A Positive Theory of Government Debt

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Abstract

A government that cannot commit to future policy choices faces a trade-off that explains the level of debt. On the one hand, there is an incentive to increase debt and delay taxation, so as to reduce current distortions. On the other hand, inflating current prices lowers the real value of nominal debt and so there is a motive to reduce it now. The size of long-run debt will depend on the interaction of these two opposing incentives. The critical determinant is the willingness of households to substitute away from goods being taxed by inflation. Numerical simulations show that the model matches some qualitative and quantitative properties of U.S. policy variables, including the fact that wars are frequently financed with a mix of instruments. The theory interprets the unusual post-World War II inflation and fast liquidation of accumulated debt as being due to higher long-run debt and expenditure in the period leading up to the war.

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1 Introduction

All governments from developed countries have positive debt. In many cases, debt is sizable relative to GDP. For example, data from the OECD (2006) shows that average central government debt in developed countries was about 50% of GDP between 1980 and 2006. In the U.S., federal government debt held by the public was $5.3 trillion by the end of the 2008 fiscal year, a bit over 37% of GDP. This paper proposes a theory that explains the level of debt. The key ingredients are nominal debt and an assumed lack of government commitment. The theory's empirical plausibility is evaluated by analyzing the reaction of debt and other policy variables to government expenditure shocks and comparing the results with data for the U.S. economy.

In a seminal contribution, Barro (1979) argues that government debt should be used to smooth distortionary taxation over time. His theory predicts that debt only reacts to temporary variations in income or government expenditure and thus, debt levels are irrelevant for current debt issue[1]. In the absence of aggregate uncertainty, debt would be constant and equal to its "initial" level. As a result, taxes depend only on the permanent component of expenditure and the level of debt, so that taxes follow a random walk. While attractive as a normative theory, Barro's model is inconsistent with U.S. data along two important dimensions. First, debt over GDP displays mean-reversion, which suggests the existence of a fundamental long-run level of debt (see Bohn, 1998). Second, evidence from war episodes (see Goldin, 1980) suggests that temporary increases in expenditure are financed with a mix of instruments, including taxes.

Lucas and Stokey (1983) formulate a Ramsey problem with state-contingent debt and taxes, and show that these instruments should not follow random walks. They still find that taxes are smoothed, but only in the sense that they are less volatile than under a balanced-budget rule. In their model debt fluctuates around a stationary value that is a monotone function of initial debt. Thus, the theory is consistent with debt exhibiting mean-reversion, but offers limited insights about the determination of long-run debt. Furthermore, Marcet and Scott (2007) show that in models of optimal fiscal policy under complete markets—as in Lucas and Stokey—debt decreases in response to a positive innovation in the deficit and features equal or less persistence than other variables. They show that both these features are not supported by post-war U.S. data[2].

In contrast, Krusell, Martin and Ríos-Rull (2006) relax the commitment assumption and analyze the time-consistent equilibrium in the basic Lucas-Stokey environment with no uncertainty and one-period bonds. They show that governments inheriting positive debt always have an incentive to increase it and find that the equilibrium features an infinite but countable number of steady states. Hence, the implied dynamics looks remarkably similar to the commitment case: the government increases debt for one or two periods and then leaves it constant.

This paper considers the case of a benevolent government that needs to finance a constant expenditure using distortionary taxes, inflation and nominal debt. The government cannot commit to future policy choices and thus, each period decides how much to distort the economy and how

[1]Aiyagari, Marcet, Sargent and Seppälä (2002) provide a microfoundation for Barro's model in a general equilibrium framework with incomplete markets. Battaglini and Coate (2008) extend this framework by incorporating inefficiencies due to pork-barrel spending. They provide a political economy explanation for the distribution of debt in the long-run.

[2]On the other hand, Shin (2007) shows that the complete market's Ramsey allocation can be implemented with active maturity management in a way that resembles British 18th century policy.

much debt to leave for the future. In other words, it trades-off current for future distortions.

To understand what determines the level of debt, note that inflation has two effects, one distortive and the other non-distortive. Since inflation distorts the consumption of cash-goods, the government has an incentive to increase debt so as to reduce current distortions. On the other hand, inflation decreases the real value of nominal debt and thus reduces the financial burden. At any given level of debt, whether the government increases or decreases debt depends on which of the two effects dominates. As debt increases, so do the gains from reducing it, since the non-distortive effect gains weight. Thus, the long-run level of debt depends on how distortive the inflation tax is. For the case of separable preferences, long-run debt is a decreasing function of the intertemporal elasticity of substitution for the cash-good. In particular, long-run debt is positive if and only if the intertemporal elasticity of substitution is less than one. Hence, the theory identifies a relation between economic fundamentals—other than the initial level of debt—and long-run debt[3].

The model is extended to include serially correlated stochastic government expenditure. Simulations show that the model closely replicates the volatility and autocorrelation observed in the data for taxes, inflation and debt. The model matches qualitatively the positive, hump-shaped response of debt to a positive innovation in government expenditure. The persistence in debt is quantitatively similar to the data.

The model's predictions for wartime financing are also evaluated. The simulated response of the model to a shock similar to the Civil War or World War I matches the qualitative response observed in the data. In particular, both debt and inflation increase and part of the war is financed with contemporary taxes. Quantitatively, the model under-predicts inflation and over-predicts taxation. The response in debt is quantitatively closer, but persistence is smaller. In contrast, Barro (1979) would predict near zero war financing with taxes (they would mostly go up to repay debt), whereas complete markets model like Chari, Christiano and Kehoe (1991) would have inflation absorbing most of the shock.

World War II differed from previous wars in that the government implemented a large post-war inflation that significantly reduced the real value of accumulated debt. The data suggest that both long-run expenditure and debt were much higher in the period leading up to World War II, when compared to previous war episodes. The model is tested to see whether these changes in fundamentals can account for the differences in U.S. policy. The model correctly predicts a high initial inflation at the beginning of the war, followed by a large post-war inflation, which implies a faster reduction in government debt. Thus, the theory provides a rationale for the inflation after World War II: since long-run levels of debt and expenditure were higher, the distortions of war finance were larger than in previous episodes and so the incentives to reduce the real value of accumulated debt through inflation were higher as well.

In a complementary study, Díaz-Giménez, Giovannetti, Marimón and Teles (2008) analyze an economy as in Nicolini (1998), similar to the one analyzed in section 2 below. Their focus is on comparing economies with real vs. nominal debt, with and without commitment, and evaluate the welfare implications of these different institutional arrangements. They also consider an extension

[3] Other models that also feature a long-run debt as a function of fundamentals include Diamond (1965), Aiyagari and McGrattan (1998) and Shin (2006). However, in these cases, debt is used to reduce some dynamic inefficiency and thus plays a role that could be played by other assets. Essentially, the friction that explains the level of debt is not intrinsic to it and could be resolved by other instruments.

with taxes and show that commitment on the part of the fiscal authority may help overcome the time-consistency problem faced by the monetary authority.

2 A Basic Model of Government Debt

2.1 The economy

Consider an economy with no capital and no uncertainty. There is a benevolent government that has to finance a constant expenditure $g \geq 0$ every period. Output y is linear in labor, i.e., $y_t = n_t$, which implies the aggregate resource constraint

$$c_t + g = n_t. \tag{1}$$

The government can finance g by issuing one period nominal bonds or printing money. Hence, it has to satisfy the following period budget constraint

$$\bar{M}_t + \bar{B}_t + \bar{p}_t g = \bar{M}_{t+1} + q_t \bar{B}_{t+1}, \tag{2}$$

where $\bar{M}$ is the aggregate money stock, $\bar{B}$ is the stock of nominal bonds, $\bar{p}$ is the price of the consumption good and q is the price of a bond that pays one unit of money in the following period.

The economy is populated by a continuum of identical, infinitely lived households that derive utility from consumption and leisure. The present value of lifetime utility is given by

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t),$$

where $\beta \in (0, 1)$ is the time discount factor and u is the period utility function. Assume u is twice continuously differentiable, with $u_c > 0$, $u_{cc} < 0$, $u_\ell > 0$, $u_{\ell\ell} < 0$ and $u_{cc} + u_{\ell\ell} - 2u_{c\ell} < 0$. Further assume that there exists $\hat{c}$ that solves $u_c(\hat{c}) = u_\ell(1 - \hat{c} - g)$.

The government announces its choice for $\bar{M}_{t+1}$ and $\bar{B}_{t+1}$ at the beginning of the period. There are two subperiods: the goods market operates in the first subperiod and the securities market opens in the second. This timing is important: if the securities market were to open before the goods market, then the problem of the government would be static. To see this, note that prices and allocations at t would depend on $\bar{M}_{t+1}$ instead of $\bar{M}_t$, which would only appear in the government budget constraint together with $\bar{B}_t$. Even though individual money and bond holdings matter for the household, from an aggregate point of view the composition of the government's nominal liabilities at the beginning of the period would be irrelevant. Hence, the actions of the current government would have no effect on future government decisions.

In the first subperiod, the household divides into two agents, a shopper and a producer/seller. There is a cash-in-advance constraint (as in Svensson, 1985)

$$\bar{p}_t c_t \leq \bar{m}_t, \tag{3}$$

where $\bar{m}_t$ is the household's money balances at the beginning of the period. Thus, the shopper takes the household's money balances and sets out to buy the consumption good. The producer/seller stays at home, works n hours and sells the produced good in exchange for money. The government purchases g units of output and thus crowds-out shoppers in the goods market.

In the second subperiod, the household becomes one again and the securities market opens. Each household carries $\bar{b}_t$ bonds acquired in the previous period and money acquired from selling output in the goods market, and chooses how much money and bonds it wants to carry to the next period. The government either buys or sells bonds, depending on its decision for $\bar{M}_{t+1}$ and $\bar{B}_{t+1}$. Given the supply and demand for money and bonds, q_t adjusts to clear the securities market.

This environment implies the following period budget constraint for the household

$$\bar{p}_t c_t + \bar{m}_{t+1} + q_t \bar{b}_{t+1} = \bar{p}_t n_t + \bar{m}_t + \bar{b}_t. \tag{4}$$

2.2 The competitive equilibrium

Since we will be solving for government policy functions, it is convenient to write the problem of the household recursively. Following Cooley and Hansen (1991), redefine individual and aggregate nominal variables (except for q) by dividing them by the aggregate money stock, i.e., for any nominal variable $\bar{x}$, let $x \equiv \frac{\bar{x}}{\bar{M}}$. Furthermore, define the money growth rate $\mu_t \equiv \frac{\bar{M}_{t+1}}{\bar{M}_t} - 1$. Switching to recursive notation (where primes denote next period variables) we can rewrite the government budget constraint (2) as

$$\frac{1+B}{p} + g = \frac{(1+\mu)(1+qB')}{p}, \tag{5}$$

the household budget constraint (4) as

$$c = n + \frac{m + b - (1+\mu)(m' + qb')}{p} \tag{6}$$

and, using (6), the cash-in advance constraint (3) as

$$\frac{(1+\mu)(m'+qb') - b}{p} - n \geq 0. \tag{7}$$

Regarding the aggregate state of this economy, what is going to matter is what fraction of total nominal assets at the beginning of the period are in the form of money. The reason is that only money can be used to make purchases in the goods market. Hence, the aggregate state variable has to be some measure of the composition of nominal assets, say the bond-to-money ratio, B.

Given some government debt policy $B' = \mathcal{B}(B)$ and associated money growth rate μ that satisfies the government budget constraint (5), the problem of the household can be written as follows

$$v(m,b,B) = \max_{n,m',b'} \; u\Big(n + \frac{m + b - (1+\mu)(m'+qb')}{p}, 1-n\Big) + \beta v(m',b',B'),$$

subject to (7). The first-order conditions are

$$\begin{aligned} u_c - u_\ell - \xi &= 0 && (8)\\ -\frac{(1+\mu)(u_c - \xi)}{p} + \beta v'_m &= 0 && (9)\\ -\frac{(1+\mu)(u_c-\xi)q}{p} + \beta v'_b &= 0, && (10) \end{aligned}$$

where ξ is the Lagrange multiplier associated with the cash-in-advance constraint. Use (8) and the envelope conditions to rewrite (9) and (10) as

$$-\frac{(1+\mu)u_\ell}{p} + \frac{\beta u_c'}{p'} = 0 \tag{11}$$

$$-\frac{(1+\mu)q u_\ell}{p} + \frac{\beta u_\ell'}{p'} = 0. \tag{12}$$

Note that p' is a function of tomorrow's government policy, which in turn is a function of the aggregate state. So let $p' = \mathcal{P}(B')$, where $\mathcal{P}$ indicates the price level implemented by future government policy as a function of the bond-money ratio.

Using (11) and (12) we get a simple expression for the nominal price of bonds

$$q = \frac{u_\ell'}{u_c'}. \tag{13}$$

This expression can also be written as $q = 1 - \xi'/u_c'$, where ξ' is the Lagrange multiplier associated with the cash-in-advance constraint in the following period. Basically, if the agent is expecting to be cash-constrained tomorrow (i.e., $\xi' > 0$), he is going to request to be compensated in order to accept bonds.

