# Matrix Methods in Optics

- For more complicated systems use Matrix methods & CAD tools
- Both are based on Ray Tracing concepts
- Solve the optical system by tracing may optical rays
- In free space a ray has position and angle of direction
  $y_1$ is radial distance from optical axis
  $V_1$ is the angle (in radians) of the ray
- Now assume you want to a Translation:
  find the position at a distance t further on
- Then the basic Ray equations are in free space
  making the parallex assumption

$$y_2 = y_1 + V_1 t$$

$$V_2 = V_1$$

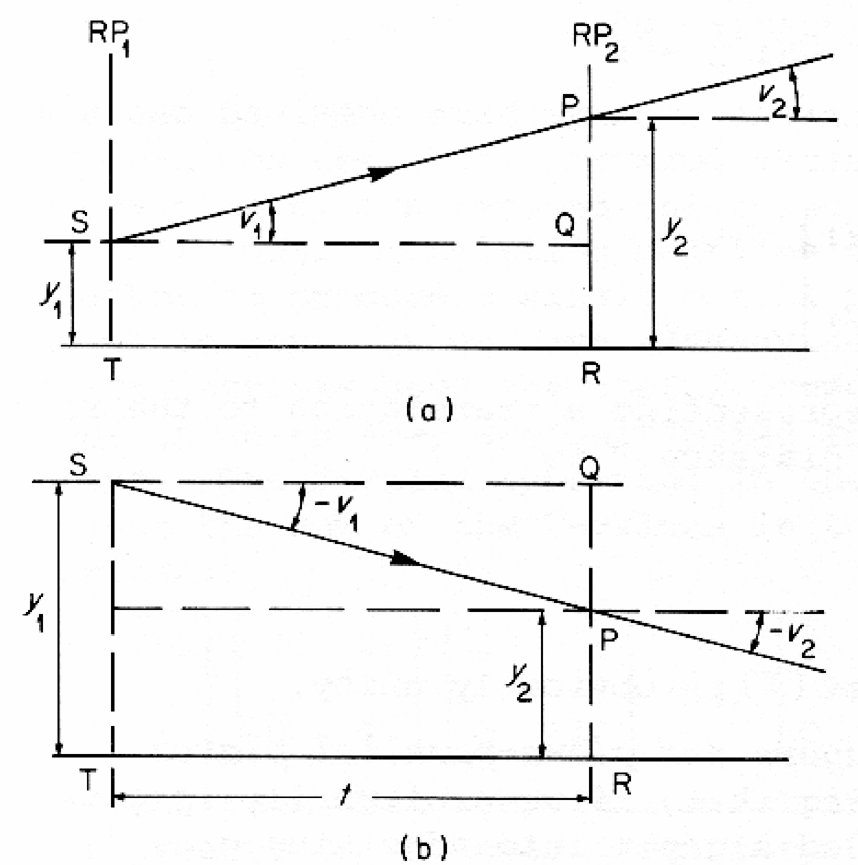



Figure II.2

# Matrix Method: Translation Matrix

- Can define a matrix method to obtain the result for any optical process
- Consider a simple translation distance t
- Then the Translation Matrix (or *T* matrix)

$$\begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

- The reverse direction uses the inverse matrix

$$\begin{bmatrix} y_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

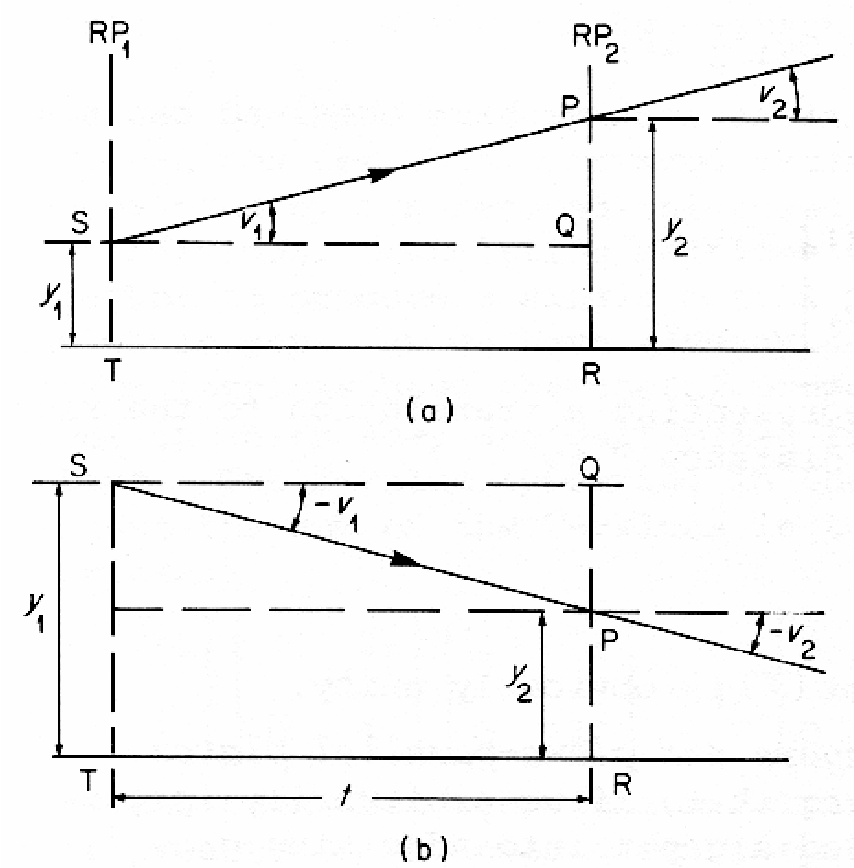
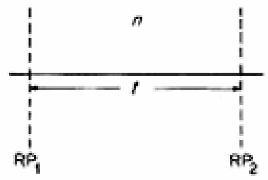
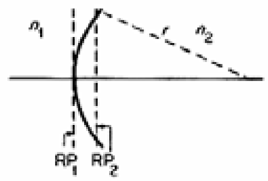
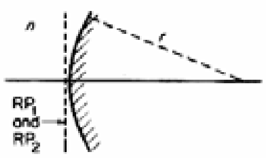
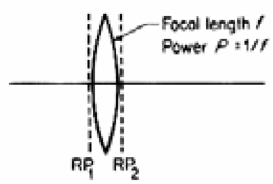



