Solving the Second Order Systems

• Continuing with the simple RLC circuit

(4) Make the assumption that solutions are of the exponential form:

\[ i(t) = A \exp(st) \]

• where \( A \) and \( s \) are constants of integration.
• Then substituting into the differential equation

\[ L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{1}{C}i = 0 \]

\[ L \frac{d^2A \exp(st)}{dt^2} + R \frac{dA \exp(st)}{dt} + \frac{1}{C}A \exp(st) = 0 \]

\[ Ls^2A \exp(st) + RsA \exp(st) + \frac{A}{C} \exp(st) = 0 \]

• Dividing out the exponential for the characteristic equation

\[ Ls^2 + Rs + \frac{1}{C} = 0 \]

• Also called the Homogeneous equation
• Thus quadratic equation and has generally two solutions.
• There are 3 types of solutions
• Each type produces very different circuit behaviour
• Note that some solutions involve complex numbers.
General solution of the Second Order Systems

- Consider the characteristic equations as a quadratic
- Recall that for a quadratic equation:

\[ ax^2 + bx + c = 0 \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

- The solution has two roots

- Thus for the characteristic equation

\[ Ls^2 + Rs + \frac{1}{C} = 0 \]

- or rewriting this

\[ s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \]

Thus

\[ a = 1 \quad b = \frac{R}{L} \quad c = \frac{1}{LC} \]

- The general solution is:

\[ s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \]
General solution Second Order Systems Cont'd

- Second order equations have two solutions
- Usually define

\[ s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \]

\[ s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \]

- The type of solutions depends on the value these solutions
- The type of solution is set by the **Describeinant**

\[ D = \left[ \left(\frac{R}{2L}\right)^2 - \frac{1}{LC} \right] \]

- Recall \( \frac{R}{L} \) is the time constant of the resistor inductor circuit
- Clearly the description can be either positive, zero, or negative
3 solutions of the Second Order Systems

• What the **Descriminant** represents is about energy flows

\[ D = \left[ \left( \frac{R}{2L} \right)^2 - \frac{1}{LC} \right] \]

• How fast is energy transferred from the L to the C
• How fast is energy lost to the resistor
• There are three cases set by the descriminant

• D > 0 : roots real and unequal
  • In electronics called the overdamped case

• D = 0 : roots real and equal
  • In electronics the critically damped case

• D < 0 : roots complex and unequal
  • In electronics: the underdamped case: very important
Second Order Solutions

• Second order equations are all about the energy flow
• Consider the spring case
• The spring and the mass have energy storage
• The damping pot loses the energy
• The critical factor is how fast is energy lost

• In **Overdamped** the energy is lost very fast
• The block just moves to the rest point
• **Critically Damped** the loss rate is smaller
• Just enough for one movement up and down
• For **Underdamped** spring moves up and down
• Energy is transferred from the mass to the spring and back again
• Loss rate is smaller than the time for transfer
Complex Numbers

• Imaginary numbers necessary for second order solutions
• Imaginary number $j$

$$j = \sqrt{-1}$$

• Note: in math imaginary number is called $i$
• But $i$ means current in electronics so we use $j$
• Complex numbers involve Real and Imaginary parts

$$\vec{W} = R_w + jI_w$$

• May designate this in a vector coordinate form:

$$\vec{W} = (R_w, I_w)$$

• Example:

$$\vec{W} = 1 + j2 = (1,2)$$

$\text{Re}(\vec{W}) = 1$ $\text{Im}(\vec{W}) = 2$
Complex Numbers Plotted

- In electronics plot on X-Y axis
- X axis real, Y axis is imaginary
- A vector represents the imaginary number has length
- Vector has a magnitude $M$
- Vector is at some angle $\theta$ to (theta) the real axis
- Then the real and imaginary parts are

  $$\text{Real}(W) = R_w = M \cos(\theta)$$

  $$\text{Imaginary}(W) = I_w = M \sin(\theta)$$

  $$\vec{W} = M \left[ \cos(\theta) + j \sin(\theta) \right]$$

- The magnitude

  $$\text{Mag}(\vec{W}) = |\vec{W}| = \sqrt{R_w^2 + I_w^2}$$

- The angle

  $$\theta = \arctan \left( \frac{I_w}{R_w} \right)$$

Thus can give the vector in polar coordinates

$$\vec{W} = (M, \theta) = M \angle \theta$$

Fig. 25-8  Magnitude and angle of a complex number. (a) Rectangular form. (b) Polar form.
Complex Numbers and Exponentials