In equilibrium, $m' = M' = 1$, $b' = B'$ and the cash-in-advance constraint holds with equality, i.e.,

$$c = \frac{1}{p}. \tag{14}$$

2.3 The problem of the government

Assume that the government cannot commit to future policy choices and that reputation mechanisms are not operative. Throughout the paper we will study the Markov-perfect equilibrium of the economy. Thus, we analyze policies where the government bases its decisions solely on fundamentals—B in this case—, taking as given the policy implemented by future governments and that households behave competitively. The government is benevolent and thus will choose its policy so as to maximize the discounted utility of the representative household.

Typically, time-consistency problems arise when successive governments disagree on what policies to implement. In the recursive representation, this disagreement is formalized by the appearance in the current government's problem, of the policy rules of future governments, which are functions of the inherited state. This reflects the fact that the current government understands its actions affect the decisions to be made by future governments, which in turn affect the decisions taken by agents today. However, future governments will not internalize this, just as the current government does not consider how its policy affected past decisions. Thus, if the current government were to choose all future actions, it would change its mind in the future about what policy to implement. From equations (11) and (13) we can see that in this case the time-consistency problem comes from the fact that current choices depend on the price level tomorrow, which in turn depends on tomorrow's government policy, i.e., they depend on $\mathcal{P}(B')$. But the government tomorrow will not take into account that its policy affects prices today and thus the time-inconsistency.

Since the cash-in-advance constraint (14) holds in every period, we can write $p = \frac{1}{c}$ and $\mathcal{P}(B') = \frac{1}{\mathcal{C}(B')}$, where $\mathcal{C}$ is the consumption function implemented by future government policy. From the aggregate resource constraint (1) we have $\ell = 1 - c - g$. Thus, using (11) we can write μ as a function of c and $\mathcal{C}(B')$

$$\mu = \frac{\beta \mathcal{C}(B') u_c'}{c u_\ell} - 1. \tag{15}$$

Next, use (13) and (15) to write the government budget constraint (5) as

$$-u_\ell(c(1+B)+g) + \beta \mathcal{C}(B')(u_c' + u_\ell' B') = 0. \tag{16}$$

Call the left hand side $\varepsilon(B, B', c, \mathcal{C}(B'))$. Note that (16) has to be satisfied in any competitive equilibrium, i.e., for any debt function $\mathcal{B}(B)$, the equilibrium consumption function $\mathcal{C}(B)$ has to satisfy $\varepsilon(B, \mathcal{B}(B), \mathcal{C}(B), \mathcal{C}(\mathcal{B}(B))) = 0$.

One way to write the problem of the government recursively is to have it choose B' and c, given B, $\mathcal{C}(B')$ and subject to (8) and (16). Thus, given the perception that future governments will implement policies that induces $\mathcal{C}(B)$, the problem of the current government is

$$\mathcal{V}(B) = \max_{c, B'} u(c, 1-c-g) + \beta \mathcal{V}(B')$$

subject to

$$\begin{aligned} \varepsilon(B, B', c, \mathcal{C}(B')) &= 0 \\ u_c - u_\ell &\geq 0, \end{aligned}$$

where the inequality constraint is derived from the first-order condition of the household (8) and needs to be satisfied in any competitive equilibrium (otherwise, households have arbitrage opportunities).

Definition 1 *A Markov-perfect equilibrium is a set of functions $\{\mathcal{V}, \mathcal{B}, \mathcal{C}\} : \mathbb{R} \to \mathbb{R}^3$ such that for all B:*

$$\{\mathcal{C}(B), \mathcal{B}(B)\} = \operatorname*{argmax}_{c, B'} u(c, 1-c-g) + \beta \mathcal{V}(B')$$

subject to $\varepsilon(B, B', c, \mathcal{C}(B')) = 0$, $u_c - u_\ell \geq 0$; and

$$\mathcal{V}(B) = u(\mathcal{C}(B), 1 - \mathcal{C}(B) - g) + \beta \mathcal{V}(\mathcal{B}(B)).$$

Assuming the equilibrium functions are differentiable (non-differentiable equilibria are covered in Appendix C[4]), we can use the first-order conditions to further characterize the equilibrium. With Lagrange multipliers λ and ζ associated with the constraints of the government's problem, we get

$$u_c - u_\ell + \lambda \varepsilon_c + \zeta(u_{cc} + u_{\ell\ell} - 2u_{c\ell}) = 0 \tag{17}$$

$$\lambda(\varepsilon_{B'} + \varepsilon_{c'} \mathcal{C}_B') + \beta \lambda' \varepsilon_B' = 0. \tag{18}$$

[4] As explained in Appendix C, the differentiable equilibrium exists when the horizon is finite and infinite, whereas the non-differentiable equilibrium only exists when the horizon is infinite. We can use this as a selection device to focus on the differentiable equilibrium.

Equation (18) states the intertemporal trade-off faced by the government: equating the marginal effects today and (present value) tomorrow of changing the debt-to-money ratio. Due to the presence of the derivative of an equilibrium function, (18) is typically referred to as a Generalized Euler Equation (GEE). Notice that $\varepsilon_{B'} = -\beta\varepsilon'_B = \beta\mathcal{C}(B')u'_\ell$, i.e., the direct gains associated with loosening the current budget constraint through borrowing cancel out with the tightening of the budget constraint tomorrow due to the higher indebtedness. We can thus write (18) as

$$\beta\mathcal{C}(B')u'_\ell(\lambda - \lambda') + \lambda\varepsilon_{c'}\mathcal{C}'_B = 0. \tag{19}$$

The first term in (19) shows the direct effect of increasing debt: it alleviates distortions today—lower taxes and inflation—at the cost of higher distortions tomorrow, due to the increased financial burden. The second term shows the effect today of anticipated changes in policy tomorrow due to increased debt. This second effect is where time-consistency problems arise (as reflected by the presence of the derivative of the consumption function), since the future government will not internalize how its policies affected past actions.

The problem then is to understand under which circumstances governments increase or decrease debt. To answer this, note that since printing money increases the price level, it has two effects. One, the inflation tax, is distortive and acts as a consumption tax. Since the government is benevolent, it would like to minimize this distortion and so there is a motive to increase debt. The other effect, the reduction in the real value of debt, is non-distortive since bond holdings at the beginning of the period are inelastically supplied. Thus, the current government views taxing these holdings, i.e., reducing their real value through an increase in the price level, as non-distortive and so has an incentive to reduce debt now. The non-distortive effect is similar to a capital levy. Note that even though the government views a current increase in the price level as having two opposing effects, it will view all future increases in prices as purely distortionary.

The government will increase or decrease debt depending on which of the two effects described above dominates. Note that as debt increases, so do the gains from reducing it, since the non-distortive effect gains weight. Basically, the revenue that can be appropriated by the government without distortions grows with debt. Thus, how the two effects play out against each other depends on how distortive the inflation tax is. If goods bought with money have close substitutes, then the distortion of the inflation tax is low and hence, there is a large incentive to reduce debt. The opposite happens if these goods are difficult to substitute. Thus, a positive—potentially large—long-run level of debt is associated with a sufficiently distortive inflation tax, so that there is enough motive to accumulate debt. This basic argument still applies even if we add more detail to the environment, such as labor taxes.

The following proposition establishes the predictions of the model for steady state debt.

Proposition 1 *In a differentiable Markov-perfect equilibrium there exist two steady states. One is the first-best,* $\hat{B} = -1 - \frac{g}{\hat{c}(1-\beta)}$*; and the other is distortionary,* $B^* = \frac{-u^*_c + c^*(u^*_{c\ell} - u^*_{cc})}{u^*_\ell + c^*(u^*_{c\ell} - u^*_{\ell\ell})}$*, where* $\varepsilon(B^*, B^*, c^*, c^*) = 0$*.*

All proofs are relegated to Appendix A. At $\hat{B}$ the first-best is implemented. Thus, we have $u_c = u_\ell$, $\mu = \beta - 1$ and $q = 1$ (i.e., the Friedman rule). The debt-to-money ratio is less than minus one if government expenditure is positive. Thus, the first-best steady state features negative nominal debt, larger—in absolute value—than the money stock. Having large enough positive claims on the private sector allows the government to finance its expenditure, while deflating prices such that the nominal interest rate is zero.

Let us now analyze the equilibrium around the first-best steady state. For debt levels below $\hat{B}$, the first-best is still implementable, since the government has even higher assets. Hence, we get $u_c = u_\ell$ and using (16),

$$\mathcal{B}(B) = \frac{1 - \beta + \frac{g}{\hat{c}} + B}{\beta} \tag{20}$$

for all $B \leq \hat{B}$. Thus, the government simply accumulates assets over time. Note that this implies that (16) does not bind in the government's problem, i.e., $\lambda = 0$, and thus from (17), $\zeta = 0$ as well. The following result follows.

Proposition 2 *The first-best steady state is unstable.*

For $B > \hat{B}$, the first-best is not implementable since debt is too high. However, it may be possible that the government implements $u_c = u_\ell$ for debt levels close enough to $\hat{B}$. To see this, consider (17) and suppose both $u_c = u_\ell$ and $\lambda > 0$. Given $\zeta(u_{cc} + u_\ell - 2u_{c\ell}) \leq 0$, we need $\varepsilon_c \geq 0$ to satisfy (17). To ease exposition assume separable utility, i.e., $u_{c\ell} = 0$. Then, $\varepsilon_c = -(1+B)(u_\ell - cu_{\ell\ell}) + gu_\ell$. This expression may be positive if B is sufficiently below -1. Thus, it may be possible for some B close enough to $\hat{B}$ to feature $\varepsilon_c > 0$ and thus, $\zeta > 0$, i.e., $u_c = u_\ell$. The debt function would be as in (20) in a neighborhood to the right of $\hat{B}$. Note however that the first-best is not achievable since the scheme involves debt accumulation and thus, distortions in future periods. It is also possible that $u_c > u_\ell$ for all $B > \hat{B}$. For example, if $u_{c\ell} = g = 0$, then $\hat{B} = -1$ and $\varepsilon_c < 0$ for all $B > \hat{B}$; thus, from (17), we get both $\zeta = 0$ and $u_c > u_\ell$.

The next proposition provides a sharper characterization of the level of debt at the distortionary steady state.

Proposition 3 *If $u(c, \ell) = \frac{c^{1-\sigma}-1}{1-\sigma} + \nu(\ell)$, where $\sigma > 0$ and $\nu(\ell)$ is strictly increasing and concave, then $B^* > 0$ iff $\sigma > 1$, $B^* = 0$ iff $\sigma = 0$ and $B^* < 0$ iff $\sigma < 1$.*

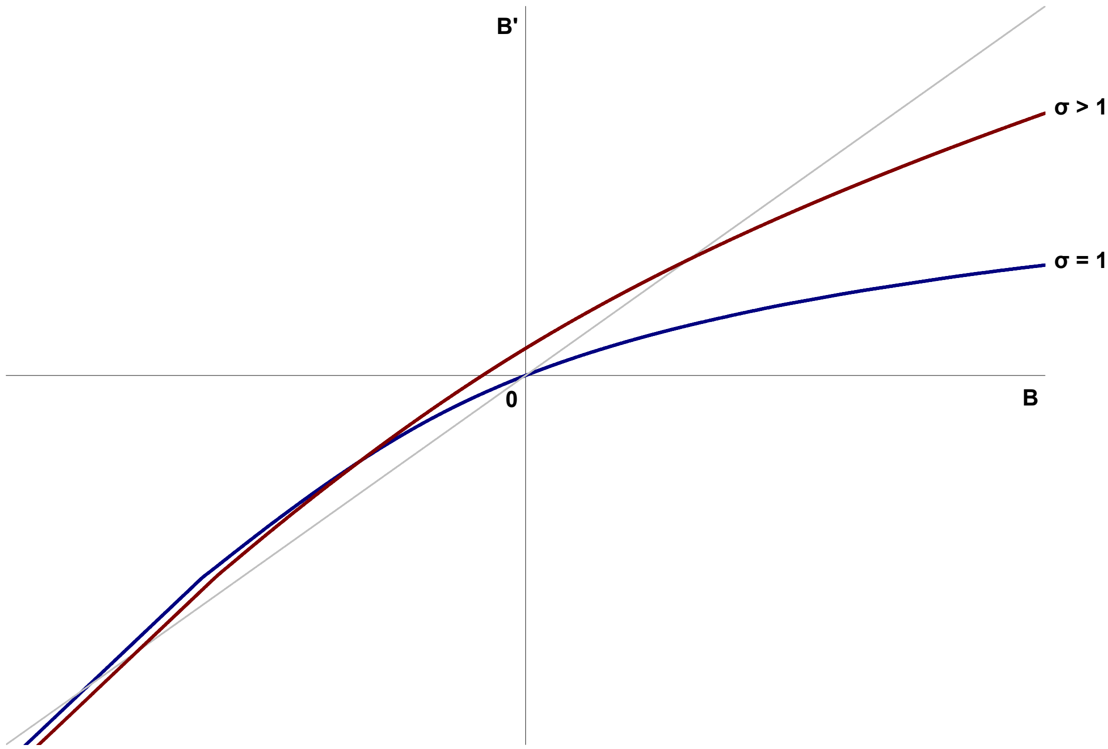
Thus, for separable utility, the sign of B^* is entirely determined by the intertemporal elasticity of substitution of consumption. For a special case, we can show two additional properties of the distortionary steady state.