Figure II.2

# General Matrix for Optical Devices

- Optical surfaces however will change angle or location
- Example a lens will keep same location but different angle

Table 1

| Number | Description | Optical Diagram | Ray-transfer matrix |
|---|---|---|---|
| 1 | Translation<br>( $\mathcal{T}$-matrix) |  | $\begin{bmatrix} 1 & t/n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$ |
| 2 | Refraction at single surface<br>( $\mathcal{R}$-matrix) |  | $\begin{bmatrix} 1 & 0 \\ \dfrac{-(n_2 - n_1)}{r} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix}$ |
| 3 | Reflection at single surface (for convention see section II.11) |  | $\begin{bmatrix} 1 & 0 \\ \dfrac{2n}{r} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix}$ |
| 4 | Thin lens in air (focal length $f$, power $P$) |  | $\begin{bmatrix} 1 & 0 \\ -(n-1)\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$ |

# General Optical Matrix Operations

- Place Matrix on the left for operation on the right
- Can solve or calculate a single matrix for the system

$$\begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = [M_{image}][M_{lens}][M_{object}]\begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

$$\begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & s' \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

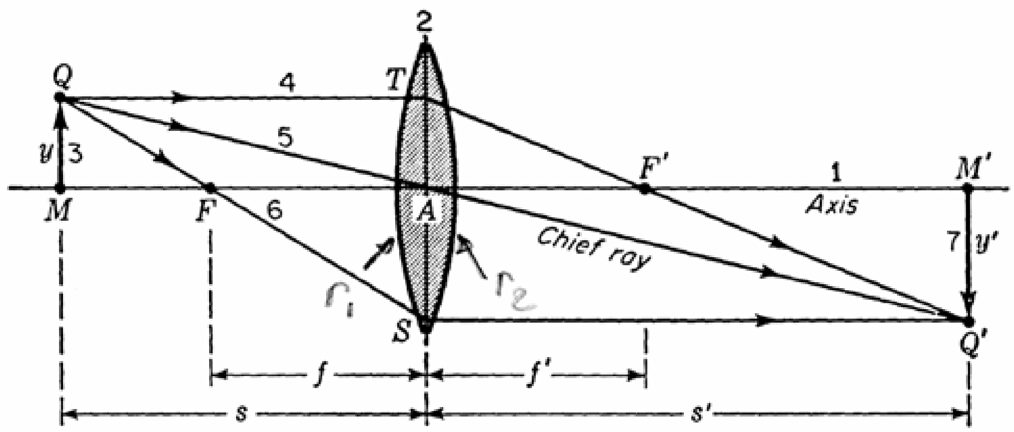



FIGURE 4D
The parallel-ray method for graphically locating the image formed by a thin lens.

# Solving for image with Optical Matrix Operations

- For any lens system can create an equivalent matrix
- Combine the lens (mirror) and spacing between them
- Create a single matrix

$$[M_n]\cdots[M_2][M_1] = [M_{system}] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

- Now add the object and image distance translation matrices

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = [M_{image}][M_{lens}][M_{object}]$$

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = \begin{bmatrix} 1 & s' \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = \begin{bmatrix} A + s'C & As + B + s'(Cs + D) \\ C & Cs + D \end{bmatrix}$$

- Image distance s' is found by solving for $B_s$=0
- Image magnification is

$$m_s = \frac{1}{D_s}$$

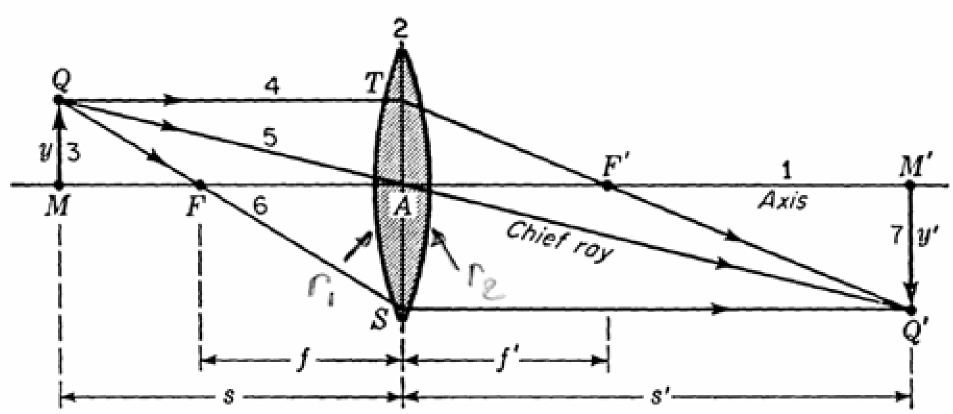



FIGURE 4D
The parallel-ray method for graphically locating the image formed by a thin lens.