- Polar coordinates are connected to complex numbers in exp
- Consider an exponential of a complex number

\[ \tilde{W} = (M, \theta) = R_w + jI_w \]

- This is given by the Euler relationship

\[ \exp(j\theta) = [\cos(\theta) + j \sin(\theta)] \]

\[ \tilde{W} = M \exp(j\theta) = M[\cos(\theta) + j \sin(\theta)] \]

- This relationship is very important for electronics
- Used in second order circuits all the time.
Overdamped RLC

• This is a very common case
• In RLC series circuits this is the large resistor
• Energy loss in the resistor much greater than energy transfers
• Two real roots to the characteristic equation

\[ s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \]

• The solution is a double exponential decay

\[ i(t) = A_1 \exp(s_1 t) + A_2 \exp(s_2 t) \]

• As R is increased damping increases
• Get the overdamped case
Overdamped RLC Energy Flows

- For the example case $L = 5 \text{ mH}$, $C = 2 \text{ µF}$, $R = 200 \Omega$
- Also assume $C$ is charged to $10 \text{ V}$ at $t = 0$
- Initially energy starts in the Capacitor $C$
- Some energy transfers to the Inductor $L$
- $C$ loses charge much faster than $L$ gains current
- So energy starts to rise in $L$ but only to a limited level
- Then energy is removed from both by the resistor
Overdamped RLC Initial Conditions

• For the overdamped case the s’s are real & different

\[ i(t) = A_1 \exp(s_1 t) + A_2 \exp(s_2 t) \]

• To solve the constants A need the initial conditions
• For second order need two conditions
• Thus both initial current & its derivative
• This varies from circuit to circuit
• In the case of a charged C switched into the circuit
• Since L acts as an open initially then \( i(0) = 0 \), thus

\[ i(0) = A_1 + A_2 \]

• Thus

\[ A_2 = -A_1 \]

• Again since the inductor acts open at time zero & \( i(0)=0 \)
• Thus voltage drop across the resistor is zero

\[ V_C(0) = V_L(0) = L \frac{di(0)}{dt} \]

\[ \frac{di(0)}{dt} = \frac{V_c(0)}{L} \]
Overdamped RLC Full Solution

• Now using substituting $A$ equation into the exp equation
  \[ i(t) = A_1 \exp(s_1 t) - A_1 \exp(s_2 t) \]

• To solve the constants $A$ with the derivative initial condition
  \[ \frac{di(t)}{dt} = A_1 [s_1 \exp(s_1 t) - s_2 \exp(s_2 t)] \]

• Now applying the initial condition derivative
  \[ \frac{di(t = 0)}{dt} = A_1 [s_1 \exp(0) - s_2 \exp(0)] = A_1 [s_1 - s_2] = \frac{V_c(0)}{L} \]

• Now solving the equations

  \[ A_1 = \frac{V_c(0)}{[s_1 - s_2]L} \]

  \[ i(t) = \frac{V_c(0)}{[s_1 - s_2]L} [\exp(s_1 t) - \exp(s_2 t)] \]
Overdamped RLC Circuit Example

- For the example case \( L = 5 \text{ mH}, \ C = 2 \mu \text{F}, \ R = 200 \Omega \)
- Solving for the roots first what are the discriminate values

\[
\left( \frac{R}{2L} \right) = \frac{200}{2 \times 0.005} = 2 \times 10^4 \text{ sec}^{-1} = \frac{1}{\tau} \quad \tau = 50 \mu \text{sec}
\]

\[
\frac{1}{LC} = \frac{1}{0.005 \times (2 \times 10^{-6})} = 10^8 \text{ sec}^{-2}
\]

\[
D = \left[ \left( \frac{R}{2L} \right)^2 - \frac{1}{LC} \right] = \left[ (2 \times 10^4)^2 - 10^8 \right] = 3 \times 10^8 \text{ sec}^{-2}
\]

- Thus gives

\[
s_1 = -\frac{R}{2L} + \sqrt{\left( \frac{R}{2L} \right)^2 - \frac{1}{LC}} = -\frac{200}{2 \times 0.005} + \sqrt{3 \times 10^8} = -2.68 \times 10^3 \text{ sec}^{-1}
\]

\[
s_2 = -\frac{R}{2L} - \sqrt{\left( \frac{R}{2L} \right)^2 - \frac{1}{LC}} = -\frac{200}{2 \times 0.005} - \sqrt{3 \times 10^8} = -3.73 \times 10^4 \text{ sec}^{-1}
\]

