Physics 347 **Assignment #6** **Due Dec 1**

Problem 1: Einstein Relation

In our discussion of the random walk we defined the diffusion constant to be $D = \delta^2/(2\Delta t)$. This depends on the step size, δ, and the time between collisions, Δt. Clearly these depend on the type of substance that the object is diffusing through and on temperature. Einstein was able to come up with a connection between the diffusion constant, the material properties and temperature. You'll now do a simple version of his derivation.

(a) Consider a particle of mass *m* which is diffusing, but also being pulled by a force, *f*. What is the acceleration of the particle due to *f* ? Use kinematics to calculate the average distance, Δx that it goes between collisions if it is being accelerated by *f* and the collision time is Δt?

(b) Calculate the average drift velocity to be $v_d=\Delta x/\Delta t$. The drift velocity for laminar flow we said was $v_d = f/c$ where *c* is the drag coefficient. Thus what is *c* in terms of *m* and Δt?

(c) Multiply the diffusion coefficient by *c*. You're result for *D c* will depend on δ, Δt and *m*.

(d) The average thermal velocity is $v_x = \delta/\Delta t$, which from kinetic theory is related to the temperature via $v_x^2 = k_B T/m$. Use this to simplify your result from (c) to arrive at Einstein's relation, $Dc = k_B T$.

(e) For a spherical object with a radius *R*, $c = 6\pi\,\eta\,R$. Use the Einstein relation to calculate the diffusion coefficient in water at room temperature of a lipid vesicle that has a radius of 50 nm where the viscosity of water is $\eta = 10^{-3}$ kg/m/s at STP.

Problem 2: Stopping time of bacteria

Consider a bacterium, idealized to be a sphere with radius $R=10^{-4}$ cm, with a density of of $\rho = 1$ g cm^{-3}, swimming in water which has a viscosity, $\eta = 10^{-2}$ g/cm/s.

(a) Use Stoke's formula $c=6\pi\eta R$ to calculate the drag coefficient, *c*. What are the units of *c*?

(b) There is a natural time scale given by $\tau = m/c$, where *m* is the mass of the bacteria. Calculate τ using your result from (a) and the mass of the bacteria, $m = \rho V$.

(c) Assume that the bacteria is initially swimming at a speed *v(0)*. It stops swimming and coasts to rest. Solve Newton's equation for the speed of the bacteria as a function of time, *v(t)*. Newton's equation, $m\,a = f$, for the bacteria can be written as $m\,(dv/dt) = c\,v$. To solve this, bring all factors involving *v* to the LHS and everything involving *t* to the RHS. Integrate the LHS from *v(0)* to *v(t)* and the RHS from $t=0$ to *t*. You'll notice that the parameter τ appears in the solution, and can be thought of as a decay time.

(d) Use your result from (c) to calculate the distance that the bacteria travels in coming to rest if $v(0) = 2 \times 10^{-3}$ cm/s. This requires integrating *v(t)* from $t=0$ to $t=\infty$. Does the bacteria coast? Moving around is a tough job for a bacteria.

Problem 3: Morphogen Gradients:

During the process of tissue development in multicellular organisms, gradients of molecules called morphogens are set up which generate complex patterns of gene expression, that ultimately lead to the vast assortment of different cell types. A common way for an organism to set up a morphogen gradient, is to have a point source (i.e. a cell (or cell cluster)) that emit morphogens, which then diffuse out into the surrounding tissue. A given morphogen molecule diffuses with a diffusion coefficient, *D*, and is degraded at some rate, γ. At steady state, the concentration of morphogen, *c(x)*, satisfies the reaction-diffusion equation $D\,d^2 c(x)/dx^2 - \gamma\, c(x) = 0$.

(a) Given the units of *D*, *c* and x (i.e. length^2/time, #/length^3, and length), what must be the units of γ?

(b) Consider the point source, that generates a flux of a morphogens at *x=0*. Show that $c(x) = A\exp(-x/\lambda)$ where $\lambda = \sqrt{(\gamma/D)}$ is a solution to the above reaction-diffusion equation. What are the units of λ?

(c) Given that there is a flux of particles, α at *x=0*, use Fick's flux equation, $j = -D\,dc(x)/dx$ at *x=0* to solve for *A* in terms of *D*, α, and γ.

(d) Developing organisms develop proportionally. That means that a larger embryo develops in the same way as a smaller embryo. In order to accomplish, morphogen's need to have the same concentration at the same fractional position in the embryo (e.g. the concentration of *c*, half way in a larger embryo needs to be the same as that in a smaller embryo). To accomplish this, the flux depends on embryo size. For a morphogen that is used to set the midpoint of the embryo, if a typical embryo has size *L* with a flux α, what does the flux α' in an embryo of size *L'* need to be such that the concentration at the midpoint is the same between the two embryos? If λ = *0.2L*, by what factor does α' increase/decrease by if *L'=3/2 L* or *L' = 2/3 L?*

Problem 4: And now for something completely different ... Lotka-Volterra

This problem is a classic predator-prey model which is used in studying food webs where one organism is feeding on another. It is also relevant to some biochemical models where one chemical species is 'eating' another. In the model there are populations of foxes, *F(t)*, and rabbits, *R(t)*. The dynamics of these two populations are given by

$$\frac{dR}{dt} = aR - bRF$$

$$\frac{dF}{dt} = e\,b\,RF - cF$$

where *a* is the natural growth rate for rabits, *c* is the natural death rate for foxes, *b* is the death rate of rabits due to an encounter with a fox, and *e* is the efficiency of turning eaten rabbits into foxes.

(a) Explain in words what each of the two terms in each of the above equations represents.

(b) Solve for the steady state populations of foxes and rabbits. You should find two steady state values for each population.

(c) Try to figure out the stability of each steady state point by doing some graphical analysis. To do this draw a 2D graph where the x axis is the population of rabbits, *R* and the y axis is the population of foxes, *F*. On this graph, label the two equilibrium coordinate positions of *(R,F)*. Now in the vicinities of each of these positions, sketch the flow vectors. The x component of the flow vector is given by *dR/dt* and the y-component of the flow vector is given by *dF/dt*. Use the parameter values *a=0.1*, *b=0.05*, *c=0.5* and *e=0.25* so that you can have some numbers. By looking at the flow around the equilibrium points, are they stable or unstable?

(d) You should arrive at a conclusion about the stability of the two steady state points that may seem confusing. Given your findings about the stability of each of the steady state points, what do you think the temporal behaviour of the populations of rabbits and foxes will look like?