# Example Solving for the Optical Matrix

- Two lens system: solve for image position and size
- Biconvex lens $f_1$=8 cm located 24 cm from 3 cm tall object
- Second lens biconcave $f_2$= -12 cm located 6 cm from $f_1$
- Then the matrix solution is

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{12} & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{8} & 1 \end{bmatrix} \begin{bmatrix} 1 & 24 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ \frac{1}{12} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 24 \\ -\frac{1}{8} & -2 \end{bmatrix}$$

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = \begin{bmatrix} 25 - 0.1042X & 12 - X \\ -0.1042 & -1 \end{bmatrix}$$

- Solving for the image position:

$$B_s = 12 - X = 0 \quad or \quad X = 12\ cm$$

- Then the magnification is

$$m = \frac{1}{D_s} = -1$$

- Thus the object is at 12 cm from 2$^{nd}$ lens, -3 cm high
- Easiest to do this with spread sheet, matlab or maple in Excel use the matrix multiply mmult function

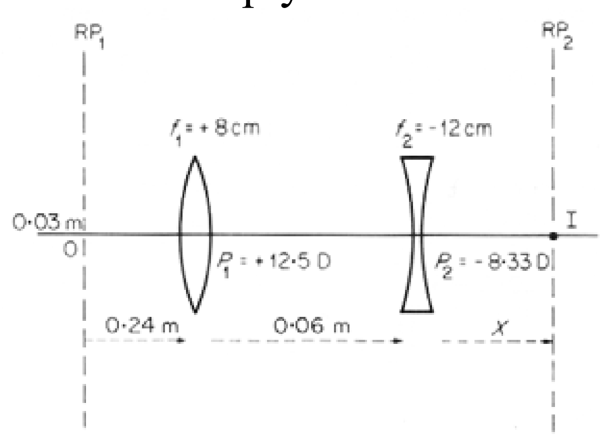



Figure II.14

# Optical Matrix Equivalent Lens

- For any lens system can create an equivalent matrix & lens
- Combine all the matrices for the lens and spaces
- The for the combined matrix

where $RP_1$ = first lens left vertex

$RP_2$ = last lens right most vertex

$n_1$=index of refraction before 1st lens

$n_2$=index of refraction after last lens

| System parameter described | Measured From | Measured To | Function of matrix elements | Special case $n_1 = n_2 = 1$ |
|---|---|---|---|---|
| First focal point | $RP_1$ | $F_1$ | $n_1D/C$ | $D/C$ |
| First focal length | $F_1$ | $H_1$ | $-n_1/C$ | $-1/C$ |
| First principal point | $RP_1$ | $H_1$ | $n_1(D-1)/C$ | $(D-1)/C$ |
| First nodal point | $RP_1$ | $L_1$ | $(Dn_1-n_2)/C$ | $(D-1)/C$ |
| Second focal point | $RP_2$ | $F_2$ | $-n_2A/C$ | $-A/C$ |
| Second focal length | $H_2$ | $F_2$ | $-n_2/C$ | $-1/C$ |
| Second principal point | $RP_2$ | $H_2$ | $n_2(1-A)/C$ | $(1-A)/C$ |
| Second nodal point | $RP_2$ | $L_2$ | $(n_1-An_2)/C$ | $(1-A)/C$ |

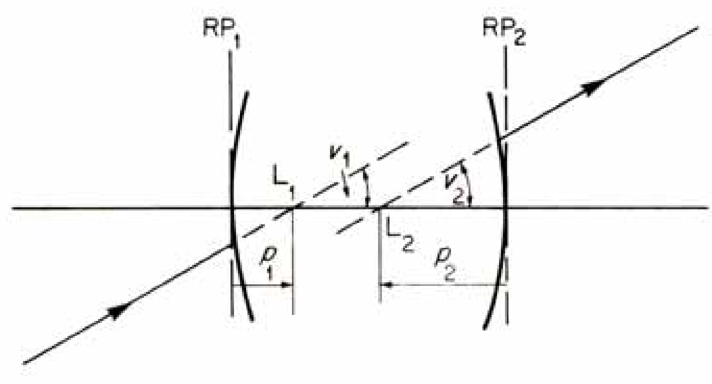



Figure II.17c

# Example Combined Optical Matrix

- Using Two lens system from before
- Biconvex lens $f_1$=8 cm
- Second lens biconcave $f_2$= -12 cm located 6 cm from $f_1$
- Then the system matrix is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{12} & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{8} & 1 \end{bmatrix} = \begin{bmatrix} 0.25 & 6 \\ -0.1042 & 1.5 \end{bmatrix}$$

