wilds?

- We know that if $\Gamma$ is a finite group (not necessarily abelian), then the class of $\Gamma$-labelled graphs of bounded branch-width is well-quasi-ordered.
- **branch-width $\leq k$**: graphs that can be built by assembling small graphs in a tree-like way.
Where is the line between civilisation and the wilds?

- For a fixed finite group $\Gamma$, is the class of $\Gamma$-labellable biased graphs well-quasi-ordered?
Where is the line between civilisation and the wilds?

- For a fixed finite group $\Gamma$, is the class of $\Gamma$-labellable biased graphs well-quasi-ordered?
- These results about biased graphs have implications for the analogous questions about certain classes of matroids
  - neither the class of lift or frame matroids is w.q.o. by minors