Matrix Methods in Optics

- For more complicated systems use Matrix methods & CAD tools
- Both are based on Ray Tracing concepts
- Solve the optical system by tracing may optical rays
- In free space a ray has position and angle of direction
  y_1 is radial distance from optical axis
  V_1 is the angle (in radians) of the ray
- Now assume you want to a Translation:
  find the position at a distance t further on
- Then the basic Ray equations are in free space
  making the parallex assumption

$$y_2 = y_1 + V_1 t$$

$$V_2 = V_1$$

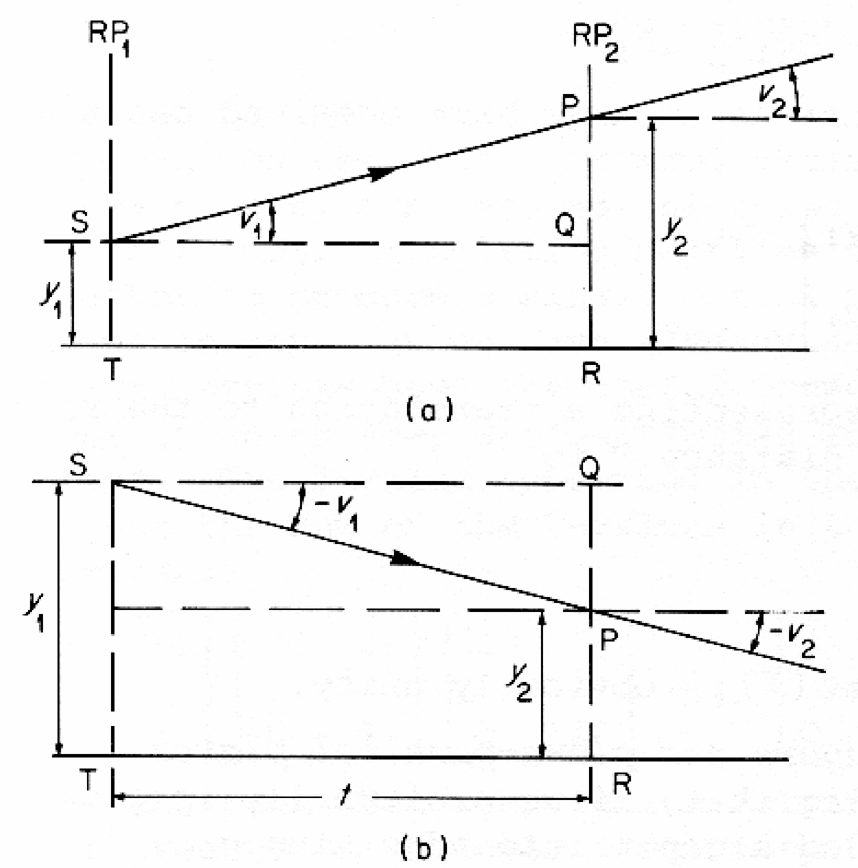


Figure II.2

Matrix Method: Translation Matrix

- Can define a matrix method to obtain the result for any optical process
- Consider a simple translation distance t
- Then the Translation Matrix (or *T* matrix)

$$\begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

- The reverse direction uses the inverse matrix

$$\begin{bmatrix} y_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

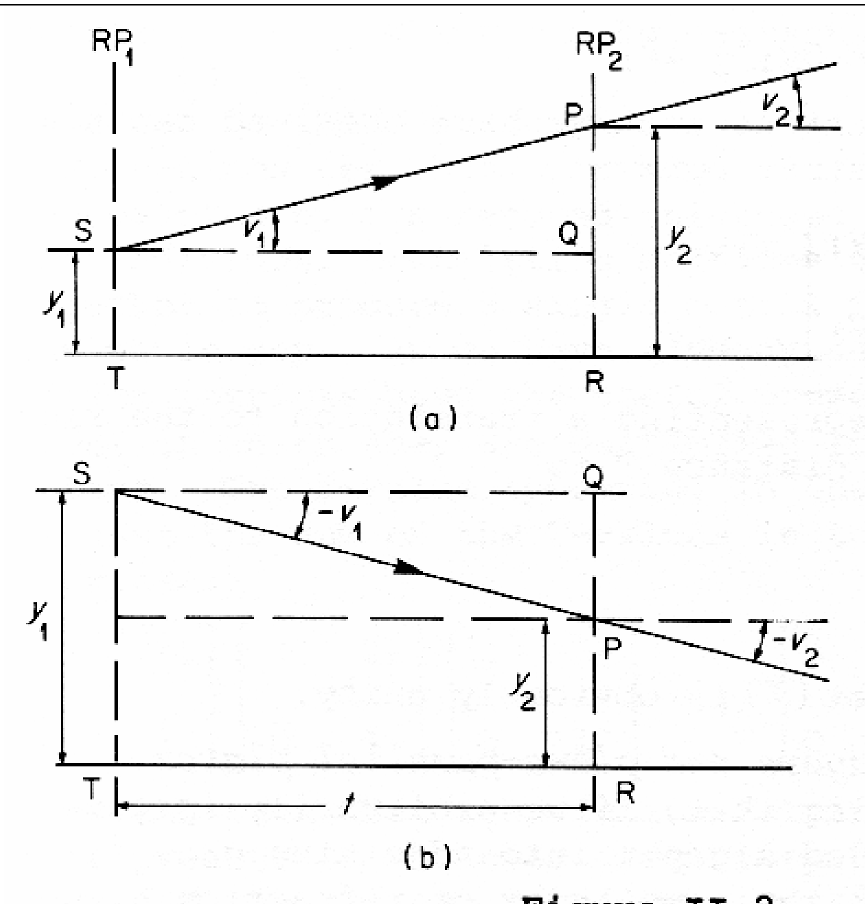
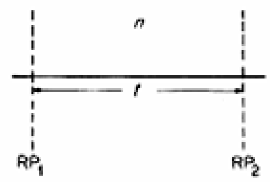
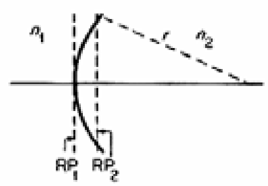
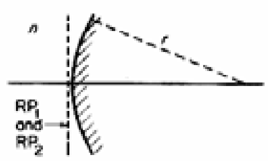
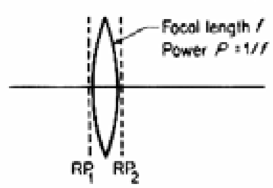


Figure II.2

General Matrix for Optical Devices

- Optical surfaces however will change angle or location
- Example a lens will keep same location but different angle
- Reference for more lens matrices & operations
  A. Gerrard & J.M. Burch,
  "Introduction to Matrix Methods in Optics", Dover 1994
- Matrix methods equal Ray Trace Programs for simple calculations