- Second focal length (relative to $H_2$) is

$$f_{s2} = -\frac{1}{C} = -\frac{1}{-0.1042} = 9.766\ cm$$

- Second focal point, relative to $RP_2$ (second vertex)

$$f_{rP2} = -\frac{A}{C} = -\frac{0.25}{-0.1042} = 2.400\ cm$$

- Second principal point, relative to $RP_2$ (second vertex)

$$H_{s2} = \frac{1-A}{C} = \frac{1-0.25}{-0.1024} = -7.198\ cm$$

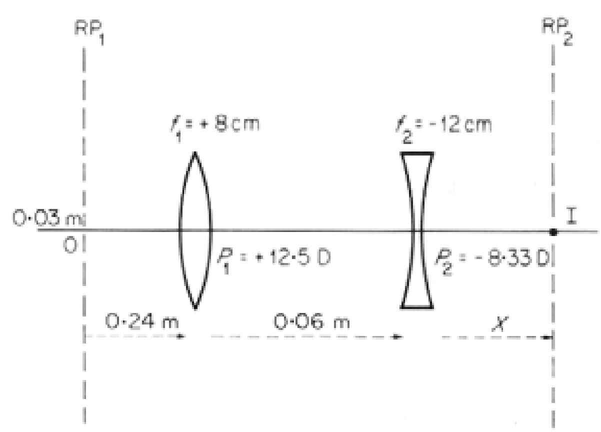



Figure II.14

# Solid State Lasers

- Was first type of laser (Ruby 1960)
- Uses a solid matrix or crystal carrier
- eg Glass or Sapphire
- Doped with transition metal or rear earth ions
- eg Chromium (Cr) or Neodynmium (Nd)
- Mirrors at cavity ends
- Typically pumped with light
- Most common a Flash lamp
- Light adsorbed by doped ion, emitted as laser light
- Mostly operates in pulsed mode (newer CW)

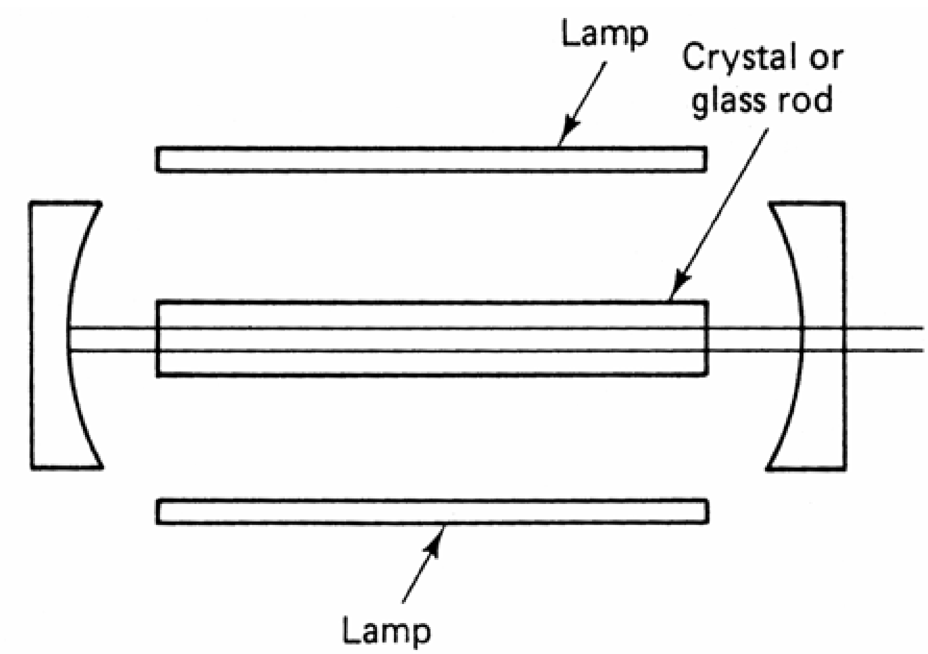



**Figure 8–4** Schematic of solid laser—for example, ruby

# Flash Lamp Pumping

- Use low pressure flash tubes (like electronic flash)
- Xenon or Krypton gas at a few torr (mm of mercury pressure)
- Electrodes at each end of tube
- Charge a capacitor bank:
  50 - 2000 μF, 1-4 kV
- High Voltage pulse applied to tube
- Ionizes part of gas
- Makes tube conductive
- Capacitor discharges through tube
- Few millisec. pulse
- Inductor slows down discharge