Proposition 4 *If $u(c, \ell) = \frac{c^{1-\sigma}-1}{1-\sigma} + \gamma\ell$, where $\sigma > 0$,$\gamma > 1$, and $g = 0$ then: (i) B^* is increasing in σ; and (ii) if $\sigma = 1$, then $\mathcal{B}_B(B^*) \in (0, 1)$.*

We can verify numerically that both results hold under more general assumptions. Figure 1 provides an example. See Appendix B for an explanation of how the equilibrium was computed.

The first result in *Proposition 4* shows that σ not only determines the sign of B^*, but also its size. Thus, a lower elasticity of intertemporal substitution implies higher steady state debt. The intuition is that with a lower intertemporal elasticity of substitution, current consumption becomes

Figure 1: Debt Functions for $u(c,\ell) = \frac{c^{1-\sigma}-1}{1-\sigma} + \frac{\ell^{1-\varsigma}}{1-\varsigma}$, $\varsigma > 1$, $g > 0$



less elastic, making it more costly for the government to reduce current consumption in exchange for higher future consumption, i.e., reducing the incentives to inflate prices today to reduce the real value of debt. Hence, there are more incentives to increase debt. Compare this result with what happens at the first-best. If the utility function is as in *Proposition 4*, then $\hat{B} = -1 - g\gamma^{\frac{1}{\sigma}}(1-\beta)^{-1}$. Since by assumption $\gamma > 1$, $\hat{B}$ is also increasing in σ if $g > 0$. Thus, as we increase σ we increase both the first-best and the distortionary steady states. This implies that debt functions with different values for σ intersect at some debt level between the two steady states (see figure 1). If $g = 0$ then debt functions with different σ coincide for all $B \leq \hat{B}$.

The second result in *Proposition 4* shows that the debt function is increasing at the distortionary steady state and that this steady state is stable. Thus, if the economy starts with some debt above $\hat{B}$, then it will converge to the distortionary steady state, B^*.

Another interesting exercise is to verify numerically what happens to long-run debt when one increases the curvature of leisure. Suppose $u(c,\ell) = \frac{c^{1-\sigma}-1}{1-\sigma} + \gamma\frac{\ell^{1-\varsigma}}{1-\varsigma}$. As ς increases, the long-run level of debt decreases; as ς approaches infinity, long-run debt approaches zero. This is in sharp contrast with what happens as we increase the curvature in consumption. The reason for this is that if households do not like to substitute current for future labor, then the motive for delaying taxation diminishes.

It is important to point out that the factor that enables the model to deliver a positive level of long-run debt, is not the intertemporal elasticity of substitution in consumption *per se*, but rather how distortionary inflation is, i.e., how difficult it is for the household to substitute current consumption for another good. In the case of the separable utility function, the relevant trade-off for the household is consumption today versus consumption tomorrow. If the utility function is non-separable then the critical trade-off will be between current consumption and leisure. To get a large, positive level of long-run debt consumption and leisure need to be complements. In this way, the household finds it difficult to substitute consumption for leisure and inflation becomes costly, thus reducing the government's incentive to decrease debt. The intertemporal elasticity of substitution still plays a role, but its effects are second-order.

3 A Model of Government Debt with Fiscal Policy

The model analyzed in the previous section highlighted the basic elements of a theory of the level of debt. The main result is that, given lack of commitment, the long-run level of nominal debt depends critically on how distortionary the inflation tax is. One significant omission so far has been fiscal policy. This is important since most government revenue is generated by taxes other than inflation. The question is then, do the results of the previous section still hold if we add fiscal policy? In particular, we are interested in understanding whether other variables matter for long-run debt and whether the cost of the inflation tax still plays a critical role.

However, one cannot just incorporate labor taxes to the simple model of the previous section. The reason is that labor taxes and inflation would distort the same margin, since from the aggregate resource constraint consumption and labor differ only by a constant[5]. A Markov government would tax the good with the higher base (labor in this case) while subsidizing the other. Why? Because in this way it wants to have both taxes behave as one lump-sum: by taxing more than it needs, it can give part of it back in order to offset the disincentive to work. The problem is that in equilibrium, the government will set the labor tax rate as high as possible and the inflation rate as low as possible (the Friedman rule in this case), so as to minimize the overall distortion[6]. Thus, we need to have government policy distort two different margins. A simple way to do it is to have credit goods, i.e., goods that are not purchased with money, as in Lucas and Stokey (1987). The aggregate resource constraint then becomes

$$c_1 + c_2 + g = n, \tag{21}$$

where c_1 is consumption of the cash good and c_2 is consumption of the credit good. The introduction of labor taxes and credit goods does not significantly modify the environment of the basic model. The economy is now described by (21) and the following equations

$$\begin{aligned}
\frac{1+B}{p} + g &= \tau n + \frac{(1+\mu)(1+qB')}{p} \\
c_1 + c_2 + \frac{(1+\mu)(m'+qb')}{p} &= (1-\tau)n + \frac{m+b}{p} \\
c_1 &\leq \frac{m}{p},
\end{aligned}$$

[5]Making g endogenous does not solve this problem since the household takes it as given.

[6]For an example of this type of scheme in a real economy with labor and capital incomes taxes, see Martin (2008).

which are the government budget constraint, the household's budget constraint and the cash-in-advance constraint, respectively.

3.1 The private sector

A household's flow utility is given now by $u(c_1, c_2, \ell)$, which is strictly increasing, strictly concave and twice differentiable in all arguments. Further assume that u is jointly concave in both consumption goods.

The first-order conditions of the household's problem imply

$$\begin{aligned} (1-\tau)\, u_2 &= u_\ell && (22)\\ \frac{(1+\mu)u_2}{p} &= \frac{\beta u_1'}{p'} && (23)\\ q &= \frac{u_2'}{u_1'} && (24)\\ u_1 - u_2 &\geq 0, && (25) \end{aligned}$$

where u_1 is the marginal utility of the cash-good, u_2 is the marginal utility of the credit-good and u_ℓ is the marginal utility of leisure. The inequality (25) reflects the tightness of the cash-in-advance constraint. In equilibrium, the cash-in-advance constraint holds with equality,

$$c_1 = \frac{1}{p}. \tag{26}$$

3.2 The problem of the government

From (21) and (26) we can write $c_2 = n - \frac{1}{p} - g$. Next, we use (22) to have $\tau = 1 - \frac{u_\ell}{u_2}$ and (23) to get $\mu = \frac{\beta u_1' p}{u_2 p'} - 1$. Hence, we can write the government budget constraint as a function of B, B', n, $n' = \mathcal{N}(B')$, p and $p' = \mathcal{P}(B')$. After some rearranging we get

$$\varepsilon(B, B', n, \mathcal{N}(B'), p, \mathcal{P}(B')) \equiv u_2\Big(n - g - \frac{1+B}{p}\Big) - u_\ell n + \frac{\beta(u_1' + u_2' B')}{\mathcal{P}(B')} = 0. \tag{27}$$

The inequality (25) is also a constraint in the government's problem since it needs to be satisfied in any monetary equilibrium. Let $\chi(n,p) \equiv u_1 - u_2$.

Given the perception that future governments will implement policies that induce $\mathcal{N}$ and $\mathcal{P}$, the problem of the government can be written as

$$\mathcal{V}(B) = \max_{n,p,B'}\; u\Big(\frac{1}{p},\; n - \frac{1}{p} - g,\; 1-n\Big) + \beta\, \mathcal{V}(B')$$

subject to

$$\begin{aligned} \varepsilon(B, B', n, \mathcal{N}(B'), p, \mathcal{P}(B')) &= 0\\ \chi(n,p) &\geq 0. \end{aligned}$$

A Markov-perfect equilibrium is a set of functions $\{\mathcal{V}, \mathcal{B}, \mathcal{N}, \mathcal{P}\}$ that solves the above problem. As in the simple model of the previous section, differentiable and non-differentiable solutions coexist. Let us here analyze differentiable equilibria using the first-order conditions to the government's problem. With Lagrange multipliers λ and ζ associated to the constraints of the problem, we get

$$u_2 - u_\ell + \lambda\varepsilon_n + \zeta\chi_n = 0 \tag{28}$$

$$-\frac{(u_1 - u_2)}{p^2} + \lambda\varepsilon_p + \zeta\chi_p = 0 \tag{29}$$

$$\lambda(\varepsilon_{B'} + \varepsilon_{n'}\mathcal{N}'_B + \varepsilon_{p'}\mathcal{P}'_B) + \beta\lambda'\varepsilon'_B = 0. \tag{30}$$

From (27) we get $\varepsilon_{B'} = -\beta\varepsilon'_B = \frac{\beta u'_2}{\mathcal{P}(B')}$, which implies (30) can be written as

$$\frac{\beta u'_2(\lambda - \lambda')}{\mathcal{P}(B')} + \lambda(\varepsilon_{n'}\mathcal{N}'_B + \varepsilon_{p'}\mathcal{P}'_B) = 0. \tag{31}$$

This is the Generalized Euler equation (GEE). The Lagrange multiplier λ measures the size of the distortions created by government policy. Like in the simple model of the previous section, the GEE states the intertemporal trade-off faced by the government in the management of its debt.

3.3 Long-run policy

As in the simple model of section 2, the differentiable Markov-perfect equilibrium features an unstable steady state that implements the first-best. The proof follows identical steps and is thus omitted.

Let us focus on the distortionary steady state. We have $\lambda = \lambda' > 0$ and thus (31) becomes

$$\varepsilon_{n'}\mathcal{N}_B + \varepsilon_{p'}\mathcal{P}_B = 0, \tag{32}$$

where

$$\varepsilon_{n'} = \frac{\beta(u_{12} - u_{1\ell} + (u_{22} - u_{2\ell})B)}{p}$$

$$\varepsilon_{p'} = -\frac{\beta(pu_1 + u_{11} - u_{12} + (pu_2 + u_{21} - u_{22})B)}{p^3}.$$

Due to the presence of the derivatives of two policy functions in (32), the steady state cannot typically be solved locally. In other words, we need to solve for the equilibrium policy functions in order to identify the distortionary steady state. However, notice that if u is separable in all arguments and linear in c_2, then $\varepsilon_{n'} = 0$ and thus (32) becomes $\varepsilon_{p'}\mathcal{P}_B = 0$. Given that $\mathcal{P}_B$ cannot be zero in a distortionary steady state (same argument as in *Proposition 1*), we get $B^* = -\frac{p^* u_1^* + u_{11}^*}{p^* u_2^*}$, which does not depend on the derivative of any policy function. We can now establish a result for long-run debt, similar to the one derived in *Proposition 3* for the basic model.

Proposition 5 *Assume* $u(c_1, c_2, \ell) = \frac{c_1^{1-\sigma} - 1}{1-\sigma} + \alpha c_2 + \nu(\ell)$, *where* $\alpha, \sigma > 0$ *and* $\nu(\ell)$ *is strictly increasing and strictly concave. Then, there exists a distortionary steady state of the differentiable Markov-perfect equilibrium which features* $B^* > 0$ *iff* $\sigma > 1$, $B^* = 0$ *iff* $\sigma = 1$ *and* $B^* < 0$ *iff* $\sigma < 1$.

With a few additional assumptions we can solve analytically for all allocations and policies at the distortionary steady state and perform some comparative statics. The following proposition highlights a central message of the paper, namely that debt, permanent expenditure and the cost of inflation all interact in the determination of government policy in the long-run.

Proposition 6 *Assume* $u(c_1, c_2, \ell) = \frac{c_1^{1-\sigma}-1}{1-\sigma} + \alpha c_2 - \frac{\gamma(1-\ell)^2}{2}$*, where* $\alpha, \gamma, \sigma > 0$*, and that agents value government expenditure*[7] *according to* ψg*, where* $\psi > 0$*,* $\frac{\alpha}{\psi} \in (\frac{\sigma-1}{\sigma}, \beta]$ *and* $\gamma > \frac{\alpha\psi}{2\psi-\alpha}$*. Then, at the distortionary steady state of the differentiable Markov-perfect equilibrium we get the following comparative statics.*

(i) An increase in the marginal utility for the public good, ψ*: (1) increases taxes and inflation; (2) decreases output; (3) increases debt if* $\sigma > 1$*, decreases debt if* $\sigma < 1$ *and has no effect on debt if* $\sigma = 1$*; (4) the effect on debt increases with* $|\sigma - 1|$*.*

(ii) A decrease in the intertemporal elasticity of substitution for the cash good, i.e., an increase in σ*: (1) increases debt and inflation; (2) has no effect on taxes and output; (3) the effects on debt and inflation increase with* ψ*.*

(iii) If $\alpha = \beta\psi$ *then government expenditure does not depend on* σ*.*

Result *(i)* of the proposition above challenges the conventional view. Barro's tax-smoothing argument establishes that permanent increases in government expenditure should be financed solely with taxes. In contrast, Proposition 6 shows that how long-run debt reacts to permanent changes in expenditure depends on how costly the inflation tax is (in this case, through the intertemporal elasticity of substitution). Result *(ii)* shows that if inflation is more distortionary, then the incentives to reduce the real value of debt through monetary policy decrease and thus, debt increases in the long-run. Inflation also increases in the long-run, since it needs to finance a larger debt. Note however, that taxes and output do not change. Result *(iii)* suggests that economies with similar macroeconomic statistics—same government expenditure, taxes and output—may have very different levels of debt. In the following section we will verify numerically that similar results hold for more general utility functions and how this findings relate to the data.