Table 1

Number	Description	Optical Diagram	Ray-transfer matrix
1	Translation ($\mathcal{T}$-matrix)		$\begin{bmatrix} 1 & t/n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$
2	Refraction at single surface ($\mathcal{R}$-matrix)		$\begin{bmatrix} 1 & 0 \\ \frac{-(n_2 - n_1)}{r} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix}$
3	Reflection at single surface (for convention see section II.11)		$\begin{bmatrix} 1 & 0 \\ \frac{2n}{r} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix}$
4	Thin lens in air (focal length f, power P)		$\begin{bmatrix} 1 & 0 \\ -(n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$

General Optical Matrix Operations

- Place Matrix on the left for operation on the right
- Can solve or calculate a single matrix for the system

$$\begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = [M_{image}][M_{lens}][M_{object}]\begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

$$\begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & s' \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

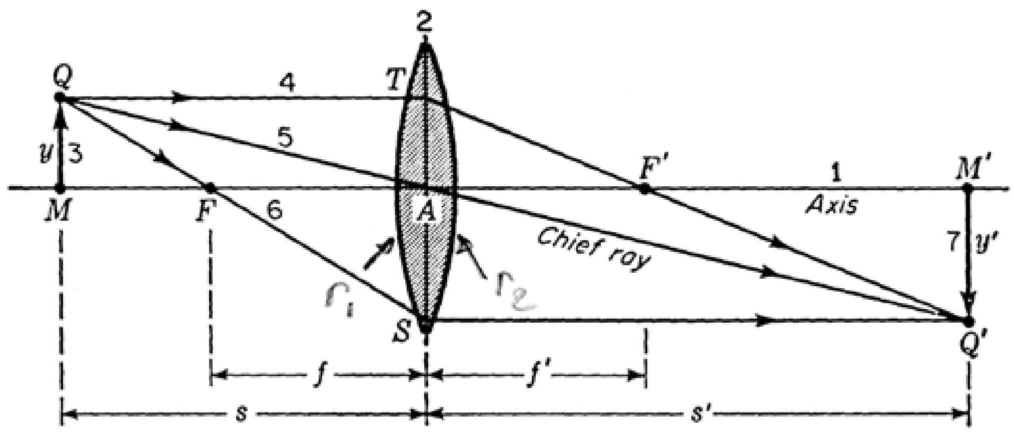


FIGURE 4D
The parallel-ray method for graphically locating the image formed by a thin lens.

Solving for image with Optical Matrix Operations

- For any lens system can create an equivalent matrix
- Combine the lens (mirror) and spacing between them
- Create a single matrix

$$[M_n]\cdots[M_2][M_1]=[M_{system}]=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

- Now add the object and image distance translation matrices

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix}=[M_{image}][M_{lens}][M_{object}]$$

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix}=\begin{bmatrix} 1 & s' \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix}=\begin{bmatrix} A+s'C & As+B+s'(Cs+D) \\ C & Cs+D \end{bmatrix}$$

- Image distance s' is found by solving for $B_s=0$
- Image magnification is

$$m_s=\frac{1}{D_s}$$

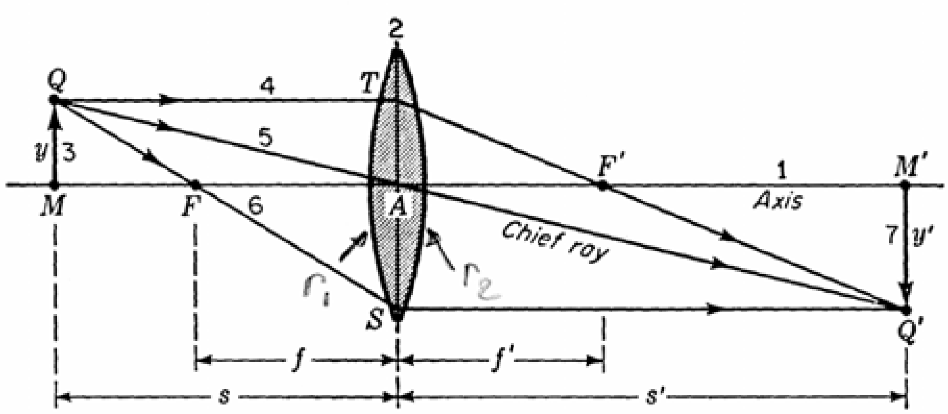


FIGURE 4D
The parallel-ray method for graphically locating the image formed by a thin lens.

Example Solving for the Optical Matrix

- Two lens system: solve for image position and size
- Biconvex lens f_1=8 cm located 24 cm from 3 cm tall object
- Second lens biconcave f_2= -12 cm located d=6 cm from first lens
- Then the matrix solution is

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{12} & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{8} & 1 \end{bmatrix} \begin{bmatrix} 1 & 24 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ \frac{1}{12} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 24 \\ -\frac{1}{8} & -2 \end{bmatrix}$$

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = \begin{bmatrix} 25 - 0.1042X & 12 - X \\ -0.1042 & -1 \end{bmatrix}$$

- Solving for the image position:

$$B_s = 12 - X = 0 \quad or \quad X = 12\ cm$$

- Then the magnification is

$$m = \frac{1}{D_s} = -1$$

- Thus the object is at 12 cm from 2^{nd} lens, -3 cm high

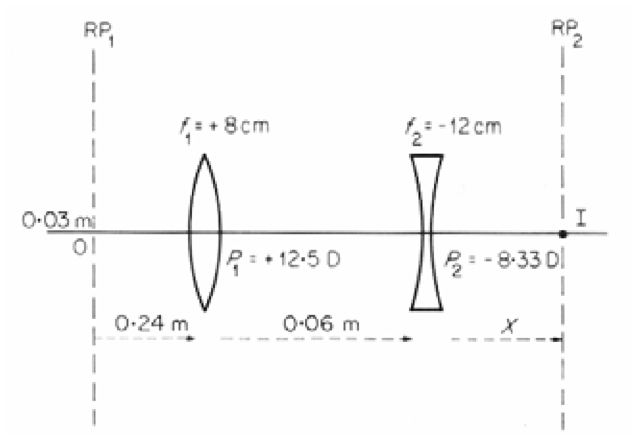


Figure II.14

Matrix Method and Spread Sheets

- Easy to use matrix method in Excel or matlab or maple
- Use mmult array function in excel
- Select array output cells (eg. matrix) and enter =mmult(
- Select space 1 cells then comma
- Select lens 1 cells (eg =mmult(G5:H6,I5:J6))
- Then do control+shift+enter (very important)
- Here is example from previous page