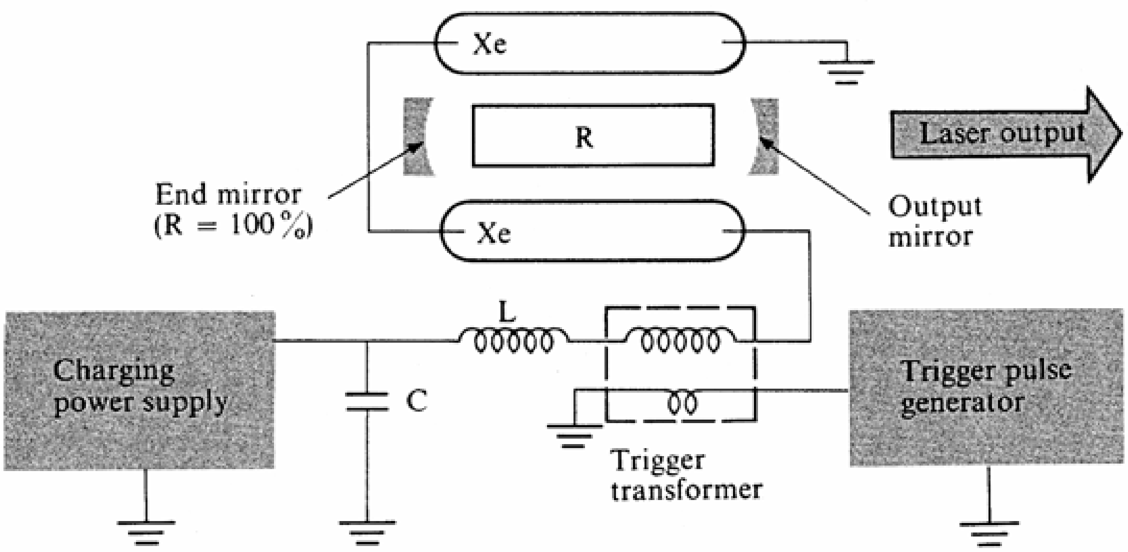



**Fig. 6.16** Schematic of a simple flashlamp-pumped laser. The trigger pulse generator and transformer provide a high-voltage pulse sufficient to cause the xenon gas in the lamps (Xe) to discharge. The ionized gas provides a low-resistance discharge path to the storage capacitor (C). The inductor (L) shapes the current pulse, maintaining the discharge. The discharge of the lamps optically pumps the laser rod (R).

# Light Source Geometry

- Earlier spiral lamp: inefficient but easy
- Now use reflectors to even out light distribution
- For CW operation use steady light sources
  Tungsten Halogen or Mercury Vapour
- Use air or water cooling on flash lamps

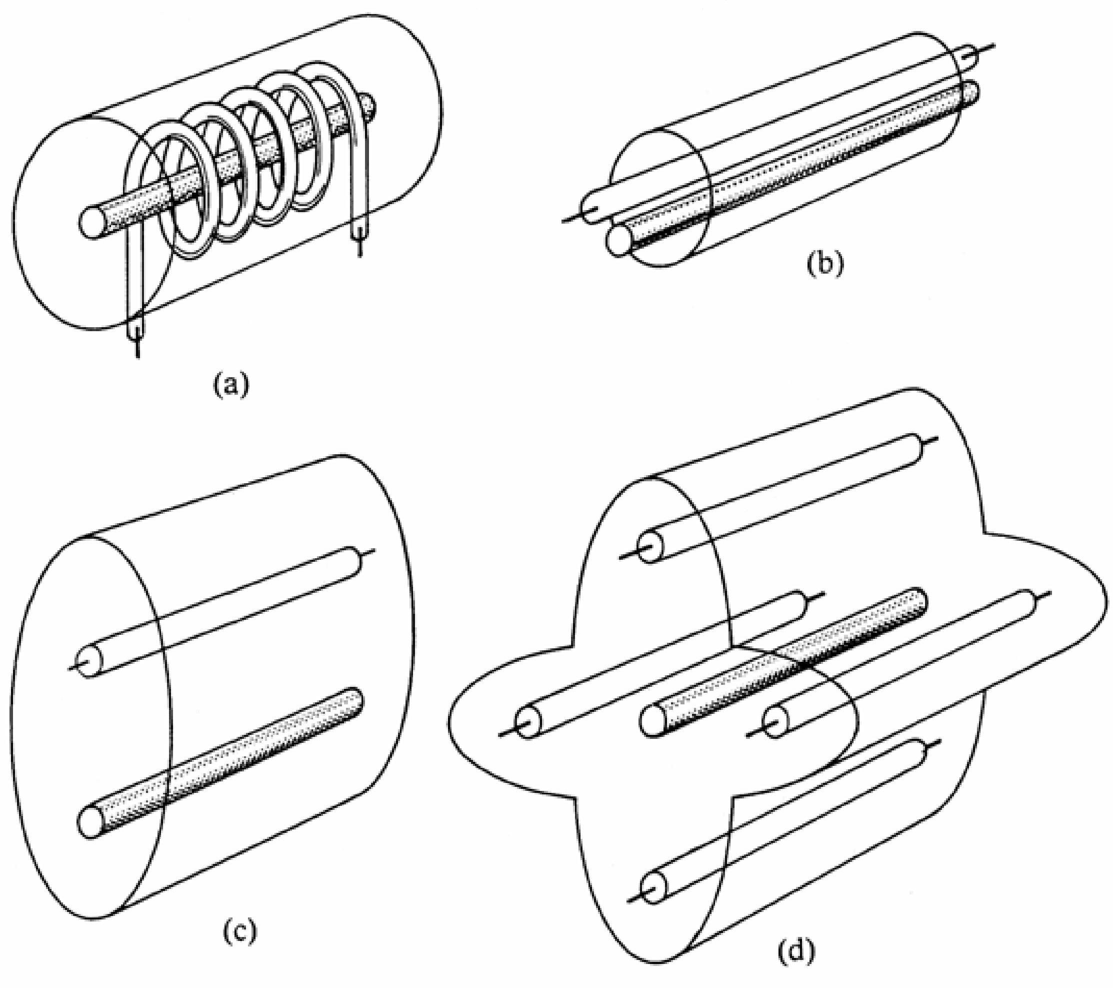


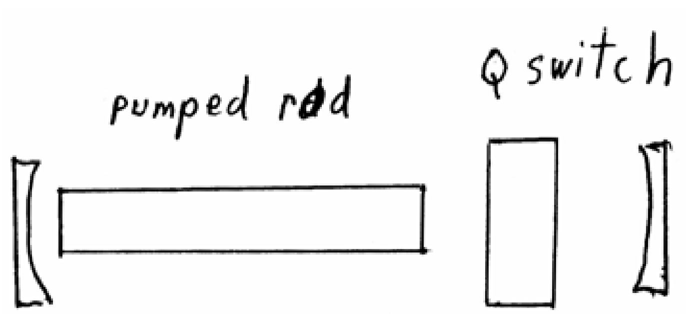

**Fig. 2.4** Some of the more common flashtube geometries used for optical pumping: (a) a helical flashtube round the laser rod; (b) close coupling between flashtube and rod; (c) flashtube and rod along the two foci of an elliptical cavity and (d) a multi-elliptical cavity.