4 Quantitative Analysis

4.1 Numerical solution

The model of debt with fiscal policy is only tractable under specific assumptions on preferences. Furthermore, the analytical results have little to say about the quantitative effect of changes in fundamentals on long-run policy. In this section, we will use numerical methods to globally solve the economy with taxes using a more general utility specification. In particular, we are interested in evaluating how long-run debt and other policy variables react qualitatively and quantitatively to changes in fundamentals, for an economy calibrated to the post-war U.S. economy.

[7] The switch to endogenous government expenditure is done for analytical tractability only. The rest of the paper focuses on exogenous government expenditure.

Let the household's utility function be

$$u(c_1, c_2, \ell) = \frac{\left((\alpha c_1^\rho + (1-\alpha)c_2^\rho)^{\frac{\gamma}{\rho}} \ \ell^{1-\gamma}\right)^{1-\sigma} - 1}{1-\sigma}, \tag{33}$$

which exhibits a constant elasticity of substitution between the cash and the credit good and is Cobb-Douglas between the consumption aggregator and leisure. Long-run debt will depend critically on ρ. In this case, to get a large, positive steady state level of debt, ρ will have to be negative, i.e., cash and credit goods have to be complements. The reason is that if the household can easily substitute cash goods for credit goods, then inflation becomes a cheap—in terms of distortions—source of funds for the government and the gains from reducing debt increase.

The parameters to calibrate are α, β, γ, ρ, σ and g. The selected target statistics are $B(1+\mu)/py$, c_1/c_2, g/y, n, π and r—the real interest rate—in the U.S. for the period 1962-2006. The value for τ is simply the tax rate that satisfies the government budget constraint given these targets. Note that the reason we use $B(1+\mu)/py$ instead of B/py as a measure of debt over GDP, is that we will take end-of-period debt from the data.

Since the inflation tax plays a key role in determining long-run debt, it is important to target the agents' exposure to it. For this purpose, the most appropriate monetary aggregate seems to be M_1. Given that M_1 is larger than quarterly consumption, the period length is set to a year. By subtracting the definition of money from total consumption we get the amount of credit goods consumed in a year. Then we get our target value for c_1/c_2, which is equal to 0.37 for the sample period.

To target debt, we can use data from the Office of Management and Budget (OMB). The series of debt held by the public—i.e., excluding holdings by federal agencies—in terms of GDP, averages 36% for the period 1962-2006. This figure includes the holdings of Federal Reserve Banks, which amount to about 5% of GDP. Hence, the target for debt over GDP is 31%.

Data from the Federal Reserve System shows that the nominal interest rate for the 1-year constant maturity Treasury Bill, averaged 6.3% annual. Next, use the annual variation of the consumption deflator as the measure for inflation, which gives an average of 3.9% annual[8]. This implies a target real interest rate of about 2.3% annual.

The fraction of time devoted to labor is set to 0.3, as is standard in the macroeconomics literature. The target for government expenditure over GDP is also taken from the OMB. Between 1962 and 2006, outlays of the Federal Government averaged 20% of GDP. Note that this figure includes transfers, both to states and households, and interest payment on debt. In the model economy, interest payments are accounted for in q, not g. Thus, we need to subtract interest payment from expenditure and thus, the target for government expenditure over GDP is 18%.

Now we need to choose parameter values that make the model match the targeted statistics. From equations (23) and (24) we have that in steady state $r = \frac{1}{\beta} - 1$, and so the value for β consistent with the target real interest is 0.9774. Given that the target for hours worked is 0.3 and for g/y is 0.18, g has to be equal to 0.054. The rest of the parameters have to be fine-tuned through successive iterations, although most of them affect primarily only one statistic.

As argued above, ρ is the main parameter determining long-run debt. The value of ρ is set to

[8] Using the CPI gives a slightly larger figure, 4.4% annual.

-2.8. The implied elasticity of substitution between cash and credit goods is around a quarter. This means that cash and credit goods are complements. The model is not alone in this respect. Papers in the inflation cost literature—see Aiyagari, Brown and Eckstein (1998) and Erosa and Ventura (2002)—actually assume perfect complementarity between cash and credit goods[9].

From equation (24) and using our targets for the nominal interest rate R and $c1/c2$ we get the following steady state condition

$$\frac{1}{1+R} = \frac{1-\alpha}{\alpha}\left(\frac{c_1}{c_2}\right)^{1-\rho},$$

which sets the value of α to 0.0237. Finally, set σ to 4.25 to match an annual inflation rate of 3.9%, and γ to 0.303 to get $n = 0.3$. Table 1 gives a summary of the parameter values chosen for the calibration exercise.

Table 1: Parameter values

Parameter	α	β	γ	ρ	σ	g
Value	0.0237	0.9774	0.303	-2.800	4.250	0.054

As mentioned above, the labor tax rate is not calibrated. Its steady state value in the artificial economy is 0.178. The equivalent measure for τ in the data is federal revenue—which does not include loans or seignorage—over GDP. Data from the OMB shows that federal revenue over the sample period averaged about 18% of GDP.

Since we solve the model globally, we can analyze government policy for any level of debt. In this sense, close enough to the steady state, both tax instruments are substitutes: the labor tax is decreasing in debt and the money growth rate is increasing in debt. The reason for this is that the government—instead of distorting all margins a little—taxes the margin with the highest return and tries to minimize the distortion of the other margin. When debt is low, consumption of the cash good and labor are both high, but labor has the largest tax base. Hence, the government taxes labor heavily and runs a deflation so as to minimize the cash-in-advance distortion. Note that for sufficiently low levels of debt, the inequality constraint (25) binds, i.e., the government runs the Friedman rule, even though it would want to contract the money supply at an even faster rate. In the range where the lower bound on monetary policy binds, taxes are increasing in debt.

When debt is high, it is still true that labor is higher than consumption of the cash good. However, with high debt the inflation tax has a larger tax base, since inflation also decreases the real value of government debt. This incentive makes inflation a more attractive source of funds and so the government inflates prices while it reduces the labor tax (which may end up negative for sufficiently large debt levels). So, with higher debt there is a shift from labor income taxation to inflation. This policy scheme implies that inflation and the nominal interest rate are increasing functions of debt, while the real interest rate is a decreasing function of debt.

[9]The money-in-utility function literature assumes complementarity between consumption and real balances. For example, Lucas (2000) sets the elasticity of substitution to one half.

4.2 Comparative statics

What happens to long-run debt when we change some parameters? Table 2 shows a summary of this exercise. Output n is not reported since it remains essentially unchanged in all cases considered.

Table 2: Comparative statics on long-run variables

	Bench mark	$\Delta g/y$ $g = 0.081$	$\nabla c_1/c_2$ $\alpha = 0.005$	$\nabla \pi$ $\sigma = 3.350$	$\Delta B/py$ $\rho = -6.000$
τ	0.178	0.252	0.180	0.182	0.164
π	0.039	0.127	0.033	0.022	0.087
g/y	0.180	0.270	0.180	0.180	0.181
c_1/c_2	0.370	0.362	0.245	0.372	0.579
$B(1+\mu)/py$	0.310	0.322	0.278	0.294	0.436

Let us first look at what happens if government expenditure were higher. Thus, let g increase by 50% to 0.081. The result—compared to the benchmark case—is an increase in labor taxes, from 17.8% to 25.2%, and annual inflation, from 3.9% to 12.7%. Debt over GDP increases slightly, about 1 percentage point. All these variations are consistent with the analytical results from *Proposition 6*. The increase in taxes is also a feature of traditional theories of debt, which predict that permanent increases in expenditure should be financed exclusively with taxes. Here, however, debt increases as well. Section 5 below analyzes what happens when the increase in government expenditure is transitory.

The next column in Table 2 evaluates the effects of a decrease in α. The parameters is lowered so that $c1/c2$ hits its 1992-2006 average of 0.245. The result is a small increase in taxes and a more important decrease in debt and inflation. A lower α reduces the cost of inflation since cash goods constitute a lower proportion of an agent's consumption. Thus, debt does not grow as much as in the benchmark parametrization and is about 3 percentage points lower in the long-run.

The third column in Table 2 shows the effect of increasing the intertemporal elasticity of substitution, i.e., decreasing σ. Under a specific utility function, result *(ii)* in *Proposition 6* predicted that decreasing σ would lower long-run debt and inflation, but would have no effect on taxes. The comparative statics exercise confirms these predictions. The annual inflation between 1992 and 2006 was 2.2% annual, i.e., 1.7 percentage points lower than in the benchmark parametrization. To hit this target, decrease σ from 4.25 to 3.35. This also results in about 1.6 percentage points decrease in debt over GDP and a small (0.4 percentage points) increase in taxes. As explained in the previous section, increasing the intertemporal elasticity of substitution raises the inflation tax motive since households are less hurt by variations in consumption over time. This decreases the motive for debt accumulation and the debt function shifts down. The variation of debt is not very dramatic since with non-separable preferences, the effects of σ are second-order.

The last column in Table 2 evaluates the effects of a decrease in ρ. This makes cash and credit goods stronger complements, which increases the cost of inflationary policy. We get sharp increases in debt and inflation: debt over GDP climbs to almost 44% and inflation reaches 8.7% annual. On the other hand, taxes are lower than in the benchmark parametrization. The behavior of taxes and inflation is related to the tax function being decreasing in debt and the money growth rate being increasing in debt. Even though both the tax and money growth rate functions shifts up due to the higher financing needs, the larger long-run debt implies higher inflation and lower taxes. Note however, that expenditure over GDP is virtually the same as in the benchmark case. This is consistent with result *(iii)* in *Proposition 6.* That is, economies can feature similar macroeconomic statistics—e.g., same government size—and still differ significantly in their long-run levels of debt.

How do these findings relate to cross-country data? Consider the U.S. between 1992 and 2006 as a benchmark[10]. Canada and Spain show similar levels of expenditure over GDP, debt over GDP and inflation. Iceland, New Zealand and the United Kingdom feature similar levels of debt and inflation, but much higher expenditure (50% to 80% more than the U.S.). In contrast, Japan and Korea have similar levels of government expenditure over GDP (plus low inflation), but widely different levels of government debt. Debt over GDP averaged 100% in Japan and 17% in Korea. Then, there are countries like Belgium and Italy, which feature high debt and expenditure, and Switzerland, which features low debt and expenditure. In sum, government debt is not predictably related to other policy variables. Furthermore, as Alesina and Perotti (1995) argue, cross-country differences in debt do not appear to be explained by other economic variables either.

The comparative statics exercise shows that the model can accommodate a large variety of macroeconomic target statistics. In principle, one could calibrate the model to various economies and determine which parameters explain the differences. For example, suppose we wanted to calibrate the model to countries like Iceland, New Zealand and the United Kingdom. On average, these three economies feature 70% more government expenditure over GDP than the U.S., with similar levels of inflation and debt. Thus, consider calibrating the model to the benchmark targets, except for g/y, which we now set to about 31%. This exercise results in the following changes in parameters: $\alpha = 0.0045$, $\rho = -4.5$, $\sigma = 1.75$ and $g = 0.092$. Notice that even though only one calibration target changed, there were significant variations in four parameters. This suggests that countries with only a few differences in government policy and other macroeconomic variables, may have profound differences in (possibly unobservable) fundamentals. It is worth remarking though, that the theory in this paper does not provide an explanation for the sources of these differences.

5 Stochastic government expenditure

5.1 A model with government expenditure shocks

Let us extend the model from the previous section to allow for stochastic government expenditure. Suppose there are two possible expenditure levels: g_L and g_H, with $0 < g_L < g_H$. Government expenditure follows a Markov process. If today the economy is in state g_L, then tomorrow the government will have to spend g_L with probability θ_L and g_H with probability $1 - \theta_L$. Likewise, if today the economy is in state g_H, then tomorrow the government will have to spend g_L with probability $1 - \theta_H$ and g_H with probability θ_H.

If today the exogenous state is g_L then the first-order conditions of the representative household

[10] Data is taken from the OECD database and corresponds to central governments.

imply

$$(1-\tau_L)u_2^L = u_\ell^L \tag{34}$$

$$\frac{u_2^L(1+\mu_L)}{p_L} = \beta E\big[\frac{u_1'}{p'}|L\big] \tag{35}$$

$$q_L = \frac{E\big[\frac{u_2'}{p'}|L\big]}{E\big[\frac{u_1'}{p'}|L\big]} \tag{36}$$

$$u_1^L - u_2^L \geq 0, \tag{37}$$

where $E\Big[\frac{u_i'}{p'}\Big|L\Big] = \theta_L \frac{u_i'^L}{p_L'} + (1-\theta_L)\frac{u_i'^H}{p_H'}$, for $i = \{1,2\}$. The subscripts "L" and "H" refer to whether a variable corresponds to state g_L or g_H, respectively. For functions, we use a superscript to denote that they are being evaluated at the appropriate state. Note that there is a corresponding set of first-order conditions when the state today is g_H.