E460 example lesson 6 Distances in cm

Lens Matrix		Lens 2		Matrx 1		Space 1		Lens 1	
		f2	-12			d	6	f1	8
0.25	6	1	0	0.25	6	1	6	1	0
-0.104167	1.5	0.083333	1	-0.125	1	0	1	-0.125	1

second focal length	-1/C	9.6
second focal point	-A/C	2.4

Image		System Matrix S		Lens Matrix		Object	
						d	24
1 X		0.25	12	0.25	6	1	24
0	1	-0.104167	-1	-0.104167	1.5	0	1

Object size	y	3
image distance	=-Bs/Cs	12
Magnification	=1/Ds	-1
Object size	=y/Ds	-3

Optical Matrix Equivalent Lens

- For any lens system can create an equivalent matrix & lens
- Combine all the matrices for the lens and spaces
- The for the combined matrix

where RP_1 = first lens left vertex

RP_2 = last lens right most vertex

n_1=index of refraction before 1st lens

n_2=index of refraction after last lens

System parameter described	Measured From	Measured To	Function of matrix elements	Special case $n_1 = n_2 = 1$
First focal point	RP_1	F_1	$n_1 D/C$	D/C
First focal length	F_1	H_1	$-n_1/C$	$-1/C$
First principal point	RP_1	H_1	$n_1(D-1)/C$	$(D-1)/C$
First nodal point	RP_1	L_1	$(Dn_1 - n_2)/C$	$(D-1)/C$
Second focal point	RP_2	F_2	$-n_2 A/C$	$-A/C$
Second focal length	H_2	F_2	$-n_2/C$	$-1/C$
Second principal point	RP_2	H_2	$n_2(1-A)/C$	$(1-A)/C$
Second nodal point	RP_2	L_2	$(n_1 - An_2)/C$	$(1-A)/C$

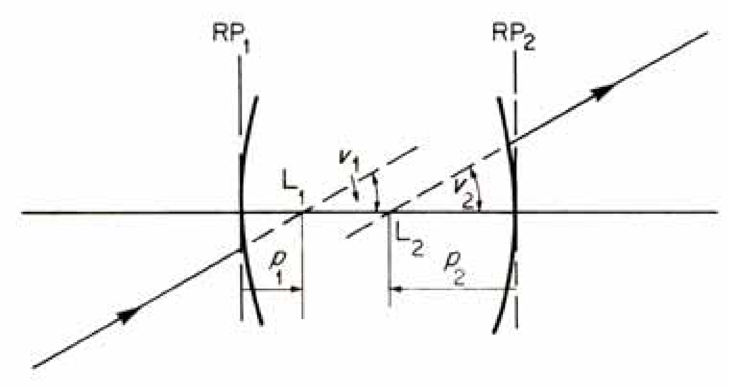


Figure II.17c

Example Combined Optical Matrix

- Using Two lens system from before
- Biconvex lens f_1=8 cm
- Second lens biconcave f_2= -12 cm located 6 cm from f_1
- Then the system matrix is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{12} & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{8} & 1 \end{bmatrix} = \begin{bmatrix} 0.25 & 6 \\ -0.1042 & 1.5 \end{bmatrix}$$

- Second focal length (relative to H_2) is

$$f_{s2} = -\frac{1}{C} = -\frac{1}{-0.1042} = 9.766\ cm$$

- Second focal point, relative to RP_2 (second vertex)

$$f_{rP2} = -\frac{A}{C} = -\frac{0.25}{-0.1042} = 2.400\ cm$$

- Second principal point, relative to RP_2 (second vertex)

$$H_{s2} = \frac{1-A}{C} = \frac{1-0.25}{-0.1024} = -7.198\ cm$$

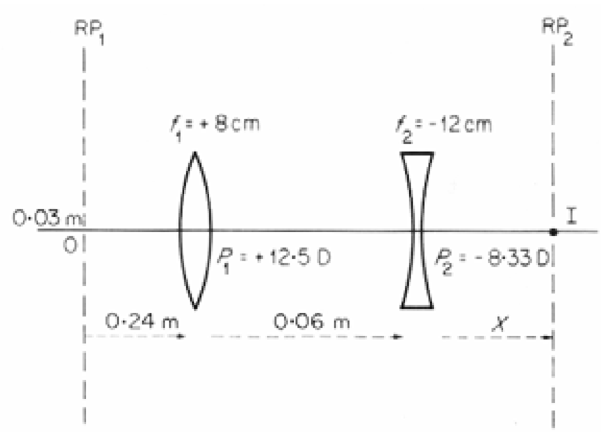


Figure II.14

Gaussian Plane Waves

- Plane waves have flat emag field in x,y
- Tend to get distorted by diffraction into
  spherical plane waves and Gaussian Spherical Waves
- E field intensity follows:

$$u(x,y,R,t) = \frac{U_0}{R} exp\left(i\left[\omega t - Kr - \frac{\left(x^2 + y^2\right)}{2R} \right]\right)$$

where ω= angular frequency = 2πf

U_0 = max value of E field

R = radius from source

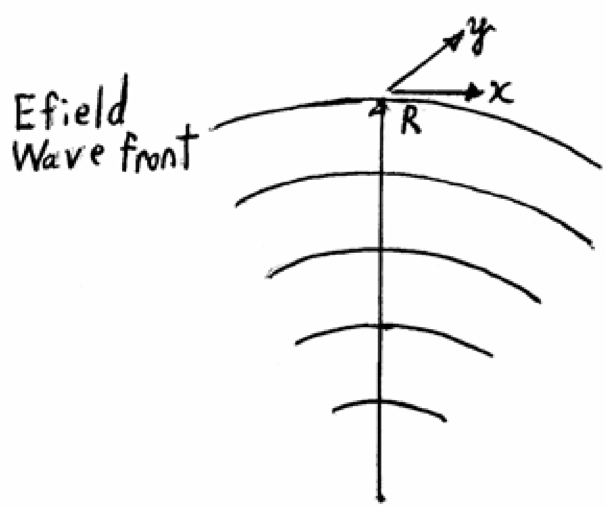
t = time

K= propagation vector in direction of motion

r = unite radial vector from source

x,y = plane positions perpendicular to R

- As R increases wave becomes Gaussian in phase
- R becomes the radius of curvature of the wave front
- These are really TEM_{00} mode emissions from laser