# Q Switch Pulsing

- Block a cavity with controllable absorber
- Like an optical switch
- During initial flash pulse switch off
- Recall the Quality Factor of resonce circuit (eg RLC)

$$Q = \frac{2\pi \textit{ energy stored}}{\textit{energy lost per light pass}}$$

- During initial pulse Q low
- Allows population inversion to increase without lasing
- no stimulated emission
- Then turn switch on
- Now sudden high stimulated emission
- Dump all energy into sudden pulse
- Get very high power level, but less energy



## Q Switch Process During Laser Pulse

- Flash lamp rises to max then declines (~triangle pulse)
- Q switch makes cavity Q switch on after max pumping
- Low Q, so little spontaneous light
- Population inversion rises to saturation
- The Q switch creates cavity: population suddenly declines due to stimulated emission
- Laser pulse during high Q & above threshold conditions

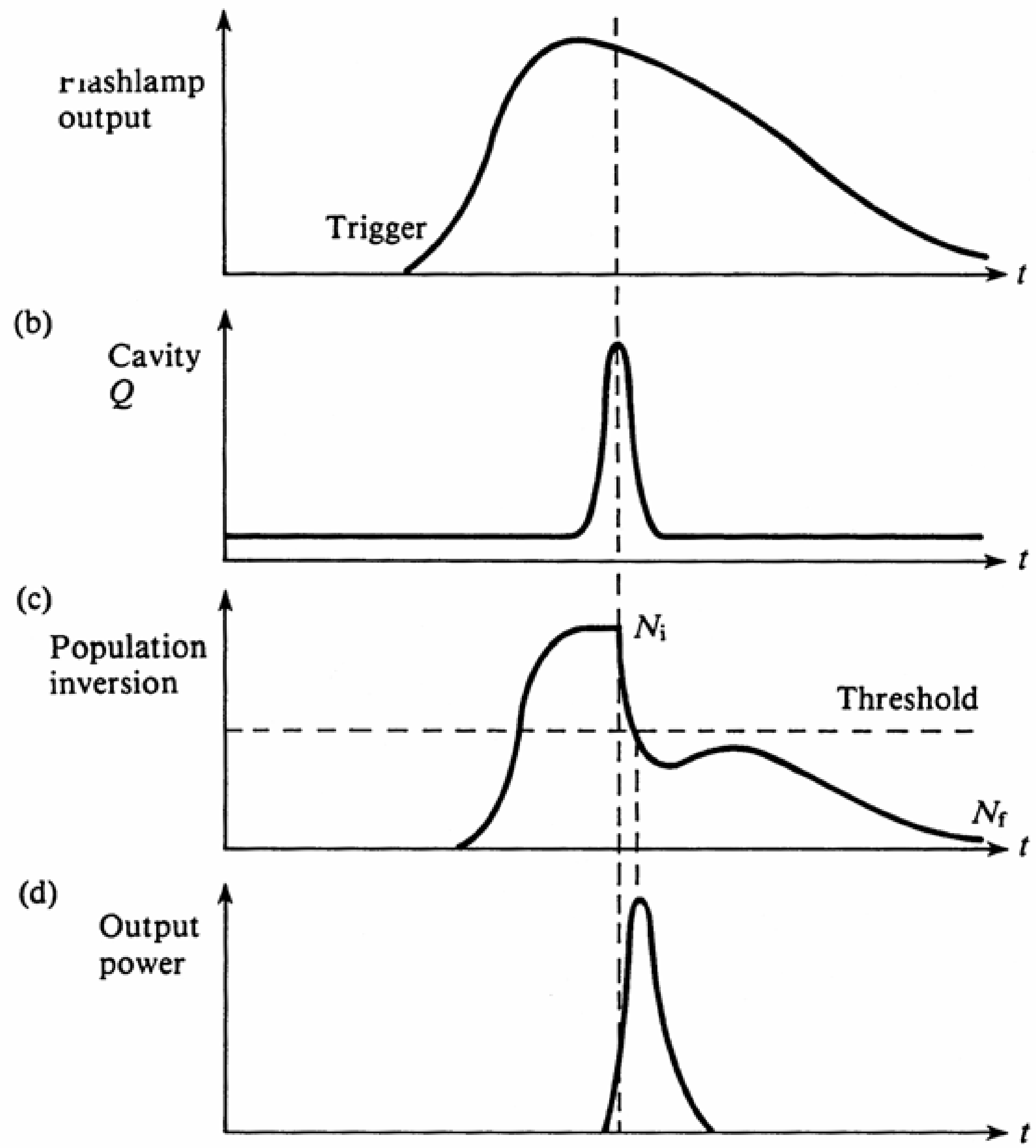



**Fig. 3.11** Schematic representation of the variation of the parameters: (a) flashlamp output; (b) cavity $Q$; (c) population inversion; (d) output power as a function of time during the formation of a Q-switched laser pulse.