As in the case of constant government expenditure, it is possible to use (34), (35) and (36) to write the government budget constraint as a function of B, B_L', n_L, $n_{LL} = \mathcal{N}^L(B_L')$, $n_{LH} = \mathcal{N}^H(B_L')$, p_L, $p_{LL} = \mathcal{P}^L(B_L')$, and $p_{LH} = \mathcal{P}^H(B_L')$:

$$\varepsilon^L\big(B, B_L', n_L, \mathcal{N}^L(B_L'), \mathcal{N}^H(B_L'), p_L, \mathcal{P}^L(B_L'), \mathcal{P}^H(B_L')\big) = 0.$$

The superscript on the ε function indicates what probabilities to use (in this case, θ_L and $1-\theta_L$).

Given that the exogenous state is g_L and that future governments will induce $\mathcal{N}^L$ and $\mathcal{P}^L$ if the state is g_L and $\mathcal{N}^H$ and $\mathcal{P}^H$ if the state is g_H, the problem of the government is

$$\mathcal{V}^L(B) = \max_{n_L, p_L, B_L'} \; u\Big(\frac{1}{p_L},\; n_L - \frac{1}{p_L} - g_L,\; 1-n_L\Big) + \beta\left(\theta_L \mathcal{V}^L(B_L') + (1-\theta_L)\mathcal{V}^H(B_L')\right)$$

subject to

$$\begin{aligned} \varepsilon^L\big(B, B_L', n_L, \mathcal{N}^L(B_L'), \mathcal{N}^H(B_L'), p_L, \mathcal{P}^L(B_L'), \mathcal{P}^H(B_L')\big) &= 0 \\ u_1^L - u_2^L &\geq 0. \end{aligned}$$

Whereas if the exogenous state is g_H then the problem of the government is

$$\mathcal{V}^H(B) = \max_{n_H, p_H, B_H'} \; u\Big(\frac{1}{p_H},\; n_H - \frac{1}{p_H} - g_H,\; 1-n_H\Big) + \beta\left((1-\theta_H)\mathcal{V}^L(B_H') + \theta_H \mathcal{V}^H(B_H')\right)$$

subject to

$$\begin{aligned} \varepsilon^L\big(B, B_H', n_H, \mathcal{N}^L(B_H'), \mathcal{N}^H(B_H'), p_H, \mathcal{P}^L(B_H'), \mathcal{P}^H(B_H')\big) &= 0 \\ u_1^H - u_2^H &\geq 0. \end{aligned}$$

A Markov-perfect equilibrium is a set of functions $\{\mathcal{V}^L, \mathcal{V}^H, \mathcal{B}^L, \mathcal{B}^H, \mathcal{N}^L, \mathcal{N}^H, \mathcal{P}^L, \mathcal{P}^H\}$ that solves the above problem.

5.2 Numerical analysis

To better understand how the model economy behaves in the presence of shocks to government expenditure, let us perform a simple numerical exercise by taking the calibration from the previous section. U.S. data for the sample period 1962-2006, shows that yearly government outlays—net of debt interest payments—averaged about 18%, with a standard deviation of 1.2 percentage points and an autocorrelation of 0.8. The distribution is very symmetric, with a skewness of -0.05. Thus, set $g_L = 0.051$, $g_H = 0.057$, $\theta_L = \theta_H = 0.9$. These values imply that expenditure over GDP will fluctuate between 17% and 19%, both states being equally likely and with a duration of 10 periods.

The model is solved numerically, using continuous global methods. To verify its time-series properties, a simulation of the model is run for $10,000$ periods[11]. The averages of variables over time match the steady state statistics of the deterministic case. The time series for government expenditure over GDP is symmetric and matches the autocorrelation from the data. Table 3 compares the standard deviation and first-order autocorrelation in the data and the model for selected variables. Sample averages are as reported in Table 2 and thus omitted. Taxes in the data are measured as federal revenue over GDP.

Table 3: Statistics for simulation of 10,000 periods

	Std. Deviation		Autocorrel.	
	Data	Model	Data	Model
τ	1.0%	0.5%	0.65	0.72
π	2.5%	2.3%	0.84	0.94
g/y	1.2%	1.0%	0.79	0.81
$B(1+\mu)/py$	7.8%	2.0%	0.96	0.98

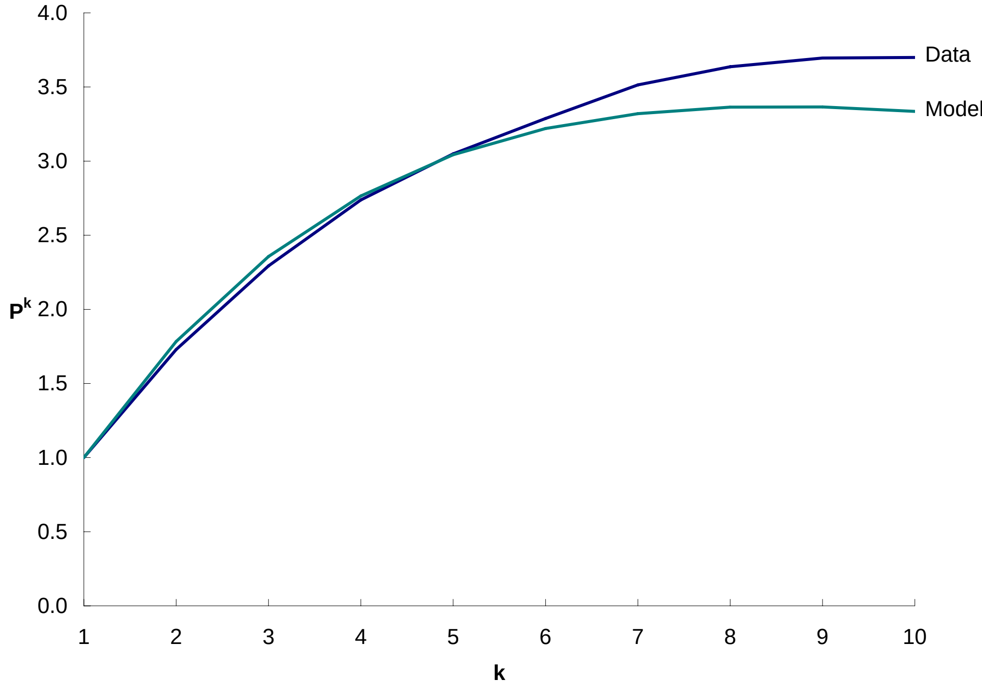
Taxes in the model show a similar volatility as in the data and a close first-order autocorrelation. Both the model and the data differ significantly from a random walk, in contrast to what Barro's tax-smoothing hypothesis predicts. Inflation in the model matches the volatility in the data but overestimates its autocorrelation, although not by much. The row for g/y has the calibration targets, so the simulation matches the data. Finally, the autocorrelation of debt matches the data, but the volatility is a bit off. This is not surprising, since the period 1962-2006 features variations in debt unrelated to expenditure shocks (e.g., the debt build-up in the 1980s).

For post-war U.S., Marcet and Scott (2007) document that innovations to government expenditure have large and persistent effects on debt[12]. They suggest using the k-variance ratio as a robust measure of persistence. This ratio is defined as $P_x^k = \frac{Var(x_t - x_{t-k})}{kVar(x_t - x_{t-1})}$; a variable is more persistent the longer it takes the ratio to converge to zero. Figure 2 compares the persistence of debt in the data and the model. In the data, the k-variance ratio for debt over GDP is increasing and only starts leveling off after 10 years at about 3.7. For k between 1 and 6, the ratios for actual and simulated debt are very close, within 3% of each other. The ratio for the simulated data peaks at $k = 9$ with a value of 3.35. Thus, the model matches debt persistence reasonably well. Marcet and

[11] Larger sample periods do not change the reported statistics significantly.

[12] Their analysis actually considers the primary deficit and the market value of debt. However, their qualitative results still hold if instead we use expenditure and debt at face value.

Figure 2: k-variance ratio for Debt over GDP



Scott also show that deficits and debt co-move in the same direction. The simulated data shows the same qualitative pattern, i.e., a positive and hum-shaped response of debt to an innovation in expenditure.

Compare the results described above with the ones reported in Chari, Christiano and Kehoe (1991) and Chari and Kehoe (1998), who solve a similar environment but under the assumption that the government can commit to future policy choices. In their simulations, taxes exhibit very low volatility (standard deviation is about 0.1%), whereas inflation is very volatile (between 10% and 60%) and features a near zero or negative autocorrelation. Furthermore, the value of end-of-period debt falls substantially following a positive expenditure shock. The lower inflation volatility reported in this paper confirms the results obtained by Nicolini (1998) who showed that discretionary governments may be tempted to chose inflation rates lower than under commitment. For optimal models of fiscal policy, Marcet and Scott (2007) show that debt increasing rather than decreasing in response to an expenditure shock is a generic feature of economies with incomplete markets. The results in this paper extend their findings to models with monetary policy.

Let us now analyze government policy in each state. The debt function in the high expenditure state is always above the one for the low expenditure state. The money growth rate functions are increasing in debt and typically higher for the high expenditure state, except for low levels of debt, when monetary policy is close to the Friedman rule. The reason is that the lower bound on μ that is consistent with a non-negative interest rate is lower for the high expenditure state. The tax rate functions are typically decreasing in debt—except for low levels of debt, when monetary policy is close to the Friedman rule—and always higher for the high expenditure state. Note that these exceptions happen well below the steady state for the low expenditure regime (i.e., the point where

$\mathcal{B}^L(B)$ crosses the 45-degree line).

In the long-run, government debt moves between the steady states for g_L and g_H. Given the discussion above, it follows that when the economy moves from the low to the high expenditure state, there is an increase in debt, inflation and taxes. When the economy returns to the low expenditure state, debt and inflation decrease gradually, whereas taxes jump down and then increase gradually.

As one would expect, the debt function for the deterministic case lies in between the two debt functions for the stochastic case. When compared to the deterministic case, the government in the low expenditure state has a lower incentive to have debt, since it internalizes the fact that it will have to distort the economy more when expenditure increases. In this sense, the steady state for g_L has debt over GDP of 28% and an inflation rate of around 0.3% annual. On the other hand, if the economy stays in the high expenditure state long enough, then debt over GDP climbs to almost 34%, with an inflation rate of 7.3% annual.

As in section 4, we can perform comparative statics to evaluate how some fundamentals affect government debt. We have four parameters to evaluate: g_L, g_H, θ_L and θ_H. All of the following have similar qualitative effects on debt: lower g_L, higher g_H, higher θ_L and lower θ_H.

Both a lower g_L or a higher g_H increase the distance between the debt functions of the two states. For example, suppose g_L is set to 0.045 so that g/y is 15% in the g_L steady state, i.e., 2 percentage points lower than in the (stochastic) benchmark case. Then, debt over GDP decreases to 25% in the g_L steady state and increases to 36% in the g_H steady state. Thus, debt becomes more volatile. Average debt over the sample of $10,000$ periods remains at 31%. The results are similar if instead we increase g_H to 0.063, so that g/y increases to 21% in the g_H steady state.

With either a lower g_L or higher g_H, the government in the high expenditure state can increase debt more since it knows that when expenditure decreases, the government will not have to distort as much as before. Since in both cases we left the states equally likely, the implications for taxes and inflation are different. With a lower g_L, average expenditure is lower and so are taxes and inflation. Taxes decrease in both states, whereas inflation decreases for the low expenditure state and increases for the high expenditure state. Conversely, a higher g_H implies higher average expenditure and thus higher average taxes and inflation. Taxes increase in both states. However, inflation behaves as with a lower g_L, i.e., decreases for the low expenditure state and increases for the high expenditure state.

Increasing θ_L or decreasing θ_H makes the low expenditure state happen more often. In terms of long-run averages, both changes have no significant effect on debt over GDP, but lower inflation and taxes since expected expenditure is lower.

Suppose we increase θ_L to 89/90, which implies the low expenditure state occurs 90% of the time, with an average duration of 90 periods. In this case, the debt functions for both states increase, although the change for the high expenditure state is negligible. Thus, debt becomes less volatile than in the benchmark case. Taxes are lower in both states, whereas inflation is higher (still, average inflation also decreases).

Suppose now that we lower θ_H to 0.1. This also implies that the low expenditure state occurs 90% of the time, but decreases the duration of the high expenditure state to about 1.1 periods. The effects on long-run averages are virtually identical to those of increasing θ_L. We also get a shift up of debt functions in both states as well as lower taxes and higher inflation. However, the short duration of g_H allows the government to increase debt much more than in the benchmark case and thus not raise taxes as much. Steady state debt for g_H climbs to 47% of GDP. Even so, volatility of debt is lower that in the benchmark case, since g_H is not very frequent.

5.3 Government policy in the United States: 1791-2006

Figure 3 shows the historical evolution of some key policy variables in the United States, from 1791 to 2006. The upper panel shows debt held by the public over GDP; the second panel shows federal outlays—net of debt interest payments—over GDP; the third panel is federal revenue over GDP. Yearly data for debt, revenue and expenditure are available from Wallis (2006*a*) and Wallis (2006*b*) for 1791-1939 and from the Office of Management and Budget for 1901-2006. The series for GDP is taken from Johnston and Williamson (2008). The bottom panel shows the annual inflation rate using the CPI historical series from Lindert and Sutch (2006) and the Bureau of Labor Statistics.