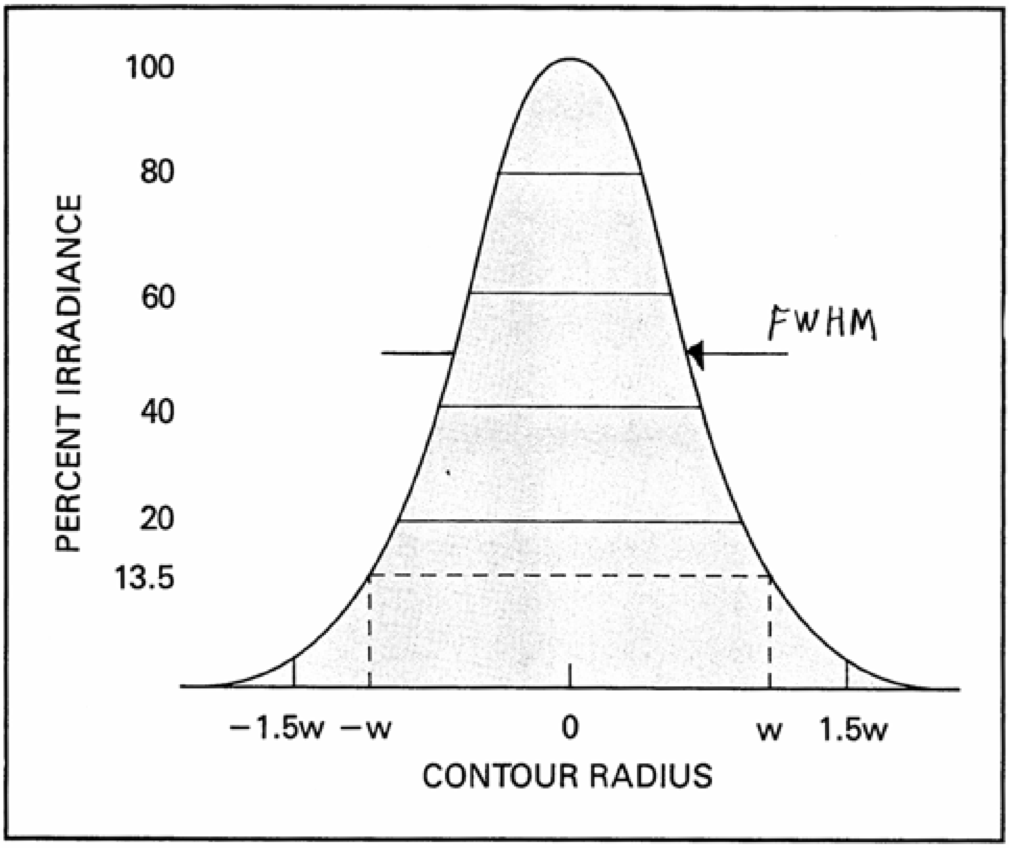
Gaussian Beam

- Assumes a Gaussian shaped beam

$$I(r) = I_0 \exp\left(\frac{-2r^2}{w^2}\right) = \frac{2P}{\pi w^2} \exp\left(\frac{-2r^2}{w^2}\right)$$

Where P = total power in the beam

w = 1/e beam radius at point w(z)



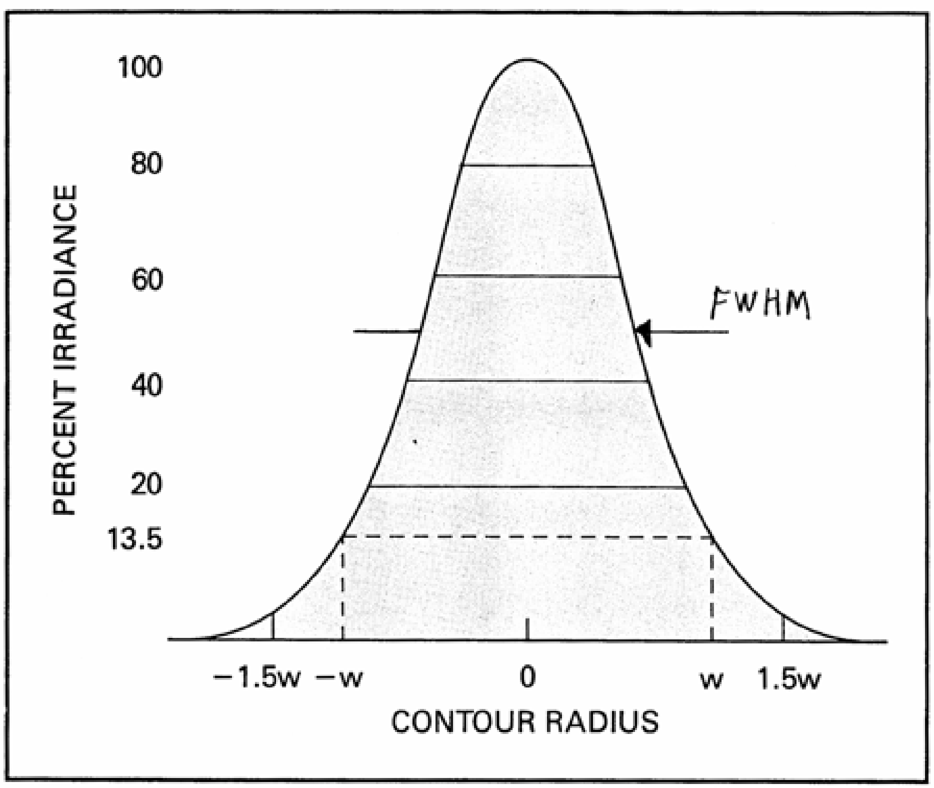
GAUSSIAN IRRADIANCE PROFILE for TEM$_{00}$ mode, showing definitions of beam radius w.

Measurements of Spotsize

- beam spot size is measured in 3 possible ways
- 1/e radius of beam
- $1/e^2$ radius = w(z) of the radiance (light intensity)
  most common laser specification value
  13% of peak power point
  point where emag field down by 1/e
- Full Width Half Maximum (FWHM)
  point where the laser power falls to half its initial value
  good for many interactions with materials
- useful relationship

$$FWHM = 1.386 r_{1/e}$$

$$FWHM = 0.693 r_{1/e^2}$$



GAUSSIAN IRRADIANCE PROFILE for TEM_{00} mode, showing definitions of beam radius w.

Gaussian Beam Changes with Distance

- The Gaussian beam radius of curvature with distance

$$R(z) = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right]$$

- Gaussian spot size with distance

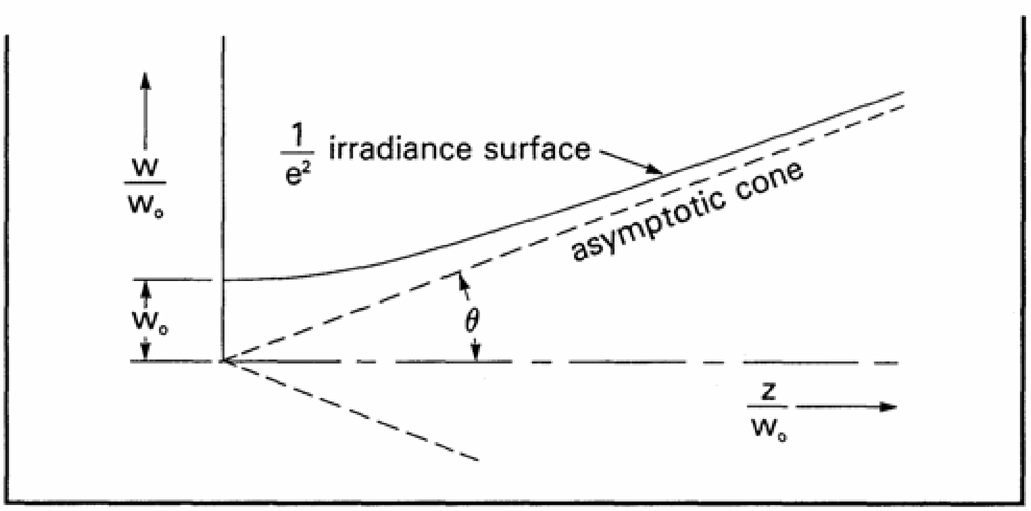
$$w(z) = w_0\left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right]^{1/2}$$