## Energy Loss due to Mirrors & Q

- Q switching can be related to the cavity losses
- Consider two mirrors with reflectance $R_1$ and $R_2$
- Then the rate at which energy is lost is

$$\frac{E}{\tau_c} = \frac{(1 - R_1 R_2)E}{\tau_r}$$

where $\tau_c$ = photon lifetime

$\tau_r$ = round trip time = 2L/c

E = energy stored in the cavity

- Average number of photon round trips is the lifetime ratio

$$\frac{\tau_c}{\tau_r} = \frac{1}{(1 - R_1 R_2)}$$

# Q Equations for Optical Cavity

- Rewrite energy equation in terms of photon lifetime $\tau_c$
- First note the energy lost in the time of one light cycle $t_f = 1/f$

$$E_{lost/cycle} = \frac{Et_f}{\tau_c} = \frac{E}{f\tau_c}$$

where f = frequency

- Thus the cavity's Q is

$$Q = \frac{2\pi E}{E_{lost/cycle}} = \frac{2\pi E}{\left(\frac{E}{f\tau_c}\right)} = 2\pi f\tau_c$$

- Thus for a laser cavity:

$$Q = 2\pi\ f\tau_c = \frac{2\pi\ f\tau_r}{\left(1 - R_1R_2\right)} = \frac{4\pi\ fL}{c\left(1 - R_1R_2\right)} = \frac{4\pi\ L}{\lambda\left(1 - R_1R_2\right)}$$

- Q switch: go form high reflectivity to low reflectivity on one mirror
- Also Q is related to the bandwidth of the laser

(from resonance cavity circuits).

$$Q = \frac{f}{\Delta f}$$

- Thus lifetime relates to the bandwidth

$$\Delta f = \frac{1}{2\pi\tau_c}$$

# Transition Metal Impurity Ion Energy levels

- Chromium $Cr_{3+}$ ion
- Atom has energy levels (shells)
  (orbit)(shell)$_{(\text{no. electrons})}$
  $1s_2\ 2s_2\ 2p_6\ 3s_2\ 3p_6$
- In ions unfilled orbital electrons interact
- inter-electron coulomb interaction split the energies
  (capital letter the L quantum)$_{(\text{spin quantum})}$
- Ion then interacts with crystal field
  splits energy levels more