Some features of the data are worth describing[13]. First, the War of 1812, the Civil War and the two World Wars are quite evident in the series of government expenditure. All these episodes triggered significant increases in government debt, revenue and inflation. In this regard, Goldin (1980) estimates that the contribution of debt and seignorage to war financing was 79% for the War of 1812, 91% for the Civil War, 76% for World War I and 59% for World War II, with the rest being financed with contemporaneous taxes. One notable exception is the Korean War, which was almost exclusively financed with contemporaneous taxes (see Ohanian, 1997 for a discussion and analysis). The effects of temporary increases in expenditure due to wars appear to be persistent on debt. In contrast, as Ohanian (1998) points out, a recurring pattern in U.S. war financing is a "significant inflation during wartime, followed by a return to the prewar price level". World War II stands out in that there was a significant postwar inflation (1946-1948) which helped reduce the real value of the accumulated debt. Ohanian estimates that the reduction of the real value of debt due to inflation is equivalent to a repudiation of debt worth 40% of GNP.

Second, there is a clear structural break in the series of federal government revenue and expenditure around the Great Depression. Even including wartime expenditure, federal outlays averaged only about 2.3% of GDP between 1791 and 1929. Starting with the "New Deal" policies in 1933, federal outlays grew steadily, stabilizing after World War II at about 17% of GDP.

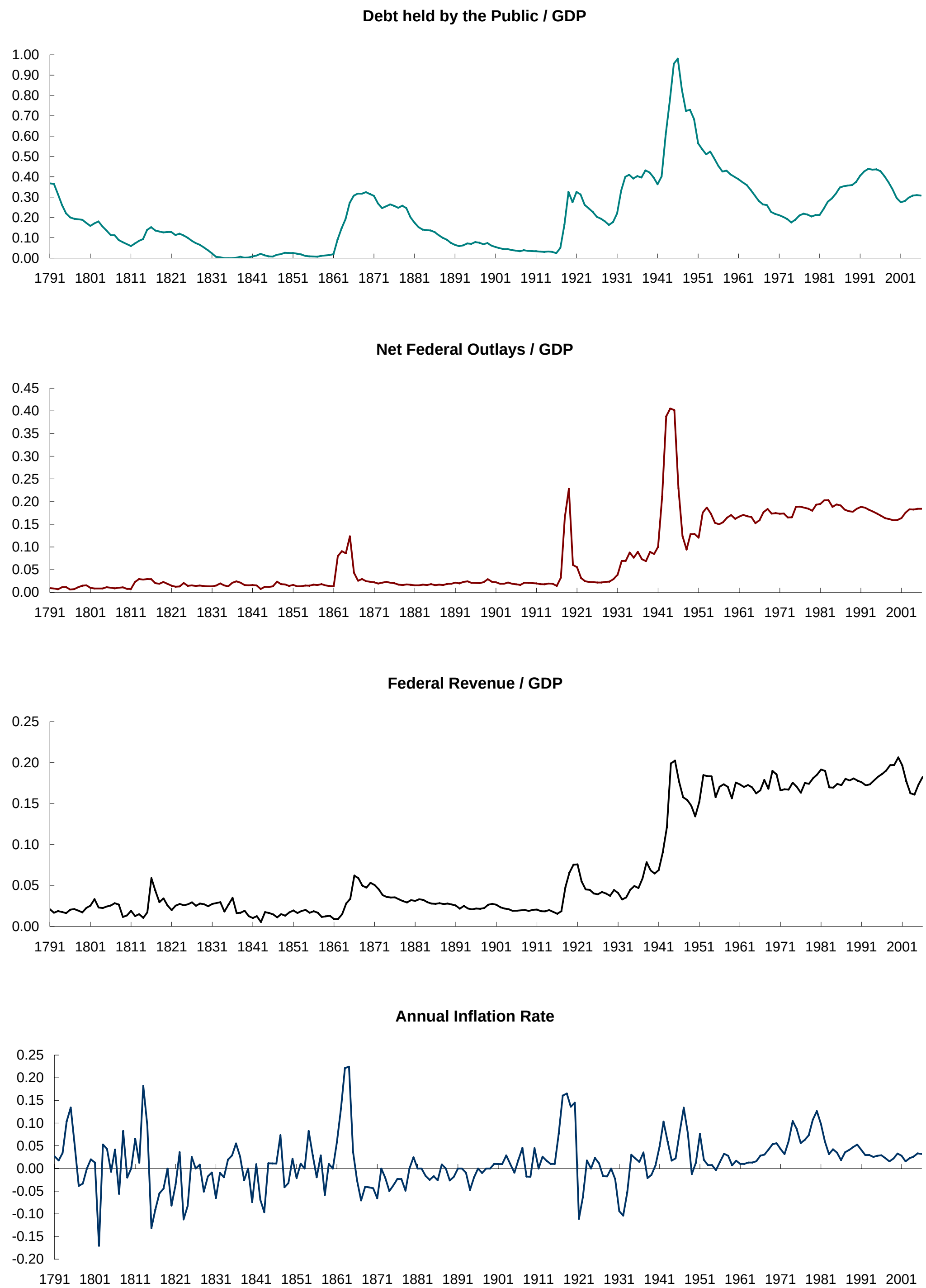
Lastly, there are two non-war-related debt buildups: one between the Great Depression and World War II, and the other in the 1980s. This second episode—which also manifested itself in several other industrialized countries around roughly the same time—has spawned an extensive literature which suggest political economy frictions are the source of the debt increase (see Persson and Tabellini, 1998).

5.4 Simulations

Consider the period 1832-1929, which is roughly the period Wallis (2000) coins "The Era of Property Finance and Local Government". It features very low peacetime federal expenditure and two important wars, the Civil War and the first World War, which triggered increases in debt, taxes and

[13] See Wallis (2000) for a thorough description and analysis of U.S. government data and policy since 1790.

Figure 3: United States Historical Series: 1791-2006



Note: inflation is shown as differences in log price.

inflation. Both end points are important: by 1832 most of the debt—including that of the Revolutionary War and the War of 1812— had been repaid and 1929 is the year of the Great Depression, right before the federal government started expanding. The two wars created corresponding periods of high temporary expenditure: 1862-1866, averaging 8.5% of GDP and 1918-1921, averaging 12.7% of GDP. During peacetime, expenditure over GDP averaged 1.9%. Thus, we have 9 out of 98 years with high expenditure, with an average duration of 4.5 years. Debt over this whole period averaged about 11% of GDP and inflation was 0.9% annual. Note that during peacetime, average annual inflation was negative, at -0.2%.

Let us calibrate the model to match the moments described for this period and then evaluate how the model economy reacts to war shocks. Table 4 summarizes the parameter choice. The procedure is similar to the one described in previous sections. The calibration targets are debt over GDP, inflation, interest rate, cash-credit good ratio, hours worked, average peacetime expenditure over GDP, average wartime expenditure over GDP, frequency and duration of wartime expenditure. The nominal interest rate (which, given inflation, pins down the real interest rate and thus, β) was calculated using the series of interest paid on debt. The ratio of cash to credit goods was calculated for the subperiod 1915-1929 at 0.6, since estimates of M_1 are not available for earlier years. Data for M_1 is taken from Friedman and Schwartz (1970) and for total consumption from Craig (2006). The target for hours was left at 0.3 and thus γ is kept at its benchmark value. The value for σ typically targets inflation; however, it cannot be calibrated in this case given the small government expenditure. Basically, with a very low expenditure, distortions are very low and thus the government wants on average to run the Friedman rule. Varying σ does not produce significant changes in the results (except where noted below), so it is left at is benchmark value. Over a sample of simulated $10,000$ periods, annual inflation averages -2.2%, a bit lower than the actual average. All other targets are matched.

Table 4: Parameter values for 1832-1929 simulation

Parameter	α	β	γ	ρ	σ	g_L	g_H	θ_L	θ_H
Value	0.3150	0.9714	0.303	-0.500	4.250	0.0057	0.031	0.9775	0.7778

The left panel of Figure 4 shows the simulated response of an expenditure shock similar to the one experienced by the U.S. during the Civil War. The initial value of debt was set to match debt over GDP in 1860[14]. The policy response during World War I was very similar, so the results mostly apply for that case as well. Starting in 1862, we get 5 years of expenditure approximately 5.5 times higher than normal. There are some important similarities between the simulated and actual policies. Tax revenue over GDP increases temporarily and partially covers the cost of the war[15]. Debt over GDP increases sharply, peaking in 1867 at 22%; about the same time debt plateaus in the data at 32%. Debt is slowly repaid, although at a faster rate than in the data. Inflation goes up briefly the year of the shock; there is also a large deflation in the first period of low expenditure

[14]Starting at the low expenditure steady state would not alter the qualitative results.

[15]In this dimension, the Civil War and World War I differ. The increase in taxes during World War I starts sooner and finances a higher percentage of the war expenditure (about twice as much according to Goldin, 1980), similar to the policy implied by the model. The slower response of taxes during the Civil War is likely associated with the fact that the federal income tax was first implemented in the U.S. with the Revenue Act of 1861 to finance the cost of the war; the tax rate was subsequently increased by the Revenue Acts of 1862 and 1864.

after the war. The data shows a similar qualitative pattern (first inflation, then deflation), but the increase in inflation is much higher. Again, the reason for inflation in the model being too low, is that the distortions are not large enough to warrant the use of monetary policy.

As described above, World War II differed from previous experiences in that the government implemented a large post-war inflation that significantly reduced the real value of accumulated debt. Figure 3 reveals that, compared to the Civil War and World War I, there were two important differences in the state of the economy leading up to World War II: both federal expenditure and debt were significantly higher. In the period 1930-1941, federal outlays averaged 7.3% of GDP and debt averaged 36% of GDP. An interesting exercise is to evaluate how the model economy would react if calibrated to these different targets. Thus, set g_L to 0.022 and $g_H = 0.096$, which target g/y of 7.3% during peacetime and 33% between 1942 and 1946, respectively. Next, set ρ to -4.5 to match the average pre-war level of debt. This would also change the implied long-run $c1/c2$; to leave this value unaltered, adjust α to 0.0561. Monetary policy in the low expenditure state is still at the Friedman rule (g is still too low), but inflation is now closer to the data: the average between 1930 and 1941 was -1.1% annual; the parametrization delivers -2.1% annual. Initial debt is set to match debt over GDP in 1940.

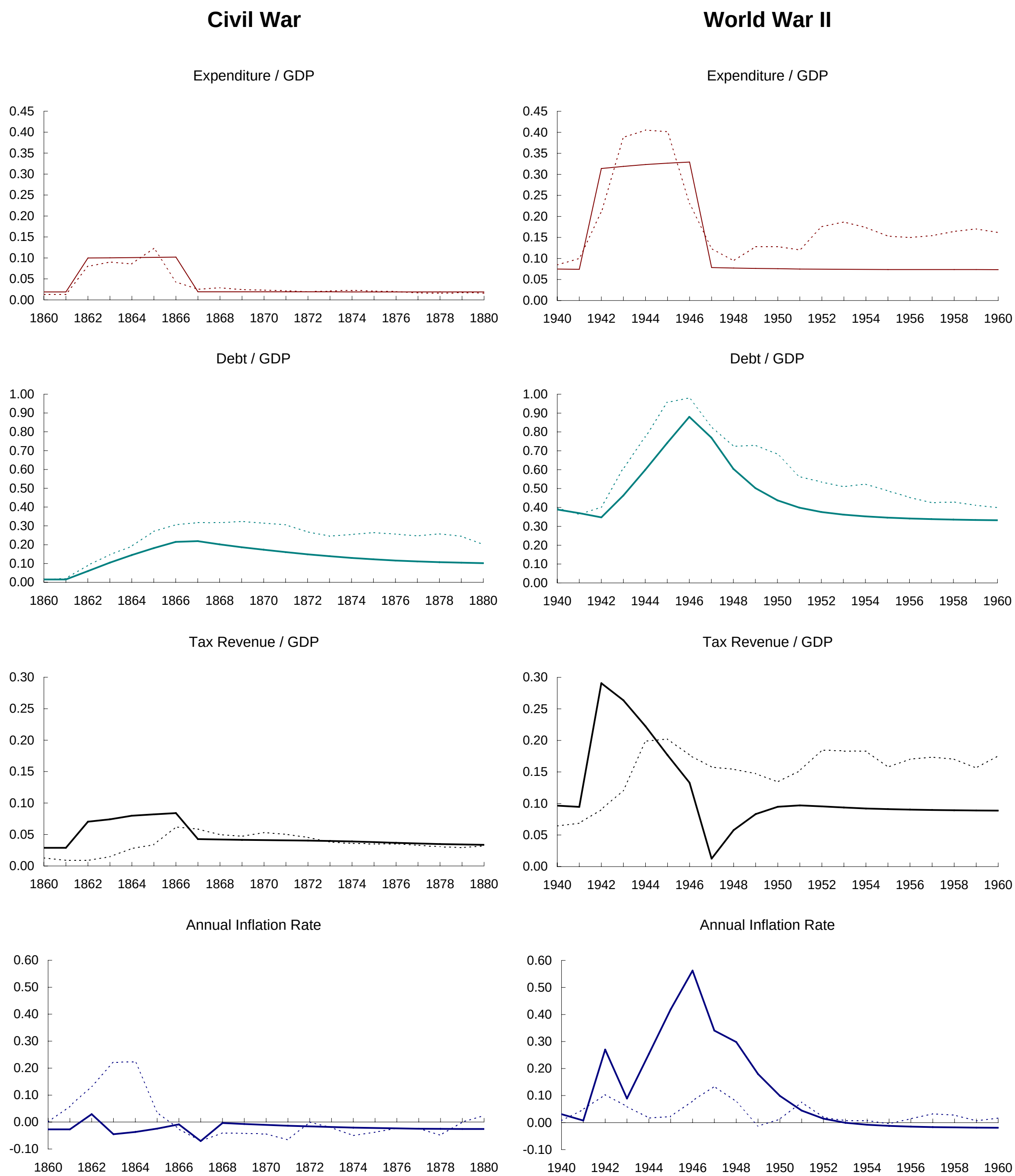
The right panel of Figure 4 shows the policy response to a World War II-type expenditure shock. Debt increases steadily and peaks at 88% of GDP in 1946 (the last year with high expenditure), as in the data. Note that for the Civil War, both simulated and actual debt peaked the year expenditure returned to normal (i.e., one period after, when compared to World War II). In the data, debt over GDP returned to average pre-war levels much faster than after the Civil War; the modeled economy shows the same relative behavior. The reason in both cases is inflation. In the data, the inflation rate climbs in 1942 and then again between 1946 and 1948. The simulated economy overshoots inflation quantitatively[16], but gets the same qualitative pattern: high inflation in 1942 and then again between 1945 and 1949. In both cases, inflation is used first to finance part of the expenditure shock and then to reduce the real value of accumulated debt. The simulated tax rate behaves similar to revenue over GDP in the data, although it peaks a few years earlier. Note that taxes in the data fail to come back down enough, since they were subsequently used to finance the Korean War in the early 1950s.