- Note: for lens systems lens diameter must be $3w_0$.= 99% of power
- Note: some books define w_0 as the full width rather than half width
- As z becomes large relative to the beam asymptotically approaches

$$w(z) \approx w_0\left(\frac{\lambda z}{\pi w_0^2}\right)$$

- Asymptotically light cone angle (in radians) approaches

$$\theta \approx \frac{w(z)}{Z} = \left(\frac{\lambda}{\pi w_0}\right)$$



GROWTH IN 1/e² CONTOUR RADIUS with distance propagated away from Gaussian waist.

Rayleigh Range of Gaussian Beams

- Spread in beam is small when width increases < $\sqrt{2}$
- Called the Rayleigh Range z_R

$$z_R = \frac{\pi w_0^2}{\lambda}$$

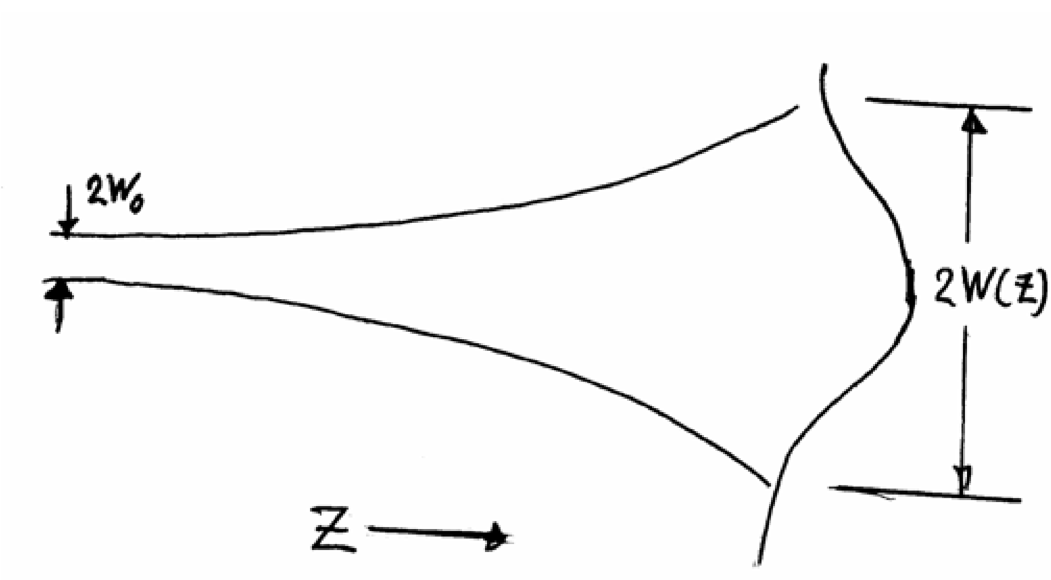
- Beam expands $\sqrt{2}$ for - z_R to $+z_R$ from a focused spot
- Can rewrite Gaussian formulas using z_R

$$R(z) = z\left[1 + \frac{z_R^2}{z^2}\right]$$

$$w(z) = w_0\left[1 + \frac{z^2}{z_R^2}\right]^{1/2}$$

- Again for z >> z_R

$$w(z) \approx w_0 \frac{z}{z_R}$$



Beam Expanders

- Telescope beam expands changes both spotsize and Rayleigh Range
- For magnification **m** of side 2 relative side 1 then as before change of beam size is

$$w_{02} = m w_{01}$$

- Ralyeigh Range becomes

$$z_{R2} = \frac{\pi\, w_{02}^{2}}{\lambda} = m^{2} z_{R1}$$

- where the magnification is

$$m = \frac{f_2}{f_1}$$

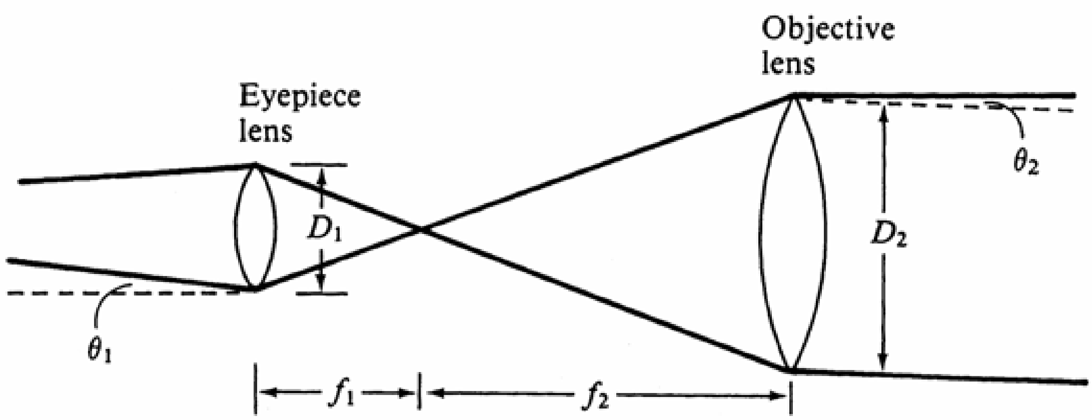


Fig. 3.7 Reduction in beam divergence using a reverse telescope arrangement. With eyepiece and objective lenses having diameters and focal lengths of D_1, f_1 and D_2, f_2 respectively, then the beam width is enlarged by the factor $D_2/D_1 = f_2/f_1$ and the divergence is decreased by the factor f_1/f_2.