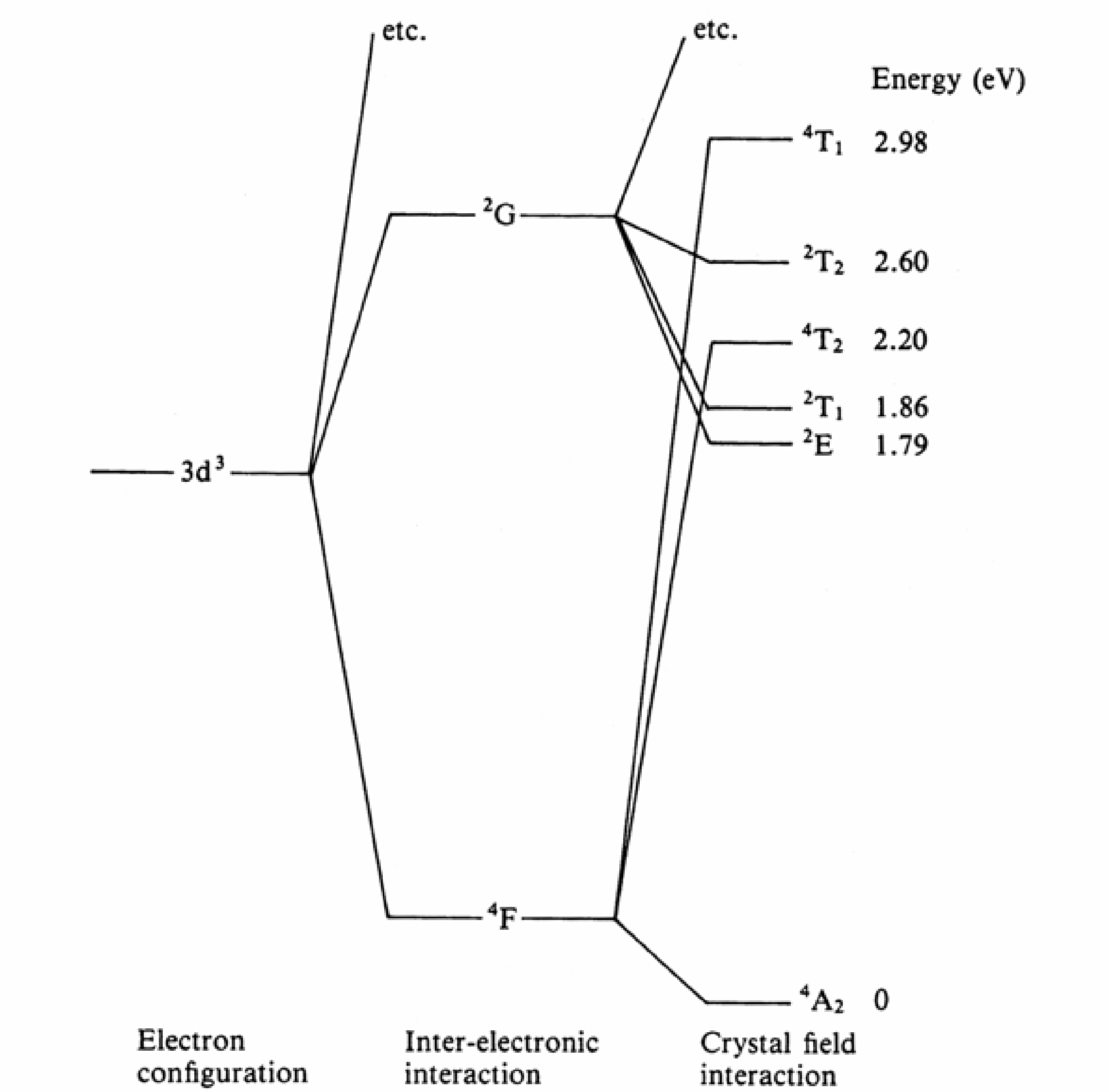



**Fig. 2.1** The origins of the low-lying electronic energy levels in $Cr^{3+}$. Successive interactions split the original $3d^3$ electron configuration into an increasing number of energy levels.

# Rare Earth Impurity Ion Energy levels

- Spin of electrons interacts with orbit
- Splits the inter-electronic levels

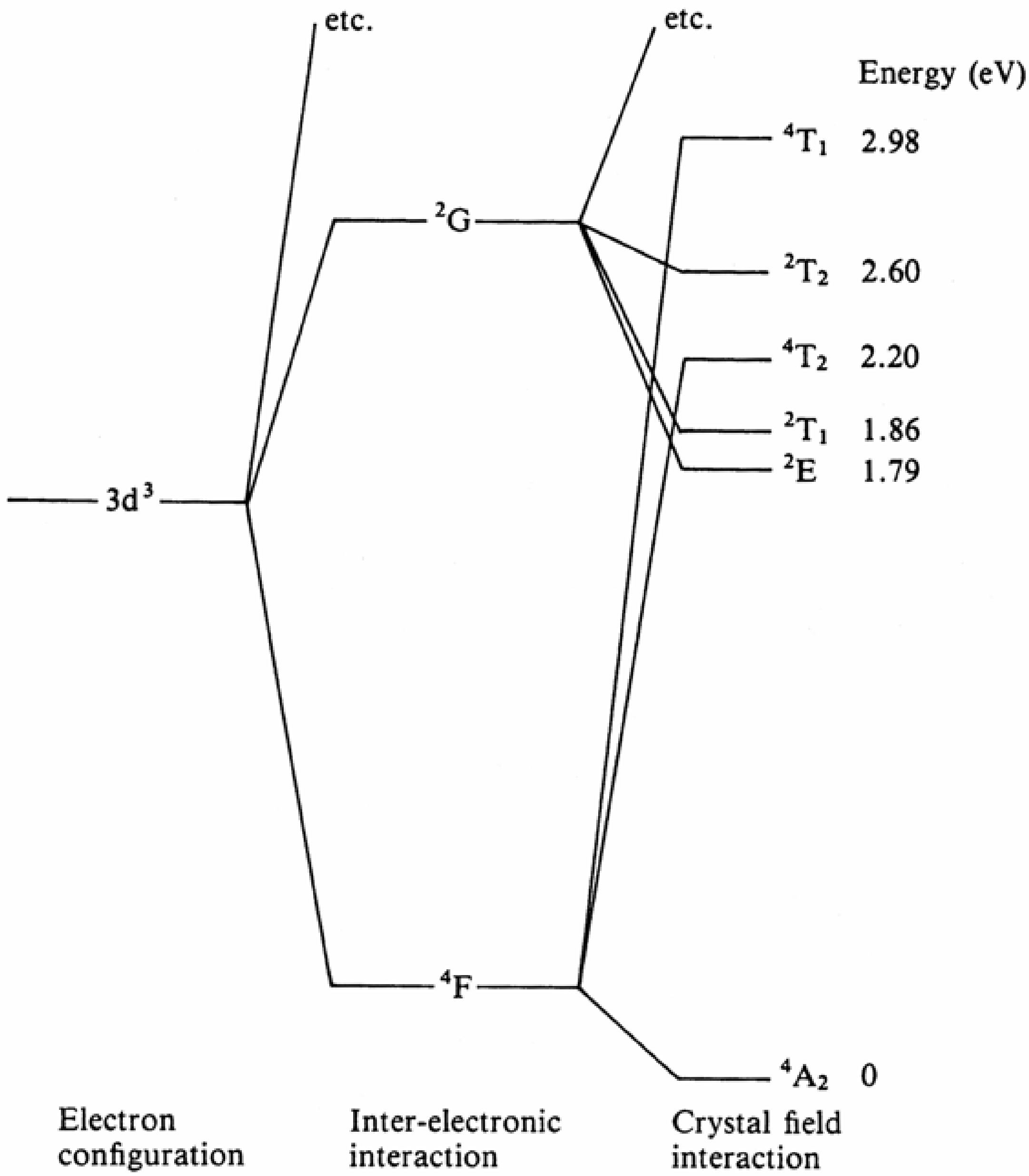



**Fig. 2.1** The origins of the low-lying electronic energy levels in $Cr^{3+}$. Successive interactions split the original $3d^3$ electron configuration into an increasing number of energy levels.