This exercise shows that the model suggests a reason for the inflation after World War II: the economy around the start of the war featured higher long-run debt and permanent expenditure; thus, the distortions the government needed to create to finance the war were higher than in previous episodes and so the incentives to reduce the real value of accumulated debt through inflation were higher as well.

For both types of war shocks considered, the model delivers good qualitative predictions for the responses of debt, inflation and taxes. Quantitatively, the model does a good job with debt, but inflation levels are off and taxes contribute too much to wartime financing.

[16] Here, σ would play role. A lower value for σ significantly reduces inflation in the high expenditure state. Consider $\sigma = 2.125$, i.e., half the benchmark value. Then, inflation behaves almost identical until 1944, but peaks at about 15 percentage points lower in 1946. The downside is that debt does not increase as much, peaking at 62% (instead of 88%) of GDP in 1946. Taxes show a similar qualitative pattern, but return to normal levels with less volatility, which would be more consistent with the data.

Figure 4: Simulated policy response to war shocks



Notes: solid lines are simulated economy; dashed lines are U.S. historical data; inflation is shown as differences in log price.

6 Concluding Remarks

Lack of commitment provides a mechanism that explains the determination of nominal debt. The high level of variance in debt levels across countries and time suggests that other mechanisms may also be relevant. What else matters for debt is still an open question: is it self-control, reputation, central bank independence, political economy reasons, aggregate wealth? The answer may not be straightforward, since any mechanism set out to explain higher levels of debt, has to provide more incentives to delay taxation, without being offset by larger incentives to inflate. The theory proposed in this paper allows us to evaluate the relative merits of these additional mechanisms within a framework that has definite predictions for long-run debt.

The focus of the paper on nominal debt is motivated by the fact that most government debt from developed countries is issued in the domestic currency and not indexed by inflation. The United States federal government only started issuing Treasury Inflation-Protected Securities (TIPS) in 1997. As of December 2008, these securities amounted to roughly $500 billion or just about 8% of total debt held by the public. The question then is why would a benevolent government issue a debt instrument that it can partially default on through inflation? Bohn (1988) suggests that it is optimal for governments to issue some nominal debt since it insures against the budgetary effects of economic fluctuations. However, it is still not well understood why governments from developed countries rely predominantly on nominal debt. This issue is left for future research.

Another relevant issue not addressed here is how to define public debt. This paper takes the standard view of defining net government liabilities as debt held by the public. Others, such as Eisner and Pieper (1984), have argued that all assets and liabilities should be considered[17]. Unfortunately, the valuation of these proves to be difficult, with problems ranging from technical to conceptual. On a positive note, the Office of Management and Budget (1996) estimated that government assets were worth roughly the same as its non-debt liabilities in 1995. Furthermore, net liabilities seem to have followed debt quite closely since 1975. Elmendorf and Mankiw (1998) provide a detailed discussion.

[17] See also Bohn (1992).

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Appendix

A Proofs

A.1 Proof of Proposition 1

In steady state, (19) simplifies to $\lambda \varepsilon_{c'} \mathcal{C}_B = 0$. One solution is $\lambda = 0$. Then, from (17) we get both $u_c = u_\ell$ (i.e., $c = \hat{c}$) and $\zeta = 0$, i.e., no constraint binds and we are at the first-best. Use $\varepsilon(\hat{B}, \hat{B}, \hat{c}, \hat{c}) = 0$ to get

$$\hat{B} = -1 - \frac{g}{\hat{c}(1-\beta)}. \tag{38}$$

Another solution to (19) in steady state is $\varepsilon_{c'} = 0$, which implies

$$B^* = \frac{-u_c^* + c^*(u_{c\ell}^* - u_{cc}^*)}{u_\ell^* + c^*(u_{c\ell}^* - u_{\ell\ell}^*)}, \tag{39}$$

where c^* solves $\varepsilon(B^*, B^*, c^*, c^*) = 0$. To show that B^* is distortionary, suppose not. Then, $u_c^* = u_\ell^* = \hat{u}_c$ and thus, $B^* = \hat{B}$ and $c^* = \hat{c}$. Using (38) and (39) we get $\frac{g}{\hat{c}(1-\beta)} = \frac{\hat{c}(\hat{u}_{cc} + \hat{u}_{\ell\ell} - 2\hat{u}_{c\ell})}{\hat{u}_c + \hat{c}(\hat{u}_{c\ell} - \hat{u}_{\ell\ell})}$, which is not generically satisfied, a contradiction.

The remaining possibility is $\mathcal{C}_B = 0$. Notice that this cannot happen in a distortionary steady state: $\mathcal{C}_B = 0$ would imply a deviation from steady state, since any actions taken by the current government would not have any effect on future governments' actions (i.e., future distortions would be fixed). Thus, the current government would minimize distortions today by accumulating debt and implementing $u_c = u_\ell$. ■

A.2 Proof of Proposition 2

From (20) we get $\mathcal{B}_B = \frac{1}{\beta} > 1$. ■

A.3 Proof of Proposition 3

From (39) we have $B^* = \frac{(\sigma-1)}{c^{*\sigma}(\nu_\ell^* - c^* \nu_{\ell\ell}^*)}$. Given $c > 0$ in any equilibrium, $v_\ell > 0$ and $v_{\ell\ell} < 0$, the statement of the proposition follows. ■

A.4 Proof of Proposition 4

Part *(i)*. Equations (16) and (39) imply $-\gamma(1 + (1-\beta)B^*) + \frac{\beta}{c^{*\sigma}} = 0$ and $B^* = \frac{\sigma-1}{\gamma c^{*\sigma}}$. Thus, $B^* = \frac{\sigma-1}{1-\sigma(1-\beta)}$ and $c^* = \left(\frac{1-\sigma(1-\beta)}{\gamma}\right)^{\frac{1}{\sigma}}$. Then, $\frac{dB^*}{d\sigma} = \frac{\beta}{(1-\sigma(1-\beta))^2} > 0$. ■

Part *(ii)*. (16) and (19) simplify to

$$-\gamma(1+B)\mathcal{C}(B) + \beta(1 + \gamma\mathcal{B}(B)\mathcal{C}(\mathcal{B}(B))) = 0 \tag{40}$$

$$\frac{(1-\gamma\mathcal{C}(B))(\mathcal{C}(\mathcal{B}(B)) + \mathcal{B}(B)\mathcal{C}'_B)}{(1+B)\mathcal{C}(B)} - \frac{1-\gamma\mathcal{C}(\mathcal{B}(B))}{1+\mathcal{B}(B)} = 0. \tag{41}$$

We can represent the debt and price functions by polynomials of arbitrarily high degree. From *Proposition 3* we have $B^* = 0$. Since we are focusing on the properties of the equilibrium at B^*, it is sufficient to assume that $\mathcal{B}$ and $\mathcal{C}$ are linear in B, i.e., $\mathcal{B}(B) = x_0 + x_1 B$ and $\mathcal{C}(B) = y_0 + y_1 B$. Note x_1 and y_1 are equal to $\mathcal{B}_B(0)$ and $\mathcal{C}_B(0)$, respectively, regardless of the order of the polynomial of the true solution. We also know $\mathcal{B}(0) = x_0 = 0$ and $\mathcal{C}(0) = y_0 = \beta/\gamma$. Equations (40) and (41) are satisfied for any B. Thus, their derivatives we respect to B are also equal to zero. Evaluating this derivatives at $B = 0$ and using the solutions to x_0 and y_0 we get

$$\begin{aligned} \beta^2 x_1 - \gamma y_1 - \beta &= 0 \\ -\beta(1-\beta)x_1 + \gamma y_1 - \gamma(2-\beta)x_1 y_1 + \beta(1-\beta) &= 0. \end{aligned}$$

This system has two roots. To choose the correct one, we can take the derivative of the GEE with respect to B' and evaluate it at $B = 0$ and at each of the solutions for (x_1, y_1). Next, we check which of the two solutions implies that this derivative is always negative, which implies we are indeed maximizing. For the two sets of solutions, we get the following expressions for the derivative of the GEE with respect to B': $-\frac{(1-\beta)(1+\sqrt{1+4\beta})}{2}$ and $\frac{(1-\beta)(-1+\sqrt{1+4\beta})}{2}$. For $\beta \in (0,1)$, the first expression is always positive, while the second expression is always negative. Therefore, we pick the second root and the solution for (x_1, y_1) is

$$\begin{aligned} x_1 &= \frac{1 + \beta - (1-\beta)\sqrt{1+4\beta}}{2\beta(2-\beta)} \\ y_1 &= \frac{(1-\beta)\gamma(3 + \sqrt{1+4\beta})}{2\beta(2-\beta)}. \end{aligned}$$

What's left to show is that $x_1 \in (0,1)$. It is easy to see that $\lim_{\beta\to 0} x_1 = 0$ and $\lim_{\beta \to 1} x_1 = 1$, so it will be sufficient to show that x_1 is strictly increasing in β. We get

$$\frac{dx_1}{d\beta} = \frac{2 + 2\beta + 2\beta\sqrt{1+4\beta} + 2\beta^3 + \beta^2\sqrt{1+4\beta} - \beta^2 - 2\sqrt{1+4\beta}}{2\beta^2(2-\beta)^2\sqrt{1+4\beta}}.$$

The denominator is always positive, so we need to verify that the numerator is positive as well. There are only two negative terms in the numerator. Clearly $\beta^2\sqrt{1+4\beta} > \beta^2$. To show that the remaining terms are also positive we only need to show that $2+2\beta+2\beta\sqrt{1+4\beta} > 2\sqrt{1+4\beta}$ (there is no need to use the remaining term $2\beta^3$). This is equivalent to showing that

$$1 + \beta + (\beta - 1)\sqrt{1+4\beta} > 0. \tag{42}$$

This expression is equal to zero when $\beta = 0$ and is positive when $\beta = 1$. Moreover, its derivative with respect to β is equal to $\frac{-1+6\beta+\sqrt{1+4\beta}}{\sqrt{1+4\beta}}$, which is strictly positive for $\beta \in (0,1)$, which implies that (42) is satisfied for $\beta \in (0,1)$. This in turn implies that $x_1 \in (0,1)$. ■

A.5 Proof of Proposition 5

At the distortionary steady state we have $B = -\frac{pu_1 + u_{11}}{pu_2}$. Under the assumptions on u we have $B^* = \frac{(\sigma-1)p^{*\sigma}}{\alpha}$, where $p^* > 0$ in any Markov-perfect equilibrium.