Example of Beam Divergence

- eg HeNe 4 mW laser has 0.8 mm rated diameter.
  What is its z_R, spotsize at 1 m, 100 m and the expansion angle
- For HeNe wavelength λ = 632.8 nm
- Rayleigh Range is

$$z_R = \frac{\pi w_0^2}{\lambda} = \frac{\pi (0.0004)^2}{6.328x10^{-7}} = 0.794\, m$$

- At z = 1 metre

$$w(z) = w_0 \left[1 + \frac{z^2}{z_R^2} \right]^{1/2} = 0.0004 \left[1 + \frac{1^2}{0.794^2} \right]^{1/2} = 0.000643\ m = 0.643\ mm$$

- At z = 100 m >> z_R

$$\theta \approx \frac{w(z)}{z} = \frac{\lambda}{\pi w_0} = \frac{6.328x10^{-7}}{\pi 0.0004} = 5.04x10^{-4}\ Radians$$

$$w(z) \approx z\theta = 100(5.04x10^{-4}) = 0.0504\ m = 50.4\ mm$$

- What if beam was run through a beam expander of m = 10

$$w_{02} = m w_{01} = 10(0.0004) = 0.004\ m = 4\ mm$$

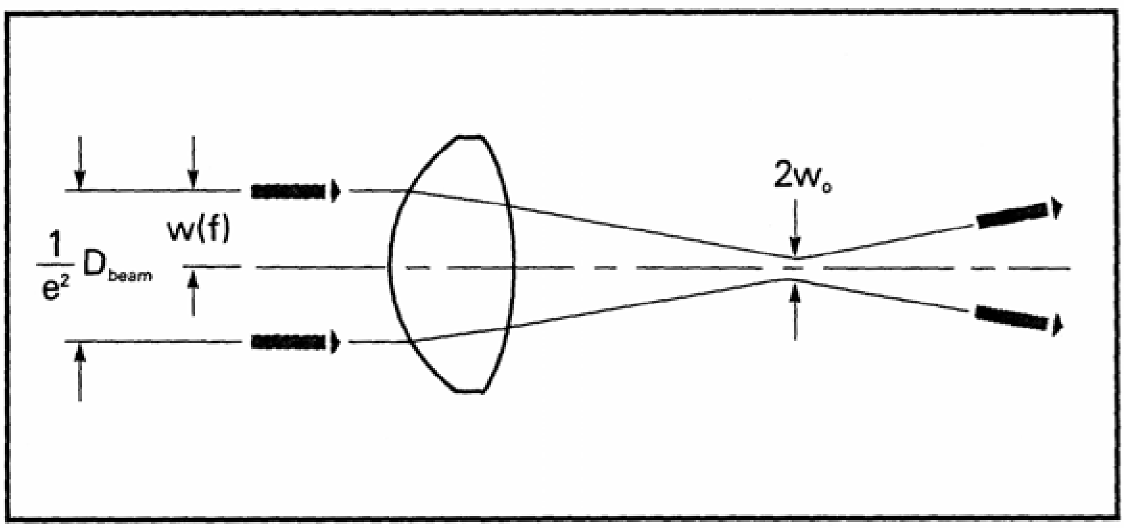
$$\theta_m = \frac{\theta}{m} = \frac{5.04x10^{-4}}{10} = 5.04x10^{-5}\ Radians$$

$$w(z) \approx z\theta = 100(5.04x10^{-5}) = 0.00504\ m = 5.04\ mm$$

- Hence get a smaller beam at 100 m by creating a larger beam first

Focused Laser Spot

- Lenses focus Gaussian Beam to a Waist
- Modification of Lens formulas for Gaussian Beams
- From S.A. Self "Focusing of Spherical Gaussian Beams"
  App. Optics, pg. 658. v. 22, 5, 1983
- Use the input beam waist distance as object distance s
  to primary principal point
- Output beam waist position as image distance s"
  to secondary principal point



CONCENTRATION OF LASER BEAM by a laser line focusing singlet. Size of the focal waist has been greatly exaggerated for illustrative purposes.

Gaussain Beam Lens Formulas

- Normal lens formula in regular and dimensionless form

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f} \quad or \quad \frac{1}{\left(\frac{s}{f}\right)}+\frac{1}{\left(\frac{s'}{f}\right)}=1$$

- This formula applies to both input and output objects
- Gaussian beam lens formula for input beams
  includes Rayleigh Range effect

$$\frac{1}{\left(s+\frac{z_R^2}{s-f}\right)}+\frac{1}{s'}=\frac{1}{f}$$

- in dimensionless form

$$\frac{1}{\left(\frac{s}{f}\right)+\frac{\left(\frac{z_R}{f}\right)^2}{\left(\frac{s}{f}-1\right)}}+\frac{1}{\left(\frac{s'}{f}\right)}=1$$

- in far field as z_R goes to 0
  (ie spot small compared to lens)
  this reduces to geometric optics equations

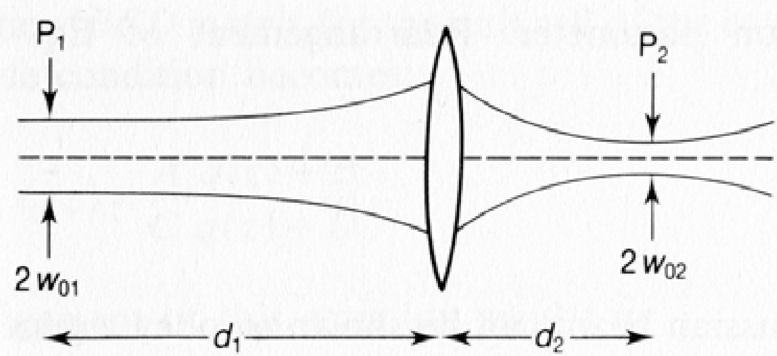
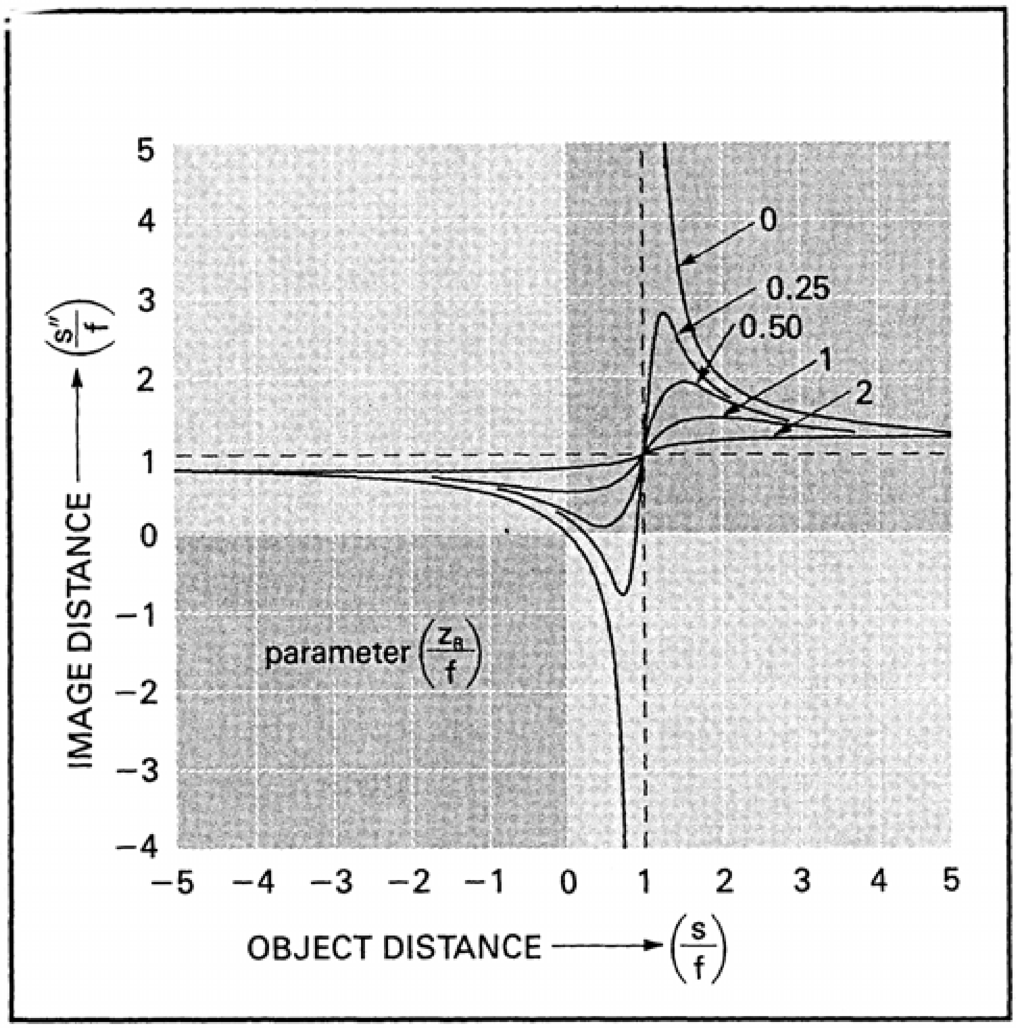