- Thus

\[
A_i = \frac{V_c(0)}{[s_1 - s_2]L} = \frac{10}{\left[ 1.87 \times 10^4 + 5.87 \times 10^4 \right] \times 0.005} = 5.77 \times 10^{-2} \text{ A}
\]

\[
i(t) = 5.77 \times 10^{-2} \left[ \exp(-2.68 \times 10^3) - \exp(-3.73 \times 10^4) \right] \text{ A}
\]
Critical Damped RLC

- If we decrease the damping (resistance) energy loss decreases
- Change the exponential decay until discriminant=0

\[
D = \left[ \left( \frac{R}{2L} \right)^2 - \frac{1}{LC} \right] = 0
\]

\[
\left( \frac{R}{2L} \right)^2 = \frac{1}{LC}
\]

- This is the point called critical damping
- Difficult to achieve: only small change in R moves from this point
- Small temperature change will cause that to occur
- Energy transfer from C to L is now smaller than loss in R
Critical Damped RLC Solutions

- The characteristic equation has two identical solutions

\[ s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \]

\[ s_1 = s_2 = -\frac{R}{2L} \]

- This is a special case, with special solution

\[ i(t) = [A_1 + A_2t] \exp\left( -\frac{Rt}{2L} \right) \]

Why this solution?
Given in the study differential equations in math
Critical Damped RLC Two Solutions

- The critically damped equation has two solutions
  \[ i(t) = [A_1 + A_2 t] \exp\left( -\frac{Rt}{2L} \right) \]

- Solution has two possible behaviours depending on A values
- The characteristic equation has two identical solutions
- Get only one oscillation (transaction)
- Or get slow approach to final value
- Difference depends on initial conditions only
- If starts with \( i(t=0) = 0 \), slow approach to rest value
- If start with \( i(t=0) \neq 0 \), get one oscillation then rest value
- Thus same circuit will have different solutions
- Depending on the initial current conditions & energy storage

![RLC Circuit Diagram]
Critical Damped RLC Example $i(0) = 0$

- Keeping $L$ & $C$ the same
- If $R$ is increased 100 ohm get critical damping
- Here $L = 5 \text{ mH}$, $C = 2 \mu\text{F}$, $R = 100 \Omega$
- Also assume $C$ is charged to 10 V at $t = 0$
- Also assume $C$ is charged to 10 V at $t = 0$ but $i(t) = 0$
- This is the no oscillation case
Critical Damped RLC, i(t=0) = 0 Energy Plot

- Energy is transferred from Capacitor to Inductor
- Inductor energy is later and higher than overdamped case
- Then both energies decline
Critical Damped RLC \(i(0)=0\) Solution

- We define the **Damping Decay Constant** \(\alpha\)

\[
\alpha = \frac{R}{2L} = \frac{100}{2 \times 0.005} = 10^4 \text{ sec}^{-1}
\]

- The damping constant gives how fast energy is decaying
- The basic Critical Damping equation is

\[
i(t) = \left[ A_1 + A_2 t \right] \exp\left( -\frac{Rt}{2L} \right)
\]

- Solving for \(A\) constants from the initial conditions
- Since current is at \(t=0\) is zero then

\[
i(t = 0) = 0 = \left[ A_1 + A_2 0 \right] \exp\left( -\frac{R0}{2L} \right) = A_1
\]

\[
i(t) = A_2 t \exp\left( -\frac{Rt}{2L} \right)
\]

- Again since the inductor acts open at time zero

\[
\frac{di(0)}{dt} = \frac{V_c(0)}{L}
\]

- Now applying this to the equation

\[
\frac{di(t = 0)}{dt} = A_2 \exp\left( -\frac{Rt}{2L} \right)\left[ 1 - \frac{Rt}{2L} \right] = A_2 \exp\left( -\frac{R0}{2L} \right)\left[ 1 - \frac{R0}{2L} \right] = A_2
\]