A.6 Proof of Proposition 6

Having agents value government expenditure only adds an extra first-order condition to the government's problem: $-u_2(1+\lambda)+\psi=0$ (the problem of the agent remains the same since they take g as given). Using the expression from *Proposition 5* and the assumed utility function, we get that debt at the distortionary steady state solves $B^* = \frac{\sigma-1}{\alpha c_1^\sigma}$. The remaining equations that characterize the steady state are (27), (28) and (29). Assume $\zeta = 0$ for now. Thus, at the distortionary steady state, we have:

$$\begin{aligned} 1+\lambda-\alpha(1+\lambda)c_1^\sigma-\sigma\lambda &= 0 \\ \alpha(1+\lambda)-\gamma(1+2\lambda)n &= 0 \\ -\alpha(1+\lambda)+\psi &= 0 \\ c_1^{1-\sigma}(1-(1-\beta)\sigma)-\alpha(c+g-n)-\gamma n^2 &= 0. \end{aligned}$$

The solution to this system of equations is

$$\begin{aligned} c_1^* &= \Big(\frac{\psi+\sigma(\alpha-\psi)}{\alpha\psi}\Big)^{\frac{1}{\sigma}} \\ n^* &= \frac{\alpha\psi}{\gamma(2\psi-\alpha)} \\ g^* &= \frac{\alpha\psi(\psi-\alpha)}{\gamma(2\psi-\alpha)^2}+\frac{\sigma(\beta\psi-\alpha)(\frac{\psi+\sigma(\alpha-\psi)}{\alpha\psi})^{\frac{1}{\sigma}}}{\psi+\sigma(\alpha-\psi)} \\ \lambda^* &= -1+\frac{\psi}{\alpha}. \end{aligned}$$

The lower bound on $\frac{\alpha}{\psi}$ is necessary for $c_1^* > 0$, while its upper bound is sufficient for $g^* > 0$. We also get $\lambda^* > 0$ and $c_1^{-\sigma} - \alpha > 0$ (thus, we verify $\zeta = 0$), i.e., the steady state is distortionary and monetary policy is above the Friedman rule. The lower bound on γ ensures leisure is less than 1. We can now solve for debt, taxes and the money growth rate:

$$\begin{aligned} B^* &= \frac{\psi(\sigma-1)}{\psi+\sigma(\alpha-\psi)} \\ \tau^* &= \frac{\psi-\alpha}{2\psi-\alpha} \\ \mu^* &= \frac{\sigma(\psi-\alpha)-(1-\beta)}{\psi+\sigma(\alpha-\psi)}. \end{aligned}$$

In steady state the inflation rate is equal to μ^*. Parts *(i)* and *(ii)* follow from taking the appropriate derivatives from the expressions above; *(iii)* follows from $\beta\psi - \alpha = 0$ implying g^* does not depend on σ. ■

B Numerical Computation

The following is a description of the numerical methods used to solve the basic model. The application of these methods to the models with fiscal policy and shocks is straightforward.

(i) Define a grid over B that includes B^* (and may also include $\hat{B}$) and guess equilibrium functions: $\mathcal{B}^0$, $\mathcal{C}^0$, $\mathcal{V}^0$.

(ii) For every B in the grid solve

$$\max_{c,B'} u(c, 1-c-g) + \beta \mathcal{V}^0(B')$$

subject to $\varepsilon(B, B', c, \mathcal{C}^0(B')) = 0$ and $u_c - u_\ell \geq 0$. Call the solution $\mathcal{B}^1(B)$ and $\mathcal{C}^1(B)$ and let $\mathcal{V}^1(B) = u(\mathcal{C}^1(B), 1 - \mathcal{C}^1(B) - g) + \beta \mathcal{V}^0(\mathcal{B}^1(B))$.

(iii) Check convergence of the decision rules and the value function. If the convergence error is not below the desired tolerance, set $\mathcal{B}^0 = \mathcal{B}^1$, $\mathcal{P}^0 = \mathcal{P}^1$, $\mathcal{V}^0 = \mathcal{V}^1$ and go to step (ii).

Note that the algorithm above presumes the use of some interpolation method (such as cubic splines) to evaluate functions at debt levels in between grid points.

If we are interested in solving for the equilibrium only around the distortionary steady state B^*, i.e., where the inequality constraint does not bind, then we can use the GEE and a projection method. This method has the advantage of not involving value function iterations and is highly accurate. First, define a grid over B that includes B^* and guess $\mathcal{B}$, $\mathcal{C}$. If using cubic splines to interpolate between grid points, then the unknowns to solve are the values of $\mathcal{B}$, $\mathcal{C}$ at the grid points. Thus, the number of unknowns is two times the number of grid points. Next, solve a system of equations, which includes the government budget constraint and the GEE, evaluated at all grid points. I.e., solve for $\mathcal{B}(B)$, $\mathcal{C}(B)$ such that $\varepsilon(B, \mathcal{B}(B), \mathcal{C}(B), \mathcal{C}(\mathcal{B}(B))) = 0$ and $\beta \mathcal{C}(B') u'_\ell (\lambda - \lambda') + \lambda \varepsilon_{c'} \mathcal{C}'_B = 0$ for every B in the grid, where from (17) $\lambda = \frac{u_c - u_\ell}{\varepsilon_c}$, $\lambda = \frac{u'_c - u'_\ell}{\varepsilon'_c}$. The use of cubic splines makes the evaluation of $\mathcal{C}_B$ straightforward. The total number of equations is two times the number of grid points. Note that after each update of the guess, the cubic splines need to be updated as well.

The default method used throughout the paper was the value function iteration. The results reported in sections 4 and 5.2 correspond to the projection method, although both methods were used for verification. As an example of the accuracy of the projection method, for the benchmark economy of section 4, the equilibrium was approximated using a grid with 10 points. The solution was then evaluated at 1,000 debt levels within the grid; the sum of squared residuals of the GEE was 5×10^{-10}. The GEE evaluated at B^* was equal to 2×10^{-7}.

Due to the coexistence of differentiable and non-differentiable equilibria, a cautionary note is in order. Along the iteration path, the discrete solution can potentially “leak” and the algorithm may not converge to the smooth solution. To avoid this problem, use a small number of grid points[18]. Once we achieve convergence we can verify the precision of the solution by evaluating the GEE over a finer grid.

C Non-differentiable equilibria

In the economies analyzed in this paper, both differentiable and non-differentiable equilibria coexist. The non-differentiable equilibrium features discontinuous policy functions. This solution solves the

[18] The equilibria shown in Figure 1 were actually calculated using 100 grid points. However, using 10 points results in debt functions which are visually undistinguishable. All other cases were solved with grids of 10 points

problem of the government but does not satisfy the GEE with equality everywhere. Krusell and Smith (2003) and Krusell, Martin and Ríos-Rull (2006) also find co-existence of continuous and discrete solutions in models with lack of commitment.

In this section, we will analyze the non-differentiable equilibrium in the basic model of section 2. The discrete solution looks like a step function (see Figure 5). For certain neighborhoods of debt levels, the government chooses the same level for tomorrow. For particular levels of debt, the government's decision rule is discontinuous, i.e., it decides to increase or decrease debt suddenly by a large amount. At some intervals, the solution is increasing and differentiable; here, the GEE is satisfied with equality.

The non-differentiable equilibrium is an artifact of the infinite horizon since it is not the limit of finite horizon economies[19]. It is also an artifact of the functional representation of the problem of the government, since we can construct equilibria that are not the limit of finite horizon equilibria, even though we are restricting policy to depend only on fundamentals. The differentiable equilibrium exists when the horizon is finite and infinite, whereas the non-differentiable equilibrium only exists when the horizon is infinite. We can use this feature as a selection mechanism to focus on the differentiable equilibrium.

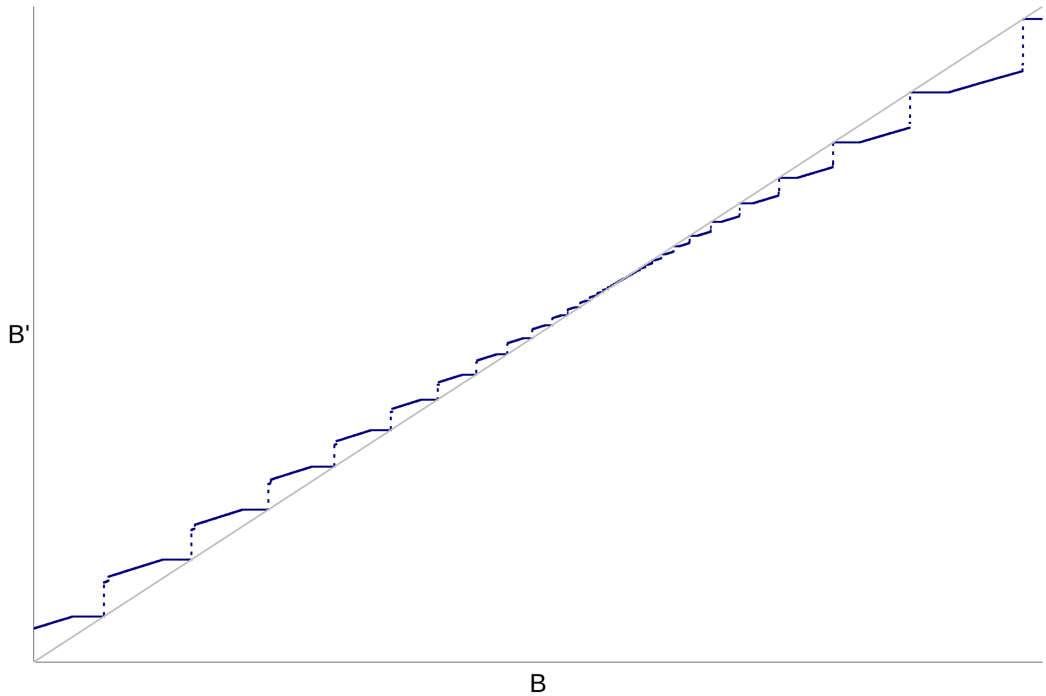
What type of behavior is being captured by the non-differentiable equilibrium? Assume government policy is such that if the government inherits debt above some $\tilde{B} < B^*$, then the increase in debt will be larger than the one prescribed by the differentiable solution $\mathcal{B}(B)$. For a government that starts just a bit below $\tilde{B}$ it is optimal not too increase debt as prescribed by $\mathcal{B}(B)$, since that would imply a too large distortion tomorrow. This government still wants to increase debt, but will do so only up to $\tilde{B}$. Thus, we get a solution where the GEE is satisfied with strict inequality. However, if the government starts with debt sufficiently below $\tilde{B}$ then it is optimal to satisfy the GEE with equality, although the decision will be different than $\mathcal{B}(B)$. In this case, the increase in debt is larger than the one prescribed by $\mathcal{B}(B)$, since the government internalizes that future governments will not increase debt beyond $\tilde{B}$. Thus, for a sufficiently low starting level of debt, the government will decide not to increase debt at all, just like it happens when the starting debt is a bit below $\tilde{B}$. As it turns out, this type of behavior is self-fulfilling and thus we get the non-differentiable equilibrium. Clearly, however this cannot be an equilibrium if the horizon is finite. The behavior of successive governments in the non-differentiable equilibrium mimics a trigger strategy. However, we cannot call it such, since we are not allowing governments to base their decisions on anything other than the aggregate state variable.

The discount factor plays an important role in how the equilibrium looks like. If β is high enough, then the equilibrium has infinitely many—but countable—steady states. This is the case shown in Figure 5. If β is low enough then we still get a step function, but one that does not touch the 45-degree line except at the smooth equilibrium steady state.

How is this solution found? Consider the case $u(c, \ell) = \frac{c^{1-\sigma}-1}{1-\sigma} + \gamma\ell$, which using (16) allows to write current consumption as $\phi(B, B', \mathcal{C}(B')) \equiv \frac{\beta(\mathcal{C}(B')^{1-\sigma}+\gamma B'\mathcal{C}(B'))-\gamma g}{\gamma(1+B)}$. Then, identify the long-run level of debt of the differentiable equilibrium, B^* and create a grid with n points of debt, where $x_i, i = 1, \ldots, n$ refers to grid point i. Let $x_n = B^*$ and choose a lower bound (say $x_1 = 0$). Since x_n is a steady state, we know the values of $\mathcal{V}(x_n)$ and $\mathcal{C}(x_n)$. Next, pick x_{n-1} and let c^*_{n-1} be

[19] To consider finite horizon versions of the model one would have to assume some appropriate terminal conditions, so that the price level does not go to infinity in the last period.

Figure 5: Discrete Debt Function



consumption if x_{n-1} was a steady state. Check which of the following expressions is higher:

$$u\left(\phi(x_{n-1}, x_n, \mathcal{C}(x_n)),\ 1 - \phi(x_{n-1}, x_n, \mathcal{C}(x_n)) - g\right) + \beta \mathcal{V}(x_n)$$

or

$$u\left(\phi(x_{n-1}, x_{n-1}, c^*_{n-1}),\ 1 - \phi(x_{n-1}, x_{n-1}, c^*_{n-1}) - g\right) \frac{1}{1-\beta}.$$

This will tell whether a government that starts at x_{n-1} prefers to increase debt to x_n or stay at x_{n-1} forever. Note that because of monotonicity of the solution, it is not necessary to check whether the government wants to increase debt beyond x_n or below x_{n-1}. Next, we move to x_{n-2} and compare the values of staying at x_{n-2} or increasing debt to x_{n-1} or x_n. Continue this process for all remaining grid points.

After solving to the left of B^*, we proceed to solve to the right of it, using a similar technique. Note that the whole procedure did not involve a single iteration, hence the only numerical errors come from the size of the grid. To verify that the obtained solution $\{\mathcal{V}, \mathcal{B}, \mathcal{C}\}$ is indeed an equilibrium, we can check the one-shot deviation to the solution found, i.e., for every B, solve for B' given $\phi(B, B', \mathcal{C}(B'))$ and $\mathcal{V}(B')$.