Figure 4.23 Focusing of a Gaussian beam by a convex lens.

Gaussain Beam Lens Behavior

- Plot shows 3 regions of interest for positive thin lens
- Real object and real image
- Real object and virtual image
- Virtual object and real image



PLOT OF THE LENS FORMULA for Gaussian beams showing normalized image distance vs normalized object distance, with normalized Rayleigh range of the input beam as the parameter.

Main Difference of Gaussian Beam Optics

- For Gaussian Beams there is a maximum and minimum image distance
- Maximum image not at s = f instead at

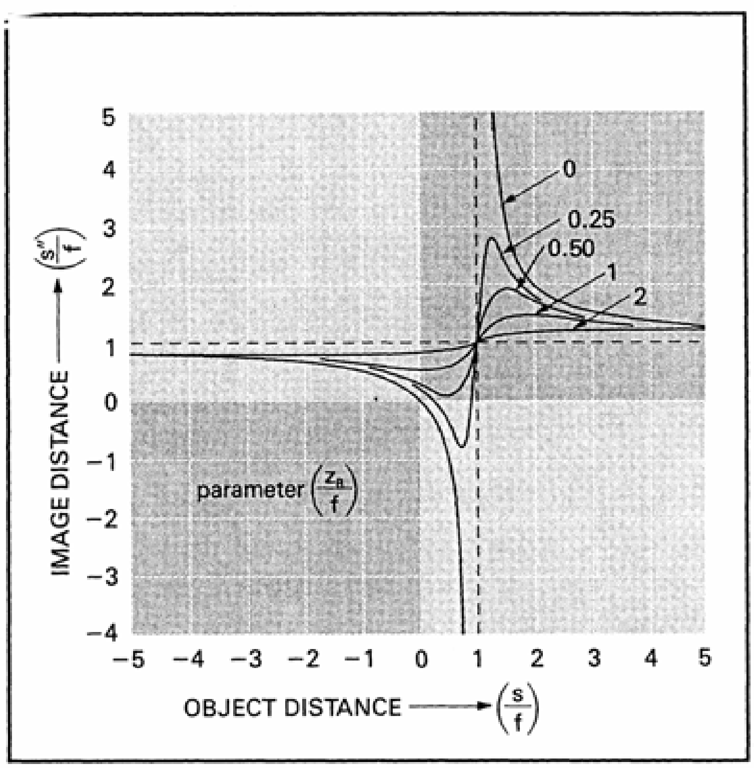
$$s = f + z_R$$

- There is a common point in Gaussian beam expression at

$$\frac{s}{f} = \frac{s''}{f} = 1$$

For positive lens when incident beam waist at front focus then emerging beam waist at back focus

- No minimum object-image separation for Gaussian
- Lens f appears to decrease as z_R/f increases from zero i.e. Gaussian focal shift



PLOT OF THE LENS FORMULA for Gaussian beams showing normalized image distance vs normalized object distance, with normalized Rayleigh range of the input beam as the parameter.

Magnification and Output Beams

- Calculate z_R and w_0, s and s'' for each lens
- Magnification of beam

$$m = \frac{w_0''}{w_0} = \frac{1}{\left\{\left[1-\left(\frac{s}{f}\right)\right]^2 + \left(\frac{z_R}{f}\right)^2\right\}^{1/2}}$$

- Again the Rayleigh range changes with output

$$z_R'' = m^2 z_R$$

- The Gaussian Beam lens formula is not symmetric
  From the output beam side

$$\frac{1}{s} + \frac{1}{\left[s'' + \frac{z_R''^2}{(s''-f)}\right]} = \frac{1}{f}$$

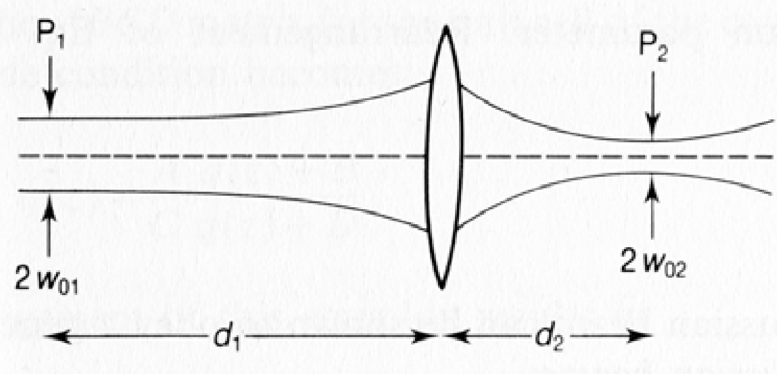


Figure 4.23 Focusing of a Gaussian beam by a convex lens.