- Thus at time \(t=0\) then

\[
A_2 = \frac{V_c(0)}{L} = \frac{10}{0.005} = 2000 \ A
\]

\[
i(t) = \left[ A_2 t \right] \exp\left( -\frac{Rt}{2L} \right) = 2000 \ t \exp\left( -10^4 t \right) A
\]
Critical Damped RLC, \(i(t=0) = 0\) Voltage Plot

- Capacitor voltage starts at 10 V and declines
- Inductor voltage starts at 10 V then reverses & declines
- Resistor voltage starts at zero, rises to peak above \(V_c\) then declines
Critical Damped RLC, \( i(t=0) = 0 \) Current Plot

- Current rises to peak due to \( A_2 t \) term
- Then current declines near exponential with \( \alpha \)

\[
i(t) = \left[ A_2 t \right] \exp\left( -\frac{Rt}{2L} \right) = 2000 \, t \exp\left( -10^4 \, t \right) \, A
\]
Critical Damped RLC, $i(t=0) \neq 0$

- Now consider the case when $i(t=0)$ is non zero
- The practical case is when capacitor is uncharged
- But inductor has current flowing in it
- The equation has both constants nonzero

\[ i(t) = [A_1 + A_2 t] \exp\left(-\frac{Rt}{2L}\right) \]

- Again the example have $L = 5$ mH, $C = 2 \mu F$, $R = 100$ Ω
- Now assume L initially carries 100 mA at $t = 0$, thus

\[ i(t = 0) = I_0 = A_1 = 100 \text{ mA} \]

- Since C acts as a short at time $t=0$ thus $V_C(t=0) = 0$
- Then the only voltage drop is across the resistance

\[ I_0 R + L \frac{di(t = 0)}{dt} = 0 \]
\[ \frac{di(t = 0)}{dt} = -\frac{I_0 R}{L} \]
Critical Damped RLC, i(t=0) ≠ 0 Equation

- Now for the derivative of the equation

\[
\frac{di(t)}{dt} = \left\{A_2 + \left[A_1 + A_2t \left(- \frac{R}{2L} \right) \right]\right\} \exp\left(- \frac{Rt}{2L} \right)
\]

- For the initial conditions

\[
\frac{di(t=0)}{dt} = \left\{A_2 + \left[A_1 \left(- \frac{R}{2L} \right) \right]\right\} \exp\left(- \frac{R0}{2L} \right) = A_2 - A_1 \left(\frac{R}{2L} \right) = A_2 - \frac{RI_0}{2L}
\]

- Relating this to the resistance

\[
\frac{di(t=0)}{dt} = A_2 - \frac{RI_0}{2L} = - \frac{I_0R}{L}
\]

\[
A_2 = \frac{RI_0}{2L} = - \frac{100 \times 0.1}{2 \times 0.005} = -1000 \ A
\]

- Thus the critically damped i(t=0)≠0 current equation is

\[
i(t) = \left[A_1 + A_2t \right] \exp\left(- \frac{Rt}{2L} \right) = \left[0.1 - 1000 \ t \right] \exp\left(- 10^4 \ t \right) \ A
\]

- Thus the current will reverse direction at t=0.1 msec.
Critical Damped RLC, \( i(t=0) \neq 0 \) Current Plot

- Current reaches zero at \( t=0.1 \) msec
- Reverses, reaches peak at \( t=0.2 \) msec then declines

\[
i(t) = [A_1 + A_2 t] \exp\left(-\frac{Rt}{2L}\right) = \left[0.1 - 1000 t\right] \exp\left(-10^4 t\right) A
\]
Critical Damped RLC, \( i(t=0) \neq 0 \) Energy Plot

- Inductor energy falls to minimum at \( t=0.1 \) msec
- Energy is transferred from L to C and back to L
Critical Damped RLC, i(t=0) ≠ 0 Energy Plot Expanded

- Capacitor energy reaches max when $E_L$ is minimum
- Then Inductor energy rises again
- Both energies decay
Critical Damped RLC, $i(t=0) \neq 0$ Voltage Plot

- Inductor voltage starts at 10 V and declines
- Resistor voltage starts at -10 V and rises to zero
Critical Damped RLC, $i(t=0) \neq 0$ Voltage Plot Expanded

- Resistor voltage reaches zero at $t=0.1$ msec
- Capacitor voltage reaches max also at $t=0.1$ msec
- Inductor voltage falls to zero at $t=0.2$ msec then goes negative
- Reason $V_L$ depends on direction of Current derivative
- Resistor voltage changes to positive, reaches peak then declines