Special Solution to Gaussian Beam

- Two cases of particular interests

Input Waist at First Principal Surface

- s = 0 condition, image distance and waist become

$$s'' = \frac{f}{1 + \left(\frac{f}{z_R}\right)^2}$$

$$w'' = \frac{\frac{\lambda f}{\pi w_0}}{\left[1 + \left(\frac{f}{z_R}\right)^2\right]^{1/2}}$$

Input Waist at First Focal Point

- s = f condition, image distance and waist become

$$s'' = f$$

$$w'' = \frac{\lambda f}{\pi w_0}$$

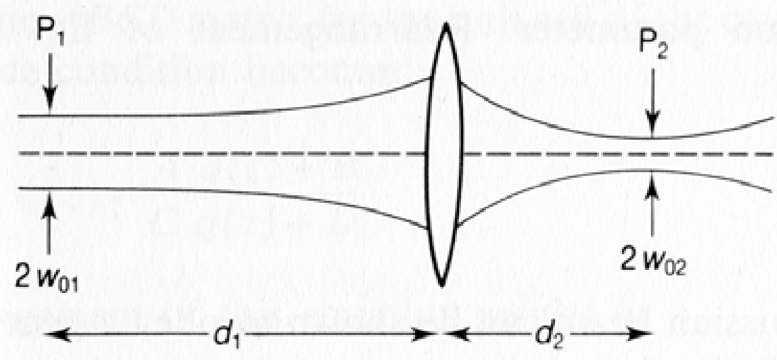


Figure 4.23 Focusing of a Gaussian beam by a convex lens.

Gaussian Spots and Cavity Stability

- In laser cavities waist position is controlled by mirrors
- Recall the cavity g factors for cavity stability

$$g_i = 1 - \frac{L}{r_i}$$

- Waist of cavity is given by

$$w_0 = \left(\frac{\lambda L}{\pi} \right)^{1/2} \left\{ \frac{[g_1 g_2 (1 - g_1 g_2)]}{[g_1 + g_2 - 2 g_1 g_2]^2} \right\}^{1/4}$$

where i=1=back mirror, i=2= front

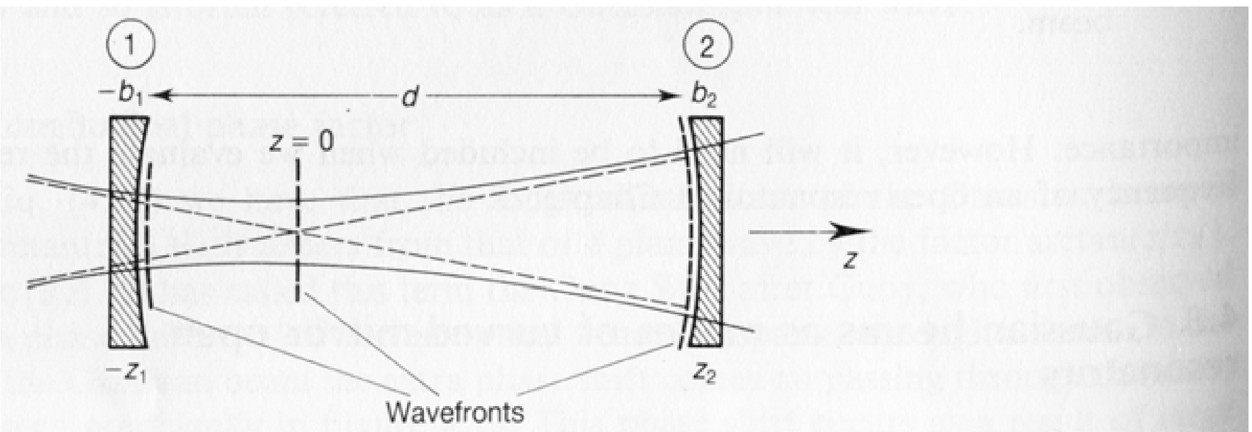


Figure 4.15 Schematic diagram of an open resonator showing the wavefront curvatures matching the mirror surfaces at the two concave mirrors. The waist occurs somewhere between them.

Gaussian Waist within a Cavity

- Waist location relative to output mirror for cavity length L is

$$z_2 = \frac{g_1(1-g_2)L}{g_1+g_2-2g_1g_2}$$

$$w_2 = \left(\frac{\lambda L}{\pi}\right)^{1/2}\left\{\frac{g_2}{g_1[1-g_1g_2]}\right\}^{1/4} \qquad w_1 = \left(\frac{\lambda L}{\pi}\right)^{1/2}\left\{\frac{g_1}{g_2[1-g_1g_2]}\right\}^{1/4}$$

- If $g_1 = g_2 = g = 0$ (i.e. r = L) waist becomes

$$w_0 = \left(\frac{\lambda L}{2\pi}\right)^{1/2} \qquad z_2 = 0.5$$

- If $g_1 = 0$, $g_2 = 1$ (curved back, plane front)
  waist is located $z_2 = 0$ at the output mirror
  (common case for HeNe and many gas lasers)
- If $g_1 = g_2 = 1$ (i.e. plane mirrors) there is no waist

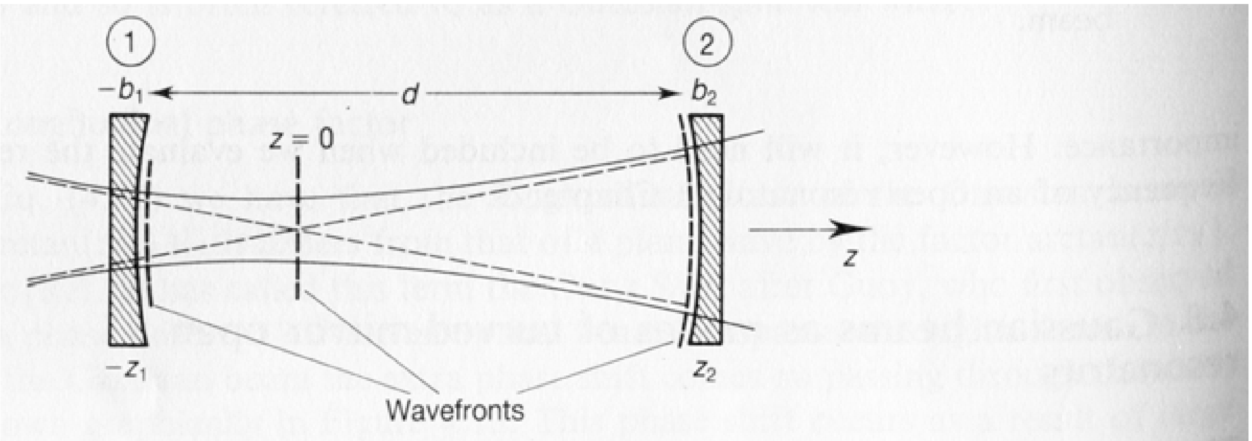


Figure 4.15 Schematic diagram of an open resonator showing the wavefront curvatures matching the mirror surfaces at the two concave mirrors. The waist occurs somewhere between them.